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Super-Twisting Sliding Mode Control for the Trajectory Tracking of Underactuated USVs with Disturbances

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Abstract: For an underactuated unmanned surface vehicle (USV), time-varying external disturbances affect the accuracy of trajectory tracking. To ensure trajectory tracking accuracy, in this paper the reduced-order extended state observer (ESO) and the super-twisting second-order sliding mode controller are adopted. The ESO is designed to address the unknown time-varying sideslip angle in the guidance law. Additionally, the super-twisting technology contributes to a reduced chattering effect of the sliding mode. The Lyapunov method is used to analyze the stability of the tracking system and prove that the proposed controllers can ensure the convergence of the tracking errors in finite time. The simulation experiments of the proposed super-twisting sliding mode control (STSMC) and adaptive sliding mode control (ASMC) methods are compared. The results show that the STSMC method enables the USV to complete the task of the reference trajectory tracking. The chattering of the STSMC is also significantly reduced compared to that of the ASMC.

Keywords: unmanned surface vehicle; trajectory tracking; super-twisting sliding mode

1. Introduction

The unmanned surface vessel (USV) is a small surface platform with the ability of autonomous planning, autonomous navigation, and the ability to independently complete environmental awareness, target detection, and other tasks. Compared with traditional marine ships, USVs are smaller in size, lower in cost, stronger in mobility, and more intelligent. They can perform dangerous tasks without the risk of casualties, and are widely used in military and civil fields. In the military field, USVs can be equipped with communication systems, weapon systems, and sensor systems to perform border patrol, target surveillance, mine clearance, and anti-submarine tasks, greatly improving the military’s combat capability. In the civil field, the USV can carry out fish detection, hydrometeorological detection, map mapping, and other work after being equipped with the corresponding equipment. At the same time, the high mobility and efficiency of the USV allows it to carry out marine survey operations, environment monitoring, underway replenishment, cooperative search and rescue, and other works.

The underactuated unmanned surface vessel, as a typical underactuated system, is disturbed by winds, waves, and currents in the absence of the control input in the sway. The underactuated system means that the number of independent control variables of the system is less than the number of degrees of freedom of the system. This greatly increases the difficulty of control for the underactuated USV. Therefore, it has an important theoretical significance and application value to seek an effective control strategy to realize the trajectory tracking of the underactuated USV.

The purpose of trajectory tracking is to force the vehicle to follow a time-varying trajectory. The trajectory tracking is considered and solved as a stabilization problem for tracking error equations by direct strategies [1]. For indirect strategies, the first step is to design the desired surge, sway speed, or yaw angle with the guidance laws. Then, by the
control methods, the actual input control laws are developed so that the surge, sway speed, or yaw angle follow the desired variables. For the underactuated USV, since the desired sway speed is not forced by the control input directly, it is difficult to guarantee the correct course of the USV [2]. Therefore, a guidance law for the surge speed and yaw angle should be developed. The yaw angle and the surge speed can be controlled directly by the control inputs [3,4].

At present, regardless of the types of trajectory tracking strategy, many research results have been achieved on the USV trajectory tracking by various control methods, including model predictive control [5,6], backstepping [7], dynamic surface control [8], sliding mode control [9], prescribed performance control [1], neural networks [10], and so on. Because of the complex nature of the environment, controllers of USVs have to be robust against different unknown external disturbances such as winds, waves, and currents. The sliding mode technique, regarded as a remarkable robust control method, has been widely used because of its fast response speed and strong robustness.

However, one disadvantage of a conventional SMC is that the tracking error converges to zero asymptotically. The asymptotic convergence of the desired variable error may make it impossible to achieve the convergence of tracking errors. To achieve the convergence of a desired variable error in a finite time, terminal sliding mode control (TSMC) [11] and fast TSMC [12] are adopted in the design of controllers for linear and nonlinear systems. However, one problem of these methods is a singularity that appears when the system state is close to zero. Fortunately, the nonsingular TSMC [13] and nonsingular fast TSMC [14] were proposed and they tackle the singularity problem for trajectory tracking. Although the nonsingular fast TSMC realizes the fast convergence, while the avoiding singularity problem, it still needs conservatively large control inputs.

Except for the fact that a tracking error converges slowly, a conventional SMC has the chattering phenomenon. By using a high-order sliding mode algorithm, the chattering can be suppressed and the control accuracy can be improved. The super-twisting method is a kind of high-order sliding mode algorithm with a simple structure [15], which can effectively suppress chattering and only requires first-order sliding mode information of the system [16]. This means that there is no need to calculate the high-order derivative of the sliding surface.

Therefore, the super-twisting method is a better sliding mode method to reduce chattering and meet the finite-time stability requirements. Except for unmanned ground vehicles (UGVs) [17] and unmanned aerial vehicles (UAVs) [18], the super-twisting algorithm is also adopted for controlling maritime autonomous vessels. The roll suppression of marine vessels subjected to harmonic wave excitations was investigated using an adaptive sliding mode control with a super-twisting algorithm to reduce the chattering phenomenon and guarantee roll control accuracy [19]. Furthermore, a combined model predictive super-twisting sliding mode control algorithm was proposed for tracking the trajectory of an autonomous surface vehicle in the presence of the time-varying external disturbances [20]. In [21], the tracking control method based on a super-twisting sliding mode is proposed for a USV. However, the virtual control laws for the desired surge and sway speeds are designed. The desired sway velocity is not suitable for designing a heading controller, since the complex problem of differential calculation is brought into the controller. In [22], an improved super-twisting sliding mode control algorithm is proposed for ship heading control, while a sideslip angle compensation is calculated by the estimations of the surge and sway speeds with a finite-time extended state observer.

Aiming at the trajectory tracking problem of an underactuated USV subject to unknown time-varying external disturbances, an effective method is to use a disturbance-observer methodology. In order to eliminate the influence of external disturbance in the navigation of a USV, a finite-time disturbance observer [23] is proposed to estimate the unknown external disturbances. However, the sliding mode control method is not sensitive to external disturbances, and the anti-disturbance characteristics of the method are realized by a large coefficient of the discontinuous term in the control law. To avoid the difficulty
in the actuator control input caused by a large coefficient, the adaptive gain technique is considered [9] when the upper bound of the disturbance is known. In [24], an adaptive integral terminal sliding mode technique is designed against bounded disturbances, and it allows a finite-time convergence of the tracking error.

To design better guidance laws for precise tracking, the drift angle has to be considered. The drift, or sideslip angle, is the angle between the heading angle and the course angle. The sideslip angle can be attributed to the sway velocity component and is caused by the lateral acceleration while turning [25]. The effect of wind and wave currents on the USV increases the variation of the sideslip angle. Therefore, if the navigation direction of the USV is accurately guided, the desired trajectory can be accurately tracked. To compensate for the sideslip angle in the guidance law, the most straightforward way is to measure it by means of sensors and add it to the guidance law. However, in many cases, it is difficult to measure sideslip angle accurately with cheap optical correlation sensors [25]. Or, the sideslip angle is calculated with the velocities of the USV according to the formula. There are other ways to alleviate the effects of sideslip angles. The integral LOS (ILOS) guidance method was presented [26], and the influence of a sideslip angle on the USV was eliminated by adaptive law [27]. Indirect adaptive control methods were proposed. For example, two predictors were developed for the estimation of the tracking error and were proposed to indirectly estimate the sideslip angle [28]. In [23–26], it is assumed that the sideslip angle is quite small (less than 5°), constant, or slowly varying. However, when the USV is disturbed by time-varying disturbances, the sideslip angle is greater than 10°. In this case, the approximate method is unreasonable. Therefore, the accurate direct estimation of the time-varying sideslip angle is very important for USV accuracy tracking.

Inspired by the above methods, in this paper, the main purpose is to design the controller for the trajectory tracking of an underactuated USV in the presence of external disturbances. The main contributions of this paper can be briefly summarized as follows:

1. A guidance law for the desired surge speed and yaw angle is proposed to obtain a simpler control structure and ensure the correct heading angle and surge speed. In the guidance law, the estimation of the time-varying sideslip angle is considered to guarantee tracking accuracy.

2. The design of surge speed and heading angle controllers uses the super-twisting second-order sliding mode method. The convergence of the tracking error is analyzed by the Lyapunov stability theory, and it is proved that the proposed controller can ensure the tracking error converges in a finite time.

3. The comparison of the proposed super-twisting sliding mode control method and adaptive sliding mode control methods is conducted in the simulation. Under the condition that the upper bound of the disturbances is known, results show that the STSMC method has a smaller tracking error and better tracking accuracy.

The paper is organized as follows: Section 2 gives preliminaries and problem formulation. In Section 3, the LOS guidance law with the reduced-order extended state observer and the proof of the convergence of trajectory tracking errors are provided. In Section 4, the procedure of the finite-time super-twisting sliding mode controller is developed, and the finite-time convergence of tracking error is proved. Simulation studies and comparisons are conducted in Section 5. Section 6 concludes this paper.

2. Problem Formulation

The body-fixed frame and earth-fixed inertial frame are two common coordinate systems for the motion control of ship, as shown in Figure 1. They can be used to describe the motion and attitude of the USVs. The earth-fixed inertial frame is used to describe the positional state of the USVs, and the body-fixed frame is used to describe the linear velocity and angular rate of the USVs [29]. The center of gravity is consistent with the origin of the body-fixed frame. \( O_bX_b \) represents the longitudinal axis from the stern to the bow, \( O_bY_b \) represents the lateral axis pointing to the starboard, and \( O_bZ_b \) represents the vertical axis pointing to the center of the earth \( O_E \). Similarly, \( O_EX_E \) is the north, \( O_EY_E \) is the east, and
**Table 1. Six-DOF motion symbols of the USVs.**

<table>
<thead>
<tr>
<th>Name of the Motion State</th>
<th>Linear Velocity and Angular Rate</th>
<th>Position and Euler Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surge</td>
<td>u</td>
<td>x</td>
</tr>
<tr>
<td>Sway</td>
<td>v</td>
<td>y</td>
</tr>
<tr>
<td>Heave</td>
<td>w</td>
<td>z</td>
</tr>
<tr>
<td>Roll</td>
<td>p</td>
<td>φ</td>
</tr>
<tr>
<td>Pitch</td>
<td>q</td>
<td>θ</td>
</tr>
<tr>
<td>Yaw</td>
<td>r</td>
<td>ψ</td>
</tr>
</tbody>
</table>

For the trajectory tracking problem on the horizontal plane, the heave, pitch, and roll motions are neglected. In addition, the three-degrees-of-freedom (3-DOF) mathematical model for the USV is simplified by applying the following assumptions:

**Assumption 1.** The whole vehicle is a rigid body with a homogeneous mass distribution and a symmetrical shape and structure.

**Assumption 2.** The origin of the body-fixed frame is located at the center of gravity of the vehicle.

**Assumption 3.** The off-diagonal terms of inertial and drag matrices are smaller than the main diagonal terms and can be neglected.

Additionally, considering external environmental disturbances, the kinematic equations and the dynamic equations for an USV can be described as follows [4]:

\[
\begin{aligned}
\dot{x} &= u \cos \psi + v \sin \psi \\
\dot{y} &= u \sin \psi - v \cos \psi \\
\dot{\psi} &= r \\
\dot{u} &= \frac{1}{m-x_p} \left[ (m-Y_v)\omega r + X_u|u|u + X_u + \tau_u \right] + \delta(u) \\
\dot{v} &= \frac{1}{m-y_p} \left[ (-m+X_u)ur \right] \frac{1}{m-y_p} \left[ Y_v|v|v + Y_v \right] + \delta(v) \\
\dot{r} &= \frac{1}{l_z-N_\psi} \left[ (-X_u + Y_v)ur \right] \frac{1}{l_z-N_\psi} \left[ N_\psi|r|r + N_\psi + \tau_r \right] + \delta(r)
\end{aligned}
\]

where the inertial coordinates of the USV are \( x \) and \( y \). The yaw angle is \( \psi \). The surge and sway velocities are \( u \) and \( v \), respectively, and the yaw velocity is \( r \). The control inputs are \( \tau_u \) and \( \tau_r \). \( X_u, Y_v, N_\psi, X_{u|u|u}, Y_{v|v|v}, X_{u|u|u}, Y_{v|v|v}, \) and \( N_{\psi|\psi} \) are the constant hydrodynamic coefficients and are computed by the equations [30]. \( \delta(u), \delta(v), \) and \( \delta(r) \) are the external disturbances. Obviously, there is no control force in the sway direction; therefore, the USV model is an underactuated system. The external disturbances \( \delta(u), \delta(v), \) and \( \delta(r) \) are continuous and differentiable, and there is a constant \( L_\delta \) such that \( \| \delta \| < L_\delta \).

In addition, Equation (2) represents the vectors of forces and moments generated by the two thrusters \( T_{\text{port}} \) and \( T_{\text{stbd}} \).
\[
\begin{cases}
\tau_u = T_{\text{port}} + \alpha_T T_{\text{stbd}} \\
\tau_r = (T_{\text{port}} - \alpha_T T_{\text{stbd}}) B / 2
\end{cases}
\]  

(2)

where \( B = 0.75 \) is the overall beam, \( \alpha_T = 0.78 \) multiplies the starboard thruster to assist a mechanical constrain of the custom vehicle [4].

According to the USV system (1), the control objective of trajectory tracking is to design control laws \( u = [\tau_u, \tau_r]^T \), which ensure that the USV tracks the desired trajectory \( p_d = [x_r(t), y_r(t)]^T \). Because of complex algorithms and calculations by the direct control strategy, in this paper, the indirect strategy is introduced, that is, the guidance law is designed to obtain the desired system variables and the controllers are used to track these desired variables. For double-propelled USVs, considering Equation (2), the practical control objective is to design control laws for the two thrusters, \( T_{\text{port}} \) and \( T_{\text{stbd}} \), which in a finite time forces the variables, \( u \) and \( \psi \), of the USV (1) to converge to the desired \( u_d \) and \( \psi_d \) obtained from the proposed guidance law. Thereby, the tracking control laws are designed and they ensure that the USV tracks a desired, time-varying, and smooth trajectory. Control objectives can be expressed as

\[
\begin{align*}
\lim_{t \to T} |x_r(t) - x(T_{\text{port}}(t), T_{\text{stbd}}(t))| &= 0 \\
\lim_{t \to T} |y_r(t) - y(T_{\text{port}}(t), T_{\text{stbd}}(t))| &= 0
\end{align*}
\]

where \( T \) is the settling time and \( T < \infty \). Moreover, a diagram of the guidance–control system is illustrated in Figure 2.

**Figure 2.** Guidance–control scheme.

### 3. Trajectory Tracking Guidance Law

For a USV located at the coordinate point \((x, y)\) and the reference trajectory \( p_d = [x_r(t), y_r(t)]^T \). The reference trajectory \( p_d \) is a known function and the first order continuous differentiable. The along-track error \( x_e \) and the cross-track error \( y_e \) can be defined in the path tangential reference frame as follows:

\[
\begin{bmatrix}
x_e \\
y_e
\end{bmatrix} =
\begin{bmatrix}
\cos \psi_r & -\sin \psi_r \\
\sin \psi_r & \cos \psi_r
\end{bmatrix}
\begin{bmatrix}
x - x_r \\
y - y_r
\end{bmatrix}
\]

(3)

where variables \( \psi_r = \arctan2(y_r'(t), x_r'(t)) \) denote the tangent angle of the trajectory, and \( \psi_r \in [-\pi, \pi] \). \( \arctan2(\cdot) \) is a function that returns the angle that has a tangent that is the ratio of two variables.
where variable \( g \) which contains the unknown sideslip angle \( \beta \).

Lemma 1. Subsystem of (3), the detailed proof is provided. Additionally, the relationship between the bound of this lemma, which satisfies \( |\hat{g}| \leq g^* \) and \( g^* \) is a positive constant. The initial values of (5) are set as \( p(t_0) = -k_o y_e(t_0) \) and \( \hat{g}(t_0) = 0 \).

At present, with the support of the ESO, the estimation of the sideslip angle \( \hat{\beta} \) can be calculated as

\[
\hat{\beta} = \arctan\left( \frac{\hat{g}}{u \cos(\psi - \psi_r)} \right)
\]

Additionally, the estimation error of \( g \) is defined as \( \tilde{g} = \hat{g} - g \). Taking the derivative of \( \tilde{g} \)

\[
\dot{\tilde{g}} = \dot{\hat{g}} - \dot{g} = p + k_o y_e - \dot{g} = -k_o p - k_o^2 y_e - k_o \left( u \sin(\psi - \psi_r) - x_e \psi_r \right) - \dot{g} = -k_o \tilde{g} - \dot{g}
\]

Remark 1. For the detailed proof of Lemma 1, using the Lyapunov function \( V = \frac{1}{2} \tilde{g}^2 \). In Section 3 of [31], the detailed proof is provided. Additionally, the relationship between the bound of this
error and the bandwidth of the reduced-order ESO is given. The estimation error \( \tilde{g} \) can be tuned arbitrarily close to zero by increasing the bandwidth of the ESO.

So far, the desired variables \( u_d \) and \( \psi_d \) are designed to make the tracking errors \( x_e \) and \( y_e \) converge to zero.

**Proposition 1.** Let the desired yaw angle and surge speed be such that

\[
\begin{align*}
\psi_d &= \psi_r + \arctan(-y_e, \Delta) - \hat{\beta} \\
u_d &= \sqrt{U_d^2 - v^2}
\end{align*}
\]

where the desired total speed \( U_d = \frac{(U_r - k_{xx}x_e)\sqrt{\dot{\psi}_e^2 + \Delta^2}}{\Delta} \), is the look-ahead distance, \( k_x \) is the positive controller gain.

**Lemma 2.** If the surge speed error \( \epsilon_u = u_d - u \) and the yaw angle error \( \epsilon_\psi = \psi_d - \psi \) converge to zero, the convergence of the position error \( x_e \) and \( y_e \) is guaranteed.

**Proof.** Consider the following Lyapunov function candidate,

\[ V_1 = \frac{1}{2} x_e^2 + \frac{1}{2} y_e^2 \]

By differentiating and substituting (4) with \( \psi = \psi_d \) and \( u = u_d \) into it, it yields the following:

\[ \dot{V}_1 = x_e \dot{x}_e + y_e \dot{y}_e = -k_x x_e^2 - \frac{U}{\sqrt{\dot{\psi}_e^2 + \Delta^2}} y_e^2 \]

By setting \( k_x > 0 \), it is obvious that \( \dot{V}_1 \leq 0 \). Therefore, it can be concluded that \( x_e \) and \( y_e \) converge to zero. \( \Box \)

### 4. Trajectory Tracking Control Law

In the previous section, the trajectory tracking guidance law is put forward by Equation (6). In this section, our mission is to design the trajectory tracking control laws, \( \tau_u \) and \( \tau_r \), which make the USV surge speed \( u \) and yaw angle \( \psi \) converge to the desired variables, \( u_d \) and \( \psi_d \), respectively. It is seen in Equation (1), that \( \delta(u) \), \( \delta(v) \), and \( \delta(r) \) are external disturbances, and it is assumed that their derivatives have an upper bound. To face such uncertainties and ensure the convergence for the tracking errors to zero in a finite time, the super-twisting sliding mode controllers are designed in the sequel.

#### 4.1. Surge Speed and Heading Control

To design an STSMC, the sliding surface could be defined as follows:

\[ s_u = \epsilon_u + \lambda_u \int \epsilon_u d\tau \]

where parameter \( \lambda_u > 0 \).

By the time derivative of (7), one achieves the following:

\[ \dot{s}_u = \dot{\epsilon}_u + \lambda_u \epsilon_u = \dot{u}_d - \dot{u} + \lambda_u (u_d - u) \]

Substituting the dynamic equation of \( \dot{u} \) form (1) in \( \dot{s}_u \). Thus, the feedback controller is presented as follows:

\[ \tau_u = (m - X_u) (f_u + \dot{u}_d + \lambda_u (u_d - u)) - u_{surge} \]
where \( f_u = -\left(\frac{1}{m-X_u}\right)\left[(m-Y_u)v+X_{u|u}u|u|+X_uu\right] \), and \( u_{\text{surge}} \) is performed by the adaptive sliding mode strategy, denoted by the following:

\[
 u_{\text{surge}} = -k_u|s|^\frac{1}{2}\sgn(s_u) - \alpha_u \int \sgn(s_u)d\tau
\]

where \( k_u \) and \( \alpha_u \) are the parameters to be designed, which should meet the following relationship [21]:

\[
k_u > \frac{3L_{\delta u}}{\sqrt{\alpha_u - L_{\delta u}}}, \quad \alpha_u > 2L_{\delta u}
\]

where \( L_{\delta u} \) satisfies the assumption \(|\delta(u)| \leq L_{\delta u}\).

Then, to design the heading control law, the following sliding surface \((\lambda \phi > 0)\) is defined:

\[
s_\phi = \dot{\phi} + \lambda \phi \phi
\]

The derivative of (9) with respect to time is as follows:

\[
\dot{s}_\phi = \ddot{\phi} + \lambda \phi \ddot{\phi} = (\dot{r}_d - \dot{r}) + \lambda \phi (r_d - r)
\]

Hence, the feedback controller is addressed as follows:

\[
\tau_r = (I_z - N_r)(f_r + \dot{r}_d + \lambda \phi (r_d - r) - u_{\text{heading}})
\]

where \( f_r = -\frac{1}{I_z - N_r}\left[(-X_u + Y_u)v + N_{\tau |r}|r| + N_r\right] \), and \( u_{\text{heading}} = -k_\phi |s_\phi|^\frac{1}{2}\sgn(s_\phi) - \alpha_\phi \int \sgn(s_\phi)d\tau \).

4.2. Closed-Loop Stability

The closed-loop stability is analyzed in this section. First, the closed-loop dynamic for surge speed is given by substituting the control input (8) in the time derivative of (7), obtaining

\[
\dot{s}_u = -k_u|s_u|^\frac{1}{2}\sgn(s_u) - \alpha_u \int \sgn(s_u)d\tau + \delta(u)
\]

According to the methods [21], the closed-loop stability is proved here.

**Proof.**

Step 1. Let \( z_1 = s_u, z_2 = -\alpha_u \int \sgn(s_u)d\tau + \delta(u) \). Then, the derivation of the \( z_1 \) and \( z_2 \) can be obtained:

\[
\begin{align*}
\dot{z}_1 &= -k_u|z_1|^\frac{1}{2}\sgn(z_1) + z_2 \\
\dot{z}_2 &= -\alpha_u \sgn(z_1) + \delta(u)
\end{align*}
\]

Step 2. Make \( \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \), where \( \xi_1 = \frac{z_1}{|z_1|^\frac{1}{2}}, \xi_2 = z_2 \). Take the derivative of \( \xi \)

\[
\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = -\frac{1}{|\xi_1|^2} \begin{bmatrix} k_\xi \\ \alpha_u - \delta(u)\sgn(\xi_1) \end{bmatrix} - \frac{1}{|\xi_2|^2} \left[ \frac{\xi_1}{\xi_2} \right] + \frac{\xi_1^2}{|\xi_2|^3} \end{bmatrix}
\]

So far, the closed-loop stability problem is transformed into the problem of proving the stability of system \( \xi \).

Step 3. Select a Lyapunov candidate function

\[
V_u = \xi^TP_u\xi
\]

where \( P = \begin{bmatrix} k_u + \frac{6}{\lambda_\phi}(\alpha_u - L_{\delta u}) & -1 \\ -1 & \frac{3}{\lambda_\phi} \end{bmatrix} \).

By differentiating (11), it can be obtained.
\[ V_u = -\frac{1}{|\xi_1^2|} \xi^T \Pi \xi \]  

(12)

where \( \Pi = \begin{bmatrix} k_u^2 + 4\alpha_u - 6L\dot{u} + 2c & -k_u + \frac{3(L\dot{u} - c)}{k_u} \\ -k_u + \frac{3(L\dot{u} - c)}{k_u} & 1 \end{bmatrix} \) and \( c = \delta(u) \text{sgn} \xi_1 \).

Step 4. Construct the conditions that satisfy the finite-time stability of the system [32].

Firstly, it should be proved that matrix \( \Pi \) is a positive definite matrix. To achieve this result, the sequential principal minor of matrix \( \Pi \)

\[
\begin{align*}
D_1 &= k_u^2 + 4\alpha_u - 6L\dot{u} + 2c \\
D_2 &= 4\alpha_u - 4c - 9(L\dot{u} - c)^2/k_u^2
\end{align*}
\]

should meet the conditions \( D_1 > 0 \) and \( D_2 > 0 \). Therefore, the conditions for the parameters \( k_u \) and \( \alpha_u \) are derived

\[ k_u > 3L\dot{u}/\sqrt{\alpha_u - L\dot{u}}, \alpha_u > 2L\dot{u} \]

Since \( \Pi \) is a positive definite matrix. So,

\[ \lambda_{\min}(\Pi) \| \xi \|^2 \leq \xi^T \Pi \xi \leq \lambda_{\max}(\Pi) \| \xi \|^2 \]  

(13)

where \( \| \cdot \| \) represents the Euclidean norm, \( \lambda_{\min}(\Pi) \) and \( \lambda_{\max}(\Pi) \) represent the diagonal matrix formed by the minimum and maximum eigenvalues of the matrix \( \Pi \), respectively.

It can be known from the definition of \( \xi \) that \( |\xi_1| \leq \| \xi \| \). Hence, form (12) and (13) can be obtained

\[ \dot{V}_u = -\frac{1}{|\xi_1^2|} \xi^T \Pi \xi \leq -\lambda_{\min}(\Pi) \xi \]  

(14)

Because of the matrix \( P \) in (11) is also a positive definite matrix, for the defined Lyapunov function \( V_u, \lambda_{\min}(P) \| \xi \|^2 \leq \xi^T \Pi \xi \leq \lambda_{\max}(P) \| \xi \|^2 \) also holds. Thus, we derive

\[ \left( \frac{V_u}{\lambda_{\max}(P)} \right)^{\frac{1}{2}} \leq \| \xi \| \leq \left( \frac{V_u}{\lambda_{\min}(P)} \right)^{\frac{1}{2}} \]  

(15)

Finally, combining (14) and (15), the relationship between \( V_u \) and \( \dot{V}_u \) can be obtained as follows:

\[ V_u \leq -\frac{\lambda_{\min}(\Pi)}{\sqrt{\lambda_{\max}(P)}} V_u \frac{1}{2} \]  

(16)

According to (16), it can be known that the origin is a finite-time stable equilibrium of system \( \xi \) [32], and the settling time

\[ T(\xi_0) \leq \frac{2\sqrt{\lambda_{\max}(P)}}{\lambda_{\min}(\Pi)} V_u \frac{1}{2}(\xi_0) \]

Consequently, \( \xi \) converges to zero in the finite time. By definition of \( \xi \), we can see that \( z = [z_1, z_2] \) also converges in finite time. Therefore, the controller design (8) for the surge speed and the proof of finite-time stability are completed. \( \square \)

Similarly, to prove the stability of the closed-loop dynamic for surge speed, the control input (10) is substituted in the time derivative of (9), obtaining

\[ \dot{s}_\psi = -k_\psi |s_\psi|^\frac{1}{2} \text{sgn}(s_\psi) - \alpha \psi \int \text{sgn}(s_\psi) d\tau + \delta(r) \]

In the same way, by the super-twisting second-order sliding mode controller, the yaw angle velocity \( \psi \) converges to the \( \psi_u \) in the finite time. Similarly, the parameters \( k_\psi \) and \( \alpha_\psi \) are selected as \( k_\psi > 3L\dot{\psi}/\sqrt{\alpha_\psi - L\dot{\psi}}, \alpha_\psi > 2L\dot{\psi} \).

Because of the discontinuity of the symbolic function \( \text{sgn}(s) \), the controllers (7) and (10) contain discontinuous terms, which means that there is chattering in the system.
To weaken the chattering phenomenon of the sliding mode control, \( sgn(s) \) is replaced by the continuous function \( \sigma = \frac{s}{|s| + \Delta_s} \), which is a parameter to be designed. Then, we have the following:

\[
\begin{align*}
\tau_u &= (m - X_n) \left[ f_u + \dot{u}_d + \lambda_u (u_d - u) + k_u \frac{s_u}{|s_u| + \Delta_s} + \alpha_u \int sgn(s_u)d\tau \right] \\
\tau_r &= (I_z - N_r) \left[ f_r + \dot{r}_d + \lambda_r (r_d - r) + k_r \frac{s_r}{|s_r| + \Delta_s} + \alpha_r \int sgn(s_r)d\tau \right]
\end{align*}
\] (17) (18)

5. Simulation Results

This section presents the simulation results to validate the effectiveness and robustness of the proposed sliding mode control strategies. The VTec S-III USV [33] is selected as the simulation USV. The value of each thruster has a saturation of \(+/-36.5\) N that represents the thruster model in the simulation.

To verify the robustness of the proposed controller, under the same external disturbance, the trajectory tracking results of straight and sinusoidal trajectories are demonstrated, respectively. According to [3], and considering the physical parameters of the USV, the external disturbances are assumed to be

\[
\begin{align*}
\delta(u) &= 0.1 \sin(0.1t) + 0.2 \cos(0.05t) \\
\delta(v) &= 0.1 \sin(0.1t) + 0.2 \cos(0.05t) \\
\delta(r) &= 0.2 \cos(0.05t)
\end{align*}
\]

The control parameters are \( k_s = 0.1, \Delta = 4, \lambda_0 = 1.3, \lambda_u = 1, k_u = 2, \alpha_u = 0.2, \lambda_\psi = 1, k_\psi = 2, \alpha_\psi = 0.2, \) and \( \Delta_s = 0.01. \)

To demonstrate the advantage of the proposed control law, the method ASMC presented in [4] is selected for comparison and the same parameters of ASMC are used.

The desired reference trajectories of the two scenarios are as follows:

(1) Straight trajectory:

\[
\begin{align*}
x_d(t) &= 0.6t \\
y_d(t) &= 0.6t
\end{align*}
\]

The initial velocities are \( [u(0) \ v(0) \ r(0)] = [100]. \) The initial position and yaw angle are \( [x(0) \ y(0) \ \psi(0)] = [400]. \)

The tracking results for a straight line are shown in Figures 3–7. As is shown in Figures 3 and 4, good performance is achieved in the case of initial errors by the two methods. Figures 3b and 4 show that the control of STSMC has smaller tracking errors. The performance of the surge speed and heading angle controllers is shown in Figure 5.

Th results show that the designed controller can quickly and accurately track the desired value of the guidance law. The observer performance for the reduced-order ESO is shown in Figure 6. It demonstrates that the designed observer can quickly and accurately estimate the lumped disturbance. The control inputs, \( T_{port} \) and \( T_{stbd}, \) are shown in Figure 7.

Figure 3. Results of straight trajectory tracking ((b) is partially enlarged detail of (a)).
Figure 4. Tracking errors ((a) the along-track error $x_e$ and (b) the cross-track error $y_e$).

Figure 5. The desired and actual ((a) the surge speed and (b) the heading angle).

Figure 6. The sideslip angle and its estimation.

Table 2 contains the metric comparison between controllers, including the mean squared error and Euclidean norm [4]. The mean squared error is the average of the squared difference between the desired and the actual value of a variable. Thus, the MSE is used to quantify the controller performance according to how small the steady-state error is. The Euclidean norm is the square root of the sum of the squares of the vector values. Hence, the Euclidean norm is used to quantify the amount of control effort used by each controller. The STSMC does not consume more control input energy and it improves the tracking accuracy.
The initial velocities are \([u(0) \, v(0) \, r(0)] = [100]\). The sideslip angle and its estimation are shown in Figure 6.

**Table 2.** Comparison results of track tracking controllers for straight trajectory.

<table>
<thead>
<tr>
<th>Controller</th>
<th>(u) (MSE)</th>
<th>(\psi) (MSE)</th>
<th>(|\tau_x|)</th>
<th>(|\tau_z|)</th>
</tr>
</thead>
<tbody>
<tr>
<td>STSMC</td>
<td>0.00015</td>
<td>0.000195</td>
<td>1998.62</td>
<td>184.69</td>
</tr>
<tr>
<td>ASMC</td>
<td>0.00210</td>
<td>0.000208</td>
<td>1998.26</td>
<td>194.30</td>
</tr>
</tbody>
</table>

(2) Sinusoidal trajectory:

\[
\begin{align*}
    x_d(t) &= 0.5t \\
    y_d(t) &= 10\sin(0.05t)
\end{align*}
\]

The initial velocities are \([u(0) \, v(0) \, r(0)] = [100]\). The initial position and yaw angle are \([x(0) \, y(0) \, \psi(0)] = [400]\).

The tracking results for a sinusoidal trajectory are shown in Figures 8–12. Using the proposed STSMC method, the tracking errors are sufficiently small and can be maintained even in the presence of external disturbances. Moreover, the tracking errors have a higher convergence speed and convergence accuracy. Table 3 contains the metric comparison between controllers for sinusoidal trajectory. Similar to straight trajectory tracking, STSMC does not consume more control input energy and the tracking accuracy is improved for sinusoidal trajectory.

**Table 3.** Comparison results of track tracking controllers for sinusoidal trajectory.

<table>
<thead>
<tr>
<th>Controller</th>
<th>(u) (MSE)</th>
<th>(\psi) (MSE)</th>
<th>(|\tau_x|)</th>
<th>(|\tau_z|)</th>
</tr>
</thead>
<tbody>
<tr>
<td>STSMC</td>
<td>0.00043</td>
<td>0.000209</td>
<td>1539.03</td>
<td>236.73</td>
</tr>
<tr>
<td>ASMC</td>
<td>0.0032</td>
<td>0.000214</td>
<td>1546.99</td>
<td>227.83</td>
</tr>
</tbody>
</table>

**Figure 7.** Control signals.

**Figure 8.** Results of sinusoidal trajectory tracking ((b) is partially enlarged detail of (a)).
Table 3. Comparison results of track tracking controllers for sinusoidal trajectory.

<table>
<thead>
<tr>
<th>Controller</th>
<th>$u$ (MSE)</th>
<th>$\psi$ (MSE)</th>
<th>$| \tau_x |$</th>
<th>$| \tau_z |$</th>
</tr>
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<tbody>
<tr>
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<td>0.00043</td>
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<td>0.0032</td>
<td>0.000214</td>
<td>1539.03</td>
<td>236.73</td>
</tr>
</tbody>
</table>
6. Conclusions

This paper addressed the problem of trajectory tracking control for an underactuated USV in the presence of disturbances. The new approach proposed in this paper divides the problem into a guidance loop and a control loop. In the guidance section, a reduced-order ESO is introduced to estimate the time-varying sideslip angle, thereby avoiding the small-angle approximation. Based on ESO, the LOS guidance law is designed to obtain the desired surge speed and heading angle. To track the desired values, the super-twisting sliding mode control is adopted to design the finite-time trajectory tracking controller. Finally, the effectiveness of the method is verified by simulations. Nevertheless, it is assumed that the change in external disturbances is known to select the controller parameters. Designing a tracking controller for unknown external disturbances will be the focus of the next research.

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References


11. Park, K.; Tsuji, T. Terminal sliding mode control of second-order nonlinear uncertain systems. *Int. J. Robust Nonlinear Control* 1999, 9, 769–780. [CrossRef]


16. Levant, A. Sliding order and sliding accuracy in sliding mode control. *Int. J. Control* 1993, 58, 1247–1263. [CrossRef]


23. Wang, N.; Zhu, Z.; Qin, H.; Deng, Z.; Sun, Y. Finite-time extended state observer-based exact tracking control of an unmanned surface vehicle. *Int. J. Robust Nonlinear Control* 2021, 31, 1704–1719. [CrossRef]


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