

Article

Uncovering Coexisting Forward and Inverse Energy Cascades in Oceanic Turbulence via an Energy Cascade Multilayer Directed Network (ECMDN)

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Abstract

Multi-scale vortex structures constitute the intrinsic skeleton of turbulent flows and govern the energy cascade process in oceanic turbulence. Elucidating their evolutionary dynamics is crucial for understanding turbulent mixing and transport. In this study, we develop an innovative Energy Cascade Multilayer Directed Network (ECMDN) framework grounded in complex network theory to directly characterize nonlinear energy coupling pathways and directional transfers among multi-scale vortices in real marine environments. By integrating multi-parameter fusion node definitions, multi-scale interaction detection, and energy transfer direction identification, the ECMDN reconstructs the nonlinear turbulent system into a topologically interpretable structure. The emergent network properties enable quantitative characterization of intermittency and inhomogeneity in the energy cascade, offering new insights into vortex interactions and cross-scale energy transfer mechanisms. Compared with conventional cascade diagnostics including spectral flux, third-order velocity structure functions, multifractal analysis and shell models that require homogeneity and local equilibrium assumptions and only output global averaged energy flux, the proposed ECMDN multilayer network retains point-wise depth coordinates of each vortex interaction, separates directed forward/inverse energy edges, and quantifies intermittency via topological metrics. Analysis of the single Shenhu thermocline shear segment demonstrates these differentiated analytical capabilities of the proposed framework. Application to shear measurements from the Shenhu Sea reveals the simultaneous occurrence of forward and inverse energy cascades, manifesting a synchronous dual-energy-cascade pattern. This indicates that vortices at a given scale can concurrently transfer energy to larger- or smaller-scale structures and receive energy from larger- or smaller-scale counterparts during the cascade process. Our findings observe a typical synchronous dual-energy-cascade pattern in the strong thermocline of the Shenhu Sea, providing a novel theoretical and methodological framework for investigating the spatiotemporal evolution of stratified ocean turbulent mixing and advancing our understanding of geophysical fluid dynamics.



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1. Introduction

Oceanic fluid flows are inherently chaotic, complex, and spatially inhomogeneous, dominated by multi-scale vortex structures. These vortices serve not only as carriers but

also as active drivers of ocean circulation, redistributing marine energy through processes such as vortex splitting, merging, and deformation. Under external forcing and dissipative mechanisms, they mediate the conversion between potential and kinetic energy, thus playing a central role in shaping large-scale circulation patterns. Despite their importance, the nonlinear coupling mechanisms that govern energy transfer across scales remain poorly understood, limiting progress in characterizing turbulent mixing and closing the oceanic energy budget. Consequently, there is a pressing need for a robust modeling framework capable of directly quantifying the pathways and directions of cross-scale energy transfer, enabling a comprehensive investigation of the intricate dynamics underlying the turbulent energy cascade.

The earliest theoretical foundation of the energy cascade dates back to Richardson's pioneering work on atmospheric turbulence [1], which proposed that turbulence consists of multi-scale eddy structures. Large-scale eddies break down into smaller ones, which further split into even smaller structures, until the energy stored within these vortices is ultimately dissipated by molecular viscosity. Based on spectral methods, Kolmogorov developed the K41 theory, providing a quantitative description of the evolution of energy input, transfer, and dissipation in three-dimensional space. However, the energy cascade mechanism proposed by Richardson lacked rigorous validation, and the K41 theory also exhibited significant deviations from scaling laws in higher-order statistics. Landau questioned the K41 theory, arguing that there was no solid basis for treating the dissipation rate as a constant. Instead, he suggested that turbulent energy dissipation rate should be considered a variable in the theoretical framework [2].

To address these limitations, several refined models were later proposed, among which Kolmogorov's 1961 (K61) refinement stands out [3]. However, the K61 model still assumed statistical homogeneity and neglected intermittency—an essential characteristic of real-world turbulence. Extensive experimental evidence now confirms that the energy cascade is far from uniform, instead exhibiting pronounced intermittency marked by sporadic, intense bursts of energy activity even under otherwise quiescent flow conditions [4]. Indeed, intermittency is now recognized as an intrinsic property of turbulence, arising from the spatially and temporally localized concentration of energy transfer events [5]. To capture the spatiotemporal heterogeneity and intermittency of turbulent energy transfer, researchers have developed multiple classical modeling frameworks including β -model, multifractal models, shell models and SL model [6–10]. These traditional methods effectively describe the statistical characteristics of turbulent cascades but still suffer from inherent limitations in spatial localization and bidirectional energy flux discrimination. In shell models [11,12], Fourier space is discretized into logarithmically spaced “shells,” each representing eddy dynamics at a specific scale. Energy transfer between adjacent shells is governed by the nonlinear interaction of velocity fluctuations, mimicking the upscale and downscale transport of energy. By partially decoupling integral (large-scale) and inertial-range dynamics, shell models offer a computationally efficient platform for investigating turbulent energy cascades. The She–Leveque (SL) model [13,14] further advanced the treatment of intermittency by proposing that turbulent fluctuations exhibit self-similarity not only in space and time but also in the hierarchical organization of burst intensities. At a given scale, intermittency manifests as non-uniform bursts with escalating amplitudes; across scales, these bursts organize into a hierarchical structure of increasing intensity, reflecting the multi-level nature of turbulent intermittency.

In real oceanic environments, vortex structures do not transfer energy along a single pathway. Research results indicate that both forward and inverse energy cascade processes coexist in the ocean. Dual energy cascades have been demonstrated in various physical flows, including stellar plasma [15–17], rotating and stratified turbulence [18], solar wind

turbulence, tropical cyclones, buoyancy-driven convection, and thermodynamically active fluids. Notably, the thermocline turbulence analyzed in this study is a typical rotating stratified turbulent flow. Although this study takes in situ shear signals as the direct analytical object without independent calculation of Coriolis parameter and buoyancy frequency, all collected shear turbulence data are inherently modulated by the local rotation and stable stratification environmental field of the Shenhua Sea thermocline. The regional hydrodynamic background of rotation and stratification acts as an implicit physical constraint for vortex evolution and bidirectional energy coupling behavior in this study. For instance, experiments on second-sound waves in superfluid helium have confirmed that although turbulent energy predominantly cascades forward, reverse transfer can occur under certain conditions. Numerical simulations of Bose–Einstein condensates, together with analytical derivations of the Kolmogorov–Zakharov spectrum, further substantiate the simultaneous presence of forward and inverse energy cascades [19]. Analytical solutions derived for Kerr media with random inhomogeneities reveal similar bidirectional energy transfer in optical turbulence [20]. Moreover, studies of modified Navier–Stokes equations show that tuning the strength of nonlinear interaction terms can switch the dominant cascade direction—from forward to inverse—highlighting a competitive interplay between the two pathways in three-dimensional isotropic flows [21].

Further support comes from direct observational studies, which have confirmed the presence of dual energy cascades in geophysical flows [22,23]. Drifter trajectory measurements in the Gulf of Mexico revealed clear signatures of both forward and inverse kinetic energy transfer, validating the coexistence of bidirectional cascades in the ocean [24]. Notably, forward energy flux dominates at scales below 1–10 km, while the overall cascade intensity varies seasonally. Satellite altimeter data analyses have further quantified the energy budgets of various eddy populations, highlighting the critical role of submesoscale vortices (1–40 km) in ocean dynamics [25]. These vortices inject energy upscale via inverse cascading, thereby reinforcing mesoscale eddies and enhancing their ability to transport heat and nutrients. Complementary numerical simulations of two-dimensional turbulence, wave turbulence, and rotating-stratified flows consistently demonstrate that turbulent energy can simultaneously propagate to both larger and smaller scales, confirming the robustness of bidirectional cascading in geophysical settings [26]. Several methodologies have been developed to investigate cross-scale turbulent energy transfer [27]. Altimetry-based analyses combined with numerical simulations by Tulloch [28] and Arbic [29] have provided partial insights, indicating that observed cascade behaviors resemble those predicted by two-dimensional geostrophic turbulence. However, these approaches depend heavily on spectral methods, which require assumptions of statistical homogeneity, isotropy, or well-separated scales—conditions rarely met in realistic oceanic environments. Numerical simulations have also contributed significantly: for example, Cheikh [30] simulated homogeneous isotropic turbulence and confirmed dual energy transfer at small scales, reporting a predominance of forward cascading and characteristic small-scale energy flux trends. Cardesa [31] performed three-dimensional simulations of homogeneous shear flows, proposing a stepwise energy cascade model. Yet, simulating high-Reynolds-number turbulence remains computationally prohibitive, as costs grow rapidly with increasing Reynolds number [32,33]. For ungridded or sparse observational data, the third-order structure function method is commonly used [34–36]. Xie [37] derived an analytical expression for the third-order velocity structure function at forced scales, extended the power-law regime, and detected dual energy transfer. Another approach leverages anomalous scaling in horizontal velocity gradients from in situ observations (2 km resolution), establishing a linear relationship between scaling exponents and gradient statistics, allowing inference of cascade variability across vertical domains via slope analysis [38]. Nonetheless, structure-

function methods require a priori assumptions of local turbulent equilibrium [39], involve delicate calibrations of statistical closures [40], and can only be validated a posteriori [41], limiting their applicability in strongly intermittent or transient regimes.

While spectral analysis, numerical simulations, and structure function methods are valuable tools for studying turbulent energy cascades, most existing frameworks suffer from fundamental limitations, including assumptions of statistical isotropy, high computational demands, and reliance on a priori tunings and a posteriori validations. Unlike these integral statistical methods that discard spatial coordinate information, the multi-layer directed network proposed herein locates each cross-scale energy coupling at specific vertical depths; based on the thermocline observation data in this paper, it independently distinguishes upscale and downscale energy transport signals that cannot be separated by traditional averaging algorithms. These constraints impede the direct quantification of nonlinear energy coupling in arbitrary spatial locations, temporal instances, and scale combinations. To overcome these challenges, this study introduces a novel analytical framework: we propose an energy cascade model based on complex network theory that relaxes traditional assumptions and enables direct characterization of cross-scale turbulent energy transfer [42]. The ECMDN pipeline integrates modules constrained by classical turbulence theories (CEEMD multi-scale decomposition, LIM intermittency calculation, wavelet three-wave coupling) and manually set empirical adjustable parameters (LIM threshold, phase synchronization cutoff, mother wavelet type). Complex network theory has proven powerful in analyzing chaotic and nonlinear systems by abstracting system dynamics into simplified graph representations, extracting essential components, and mapping dynamical rules onto topological features. A graph-theoretical study on noisy high-dimensional biomeasurement signals [43] provides methodological reference for us to transform turbulent shear time series into interpretable multilayer topological structures. Though underutilized in physical oceanography, it has found success in other branches of fluid dynamics [44,45]. For instance, Scarsoglio [46] constructed a single-layer global network to analyze spatial coherence in forced isotropic turbulence. Liu [47] adopted visualization algorithms to build topological networks and revealed self-similar structural features within the energy dissipation field of fully developed three-dimensional turbulence. Gurcan [48] modeled cross-scale energy transfer using a small-world network, revealing clustering and short-path characteristics. However, such models often rely on ad hoc assumptions or artificially imposed connectivity rules. More critically, because turbulence inherently involves multi-scale interactions, single-layer networks fail to resolve cross-layer coupling mechanisms. To address this, we map the energy cascade process onto a multi-layer directed network model. Multi-layer networks, consisting of interconnected layers with distinct functional or scale-specific attributes, are ideally suited for capturing complex, multi-mechanism interactions. For example, Gao [49] employed a multilayer network based on multivariate entropy to characterize gas–liquid flow regimes, demonstrating the framework’s capability to decode nonlinear transitions.

This paper presents a multi-layer directed network model, termed the Energy Cascade Multilayer Directed Network (ECMDN), which maps complex turbulent systems into a topological framework. The ECMDN is specifically designed to unveil hidden cross-scale energy transfer pathways and directions from observational signals, enabling direct characterization of nonlinear coupling among multi-scale vortex structures. Nodes are defined through multi-parameter fusion, integrating key physical descriptors—such as depth, scale, intermittency, and phase—to effectively simplify and reconstruct the turbulent state. Multi-scale energy interaction detection is implemented to quantify nonlinear couplings across scales, while energy transfer direction identification determines the net direction of energy flux with high precision. By exploiting the topological attributes of nodes and directed

inter-layer edges, the ECMDN provides a direct, space-time-resolved representation of cross-scale energy transfer, without relying on ergodicity or stationarity assumptions. This framework establishes a new paradigm for analyzing turbulent cascades, offering a flexible and generalizable methodology for future investigations.

2. Methodology

2.1. Study Instrumentation and Observation Area

This study focuses on the Shenhua Sea, located in the northern South China Sea (Figure 1). The observational site lies at a water depth of 1970 m, within the continental slope transition zone between the shallow shelf and the deep ocean basin. This region is characterized by a dynamic tidal environment that frequently generates internal solitary waves, which in turn induce intense turbulent mixing. Such mixing processes play a critical role in regulating material transport and shaping marine ecosystem dynamics in this area. The strong stratification and weak rotational modulation in this thermocline region jointly control the vortex interaction and energy cascade rules, which fundamentally determine the asynchronous dual-cascade characteristics of local shear turbulence. Before calculating wavelet coefficients, LIM values and phase parameters for intermittency analysis, raw shear data were preprocessed to remove low-frequency trends, internal wave disturbances and instrumental noise. This procedure eliminates non-turbulent interference and ensures reliable and accurate turbulence characteristic calculation.

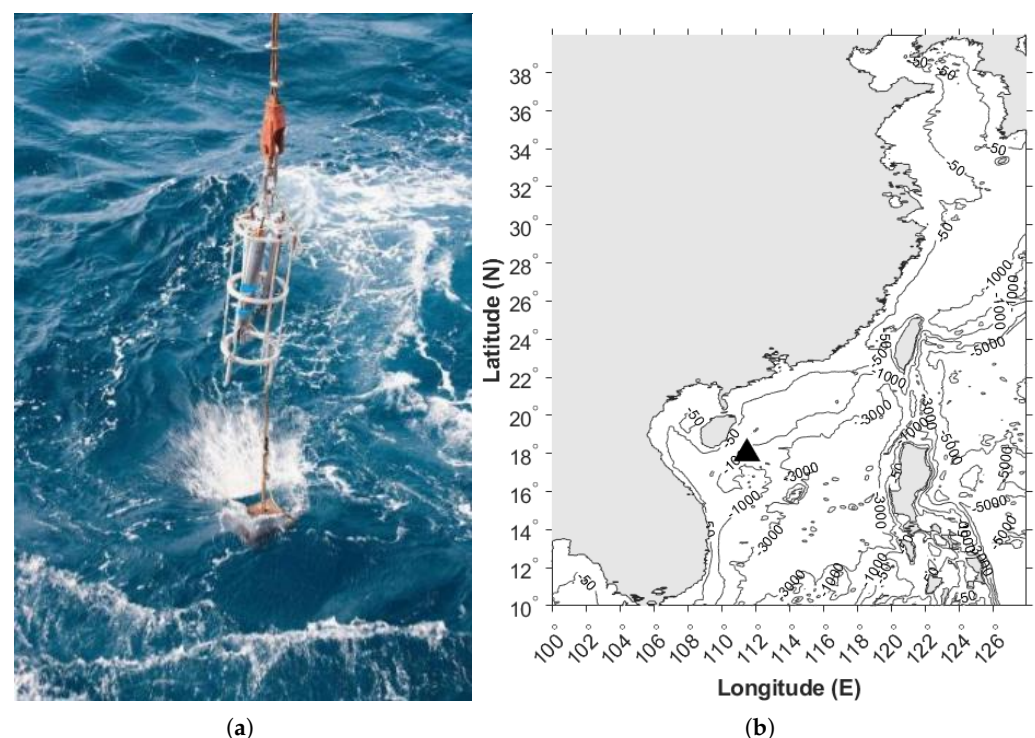


Figure 1. (a) Field operations in the northern South China Sea. (b) Location of the observation site.

Turbulence signals were collected using a micro-electromechanical systems (MEMS)-based turbulence profiler, independently developed by the research team at Xiamen University. The instrument has an overall length of 70 cm and a diameter of 7.7 cm. At its forward sensor base, a multi-sensor array is integrated (Figure 2), consisting of a pair of shear sensors, a platinum thin-film temperature sensor, a quartz-tube seven-platinum-ring conductivity sensor, a piezoresistive thin-film pressure sensor (Model 10LX/10LH, Keller AG, Winterthur, Switzerland), and a SEC290 nine-axis full-attitude magnetic sensor (BWSensing, Wuxi, China). The corresponding sampling frequencies for these sensors

are 1000 Hz (shear), 2 Hz (temperature), 2 Hz (conductivity), 2 Hz (pressure), and 2 Hz (attitude), respectively. Prior to deployment, all sensors underwent rigorous calibration to ensure high measurement accuracy. Notably, the shear sensors offer a spatial resolution of 3 mm and a response time of 1 ms. All sensors are rigidly mounted on the forward base, ensuring coherent and synchronized capture of microscale oceanic turbulence features.

The self-developed MEMS turbulence profiler was deployed from the stern deck of the research vessel and continuously recorded high-precision turbulence shear data over a full vertical range of 0–1800 m with a fixed sampling frequency of 1000 Hz. Throughout the descent, the instrument maintained a stable near-constant velocity of approximately 0.45 m/s with negligible attitude deviation, ensuring synchronous and undistorted acquisition of microscale turbulence signals. The core analytical dataset of this study is extracted from a valid 2 m depth segment of 518–520 m, located in the permanent strong thermocline of the northern Shenhu Sea. This depth zone features stable seawater density stratification, weak large-scale current fluctuations, and prominent shear-induced turbulent vortex activity, which is a typical representative area for studying stratified oceanic turbulence. The 1000 Hz high sampling rate and 3 mm spatial resolution of the sensor fully resolve the inertial subrange scales of ocean turbulence, completely capturing multi-scale vortex fluctuation signals required for cross-scale energy cascade analysis.

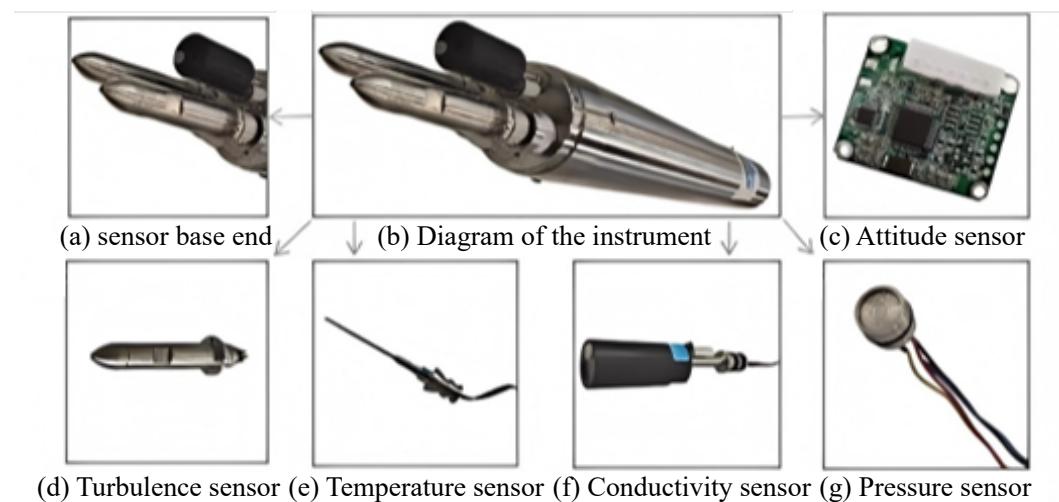


Figure 2. (a) Sensor base end; (b) full instrument schematic; (c) attitude sensor; (d) turbulence sensors; (e) temperature sensor; (f) conductivity sensor; (g) pressure sensor.

2.2. Analysis Methods

The Energy Cascade Multi-layer Directed Network (ECMDN) model is constructed through three core components: (1) definition of multi-parameter fusion nodes, (2) determination of multi-scale energy interactions, and (3) identification of energy transfer directions. Together, these components map the nonlinear energy transfer dynamics onto a topological network structure. Subsequent analysis of the resulting network's topological properties enables the extraction of cross-scale turbulent energy transfer pathways and directional fluxes. The ECMDN is constructed via multi-parameter node construction and cross-scale edge identification: core physical rules are derived from turbulence theories, while numerical cutoff values used to filter valid vortex interactions are empirical selections fitted to the thermocline turbulence case in this research. An overview of the ECMDN workflow is illustrated in Figure 3.

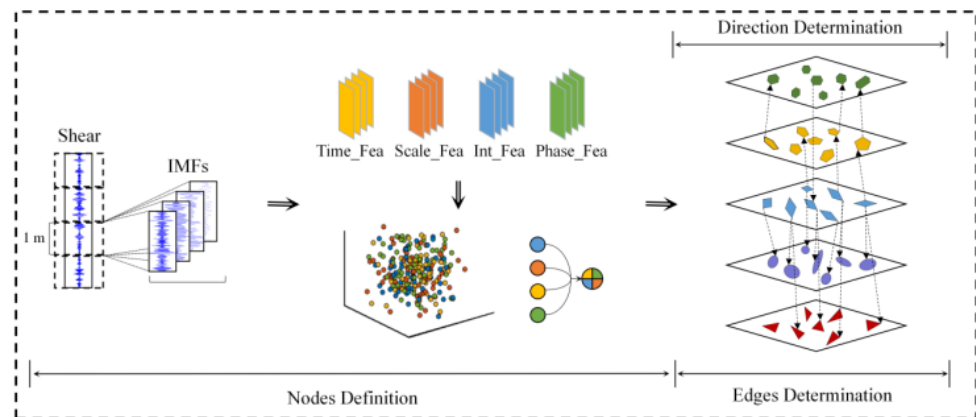


Figure 3. Schematic workflow of the ECMDN model.

2.2.1. Definition of Multi-Parameter Fusion Nodes

Mathematically, each topological node can be expressed as a quadruple vector $v_{k,x} = [z(x), S_k, LIM_{k,x}, \varphi_{k,x}]$, where “ k ” is the serial number of IMF scale layer, and “ x ” stands for vertical sampling coordinate. All four feature variables are sequentially extracted from raw shear signals through CEEMD decomposition, wavelet transform and Hilbert transformation. Evaluating energy evolution, the node v is defined as a multi-parameter fusion. Physical feature statistics, including depth, scale, local intermittency value, and phase, are extracted to construct the topological node, e.g., $v = depth, scale, lim, \varphi$. Considering the nonlinear and non-stationary characteristics, the empirical mode decomposition (EMD), the wavelet transform, the local intermittency measure (LIM), and the Hilbert transformation (HT) are introduced to evaluate the chaotic energy motions.

For a space series of signal $s(x)$, with N observation points, each sampling point corresponds to a specific sampling depth $[depth_1, depth_2, \dots, depth_N]$.

Considering the multi-scale vortex structures present in the aquatic environment, the complementary ensemble empirical mode decomposition (CEEMD), a noise-assisted optimized variant of traditional EMD, is introduced to decompose the multi-scale fused shear data $s(x)$ into K narrow-band intrinsic mode function imf_k components and one residual component Re . This method effectively suppresses EMD mode mixing and endpoint distortion, improving the stability and accuracy of multi-scale decomposition results [50].

$$s(x) = \sum_{k=1}^K imf_k(x) + Re(x) \quad (1)$$

Using the zero-mean method, the scale value $scale_k$ corresponding to each set of imf_k is computed.

Data points derived from the same imf_k share identical scale parameters.

The wavelet transform method is employed to quantify the energy fluctuations of each imf_k component [51],

$$wt_{a,x} = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \Psi^* \left(\frac{d-x}{a} \right) imf_k dx \quad (2)$$

where Ψ is the mother wavelet function, a represents the scale dilation, and b is the space parameter. The Morlet wavelet is selected as the mother wavelet for all wavelet transformation calculations in this study.

The LIM method is introduced to assess the energy accumulation fluctuations of the imf_k , predicting the prominent energy accumulation locations and the positions of intermittent energy bursts within each set of imf_k components.

$$\lim_{a,x} = \frac{|wt_{a,x}|}{\langle |wt_{a,x}|^2 \rangle_b} \quad (3)$$

where $wt_{a,x}$ represents the wavelet coefficient and the angled bracket $\langle \cdot \rangle$ denotes global average.

The Yaglom law posits that the cross-scale energy transfer process can synchronously amplify or attenuate oscillation amplitudes, resulting in phase synchronization of oscillations. Accordingly, the Hilbert transform is applied to estimate the phase fluctuations corresponding to each set of imf_k components,

$$H(s(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{s(\tau)}{x - \tau} d\tau \quad (4)$$

Thus, the essence of a topological node is a cell matrix of size $K \times N$, where each element is a quadruple containing sampling depth, scale, local intermittency measure, and phase. This study adopts multiple nonlinear signal processing methods to process raw shear data. Multiple physical characteristic parameters are quantified and fused to construct topological network nodes. This multi-parameter fusion design effectively overcomes the structural limitation of single-attribute node construction in traditional network models.

2.2.2. Determination of Multi-Scale Energy Interaction

The Batchelor experiment [4] demonstrated that energy transfer is a non-stationary and highly intermittent process. Even under relatively stable and quiescent fluid conditions, the energy transfer process is accompanied by intense bursts of energy activity. The LIM is introduced to identify strong intermittent energy bursts. Physically, the local intermittency measure quantifies the energy accumulation at specific locations and scales. Peaks in energy accumulation typically coincide with intense intermittent bursts. If $\lim > 1$, the location is highly prone to intermittent energy bursts, triggering the energy cascade process; conversely, if $\lim < 1$, it indicates insufficient energy accumulation at that scale and location to sustain the energy cascade.

Perri [52] verified through solar wind magnetic turbulence experiments that structures at different scales exhibit synchronized amplification or attenuation of oscillation amplitudes during energy transfer, accompanied by phase synchronization. Due to synchronized phase oscillations, the phase difference between the two energy-transfer structures approaches zero. Therefore, if two nodes exhibit an extremely small phase difference, there is a high probability that energy transfer is occurring.

Considering the synchronized energy accumulation, intermittent burst characteristics, and phase synchronization phenomena accompanying the energy cascade process, the criterion for determining cross-scale energy transfer connections are given as:

1. A pair of nodes v_i, L_α and v_j, L_β are selected from different layers. Node v_i is located on layer L_α , and node v_j is located on layer L_β . These two layers correspond to different scales.
2. Node v_i and v_j are selected at the same depth, e.g., $\text{depth}_i = \text{depth}_j$.
3. The local intermittency values of both nodes exceed the average level of the spatial sequence, $\lim_i > 1$ and $\lim_j > 1$.
4. Both nodes exhibit peak values in local intermittency.
5. The phase difference between the two nodes approaches zero, $|\phi_i - \phi_j| < \sigma$ ($\sigma = 1 \times 10^{-3}$). The phase threshold adopts a small radian value suitable for fine-scale phase synchronization identification of stratified turbulence, which can effectively screen valid strong-coupling vortex pairs and eliminate weak interference signals. The $\text{LIM} > 1$ and LIM peak screening criteria are classic empirical thresholds for turbulence intermittent burst identification, which are widely adopted in turbulence intermittency research.

If any pair of nodes $v_i, L_\alpha, v_j, L_\beta$ simultaneously satisfies all the above criteria, it is concluded that an edge exists between them, indicating that the vortex structures corresponding to these two scales are engaged in cross-scale energy transfer at this spatial location. Otherwise, no connection exists between the nodes. The five simultaneous screening rules rely on fixed empirical LIM and phase thresholds adopted in this paper; the network topology and detected inverse edge are obtained under the present parameter set. All five simultaneous screening rules rely on fixed empirical LIM and phase thresholds adopted in this paper; the network topology and detected inverse edge are obtained under the present parameter set. All threshold values are case-specific empirical settings for Shenhua Sea thermo-cline turbulence, without universal applicability. Threshold sensitivity analysis will be carried out in future research to test the stability of cascade observation results under different threshold combinations.

Given the core research objective of revealing cross-scale energy cascade pathways in turbulent flow, the ECMDN model focuses exclusively on energy transfer between nodes on different layers. That is, it only considers the existence of inter-layer edges $(v_i, L_\alpha), (v_j, L_\beta) \in E \mid L_\alpha \neq L_\beta$. Intra-layer same-scale connections and node self-connections are explicitly excluded in the current modeling framework. This setting is based on the physical fact that cross-scale vortex interaction is the dominant dynamic process governing turbulent energy cascade, turbulent mixing and marine material transport in thermocline turbulence. By contrast, same-scale vortex interactions mainly induce local energy redistribution and weak flow field fluctuation adjustment, with negligible contribution to the macroscopic cross-scale energy transfer cascade process focused on in this study. It is worth noting that same-scale vortex interaction is still an indispensable component of full turbulence dynamics, and its potential influence on fine-scale turbulent fluctuation can be further explored in future targeted research.

2.2.3. Identification of Turbulent Energy Transfer Direction

The wavelet bispectrum is employed to analyze the phase coupling of vibrational modes in the identified multi-scale vortex structures and to determine the energy transfer direction.

The shear signal $s(x)$ is decomposed using the wavelet transform to obtain the corresponding wavelet coefficients $wt_{a,x}$. For a given scale a , it can be converted to the actual frequency f_a according to the formula $f_a = f_c \times f_s / a$, where f_s denotes the sampling frequency of the shear data, i.e., 1000 Hz, and f_c represents the central frequency of the wavelet. Based on the above conversion, the original wavelet coefficients $wt_{a,x}$ can be transformed into $wt(f, x)$ for subsequent computations. In the scale-frequency conversion formula $f_a = f_c \cdot f_s / a$, the sampling frequency $f_s = 1000$ Hz (fixed instrument parameter), f_c denotes the central frequency of the adopted mother wavelet (dimensionless normalized parameter), a is the wavelet scale dilation factor, and the converted f_a is the characteristic frequency of each IMF component with the unit of Hz. All six IMF characteristic frequencies derived from data calculation are $f = [1.3, 2.1, 6, 12, 35, 90]$ Hz.

The wavelet bispectrum $WB(f_\alpha, f_\beta)$ is then calculated from the wavelet coefficients $wt(f, x)$ using the formula

$$WB(f_\alpha, f_\beta) = \int_0^D wt(f_\alpha, x) wt(f_\beta, x) wt^*(f_\gamma, x) dx \quad (5)$$

where the integration is performed over the depth domain, and the three frequency components must satisfy the summing condition $f_\gamma = f_\alpha + f_\beta$. Here, $D = 2$ m, corresponding to the vertical length of the valid turbulence analysis segment selected in this paper. The symbol $*$ denotes the complex conjugate. The wavelet bispectrum $WB(f_\alpha, f_\beta)$ is a complex

quantity that reflects the phase coupling among three wavelet coefficient components at frequencies f_α , f_β , and f_γ .

To estimate the degree of nonlinear phase coupling among these components, the wavelet bispectrum is normalized:

$$NWB(f_\alpha, f_\beta) = \frac{|WB(f_\alpha, f_\beta)|^2}{\left[\int_0^D |wt(f_\alpha, x) wt(f_\beta, x)|^2 dx \right] \left[\int_0^D |wt(f_\gamma, x)|^2 dx \right]} \quad (6)$$

The normalized wavelet bispectrum ranges between 0 and 1. $NWB(f_\alpha, f_\beta) = 0$ indicates that the three components fluctuate randomly with no phase correlation, while $NWB(f_\alpha, f_\beta) = 1$ signifies strong phase coupling.

Notably, the imaginary part of $WB(f_\alpha, f_\beta)$ is related to the control of energy transfer. To determine the direction of energy transfer, the imaginary part of the normalized wavelet bispectrum $b_{im}(f_\alpha, f_\beta)$ is computed as:

$$b_{im}(f_\alpha, f_\beta) = \frac{\text{im}[WB(f_\alpha, f_\beta)]}{\left\{ \left[\int_0^D |wt(f_\alpha, x) wt(f_\beta, x)|^2 dx \right] \left[\int_0^D |wt(f_\gamma, x)|^2 dx \right] \right\}} \quad (7)$$

where $\text{im}[WB(f_\alpha, f_\beta)]$ corresponds to the imaginary part of the wavelet bispectrum.

In conventional studies, the sign of $b_{im}(f_\alpha, f_\beta)$ is used to determine the direction of energy transfer. If $b_{im}(f_\alpha, f_\beta) > 0$, energy is transferred from the lower frequencies f_α and f_β to the higher frequency $f_\gamma = f_\alpha + f_\beta$. If $b_{im}(f_\alpha, f_\beta) < 0$, energy is transferred from the high frequency f_γ to f_α and f_β . However, in order to determine the direction of energy transfer connections for stratified ocean turbulent vortices, the above standard bispectrum criterion needs to be appropriately adapted. For the thermocline turbulence in the Shenhu Sea dominated by stratified shear flow, this study focuses on bidirectional energy transfer between low-frequency large-scale vortices and high-frequency small-scale vortices. The secondary frequency component f_β in the standard three-wave coupling formula does not correspond to independent vortex scale signals in stratified turbulence, and is thus defined as an auxiliary growth frequency parameter θ that assists in energy direction discrimination. In this study, θ is precisely defined as the minimum incremental frequency interval of the six IMF characteristic frequency series, which is a fixed constant auxiliary parameter independent of observational vortex signals, ensuring the reproducibility of wavelet bispectrum calculation and directional discrimination results. For directional discrimination rules, this study adopts a physically unified hierarchical logic: all inertial-scale vortex layers (Layer1–Layer5) rely on the sign of normalized wavelet bispectral imaginary part for data-driven directional judgment; the dissipation-scale Layer6 exceeds the inertial subrange cutoff frequency, and its energy evolution follows the universal turbulence dissipation law of unidirectional downscale transmission, thus adopting physical prior constraint judgment. This differentiated processing conforms to the intrinsic physical characteristics of turbulent multi-scale evolution and avoids logical conflict. Therefore, the standard discriminant $b_{im}(f_\alpha, f_\beta)$ is effectively reinterpreted as $b_{im}(f_\alpha, \delta, f_\gamma)$. In this improved criterion, a positive b_{im} value uniquely corresponds to downscale forward energy transfer from large-scale to small-scale vortices, while a negative value represents upscale inverse energy transfer from small-scale to large-scale vortices. This adaptation retains the core physical logic of wavelet three-wave coupling theory, and the modified directional discrimination rule is physically valid and targeted for the stratified shear turbulence dataset used in this study.

If $b_{im}(f_\alpha, \delta, f_\gamma) > 0$, turbulent energy is transferred from the large-scale structure at low frequency f_α to the small-scale structure at high frequency f_γ , representing a forward energy transfer. In the ECMDN, this corresponds to inter-layer edges directed from nodes

in layer L_α to nodes in layer L_γ . Conversely, if $b_{im}(f_\alpha, \delta, f_\gamma) < 0$, energy is transferred inversely from the small-scale structure at high frequency f_γ to the large-scale structure at low frequency f_α , indicating an inverse energy transfer. In the ECMDN, this is represented by inter-layer edges directed from nodes in layer L_γ to nodes in layer L_α .

3. Results and Analysis

3.1. Results and Discussion

Validation of the shear data against the standard Nasmyth spectrum (Figure 4) demonstrates excellent agreement across the inertial subrange, confirming the quality and reliability of the turbulence measurements for subsequent modeling.

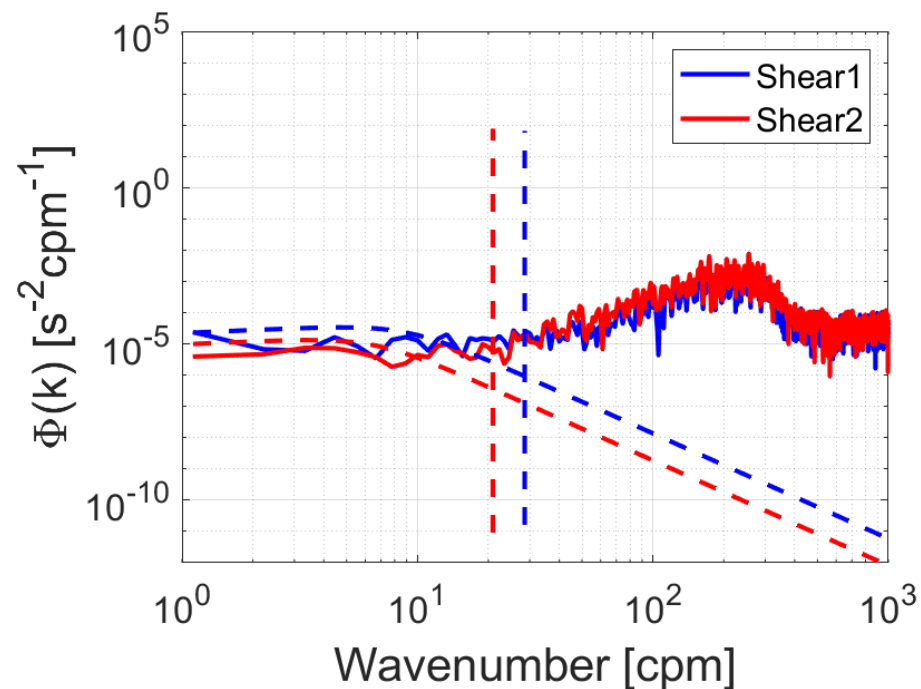


Figure 4. Observed shear spectrum (solid blue/red curves), theoretical Nasmyth spectrum (dashed curves), and cutoff wavenumber (vertical dashed lines).

Simultaneously acquired hydrographic profiles from a custom-developed CTD sensor provide contextual environmental data. As shown in Figure 5, a surface mixing layer can be found from 0 m to 125 m, temperature is maintained at a stable 23.5 °C and density is kept at nearly 1035.6 kg/m³. The surface layer is strongly influenced by the intense sea–air interaction such as wind stress stirring, wave breaking and convection, resulting in good mixing of temperature and density. A seasonal thermocline is present from 125 m to 600 m, where temperature drops sharply from 23.5 °C at the surface to 7.4 °C. The column heat obtained from solar radiation is mainly retained in the surface layer and deeper water comes to cold due to the lack of heat. All density values plotted in Figure 5 are in situ seawater density instead of potential density. Density decreases from 1035.6 kg/m³ in the upper layer to a minimum of 1030.4 kg/m³ at 600 m. The counterintuitive density inversion originates from low-salinity water mass intrusion in the study area, triggering an intense unstable fluctuating state within the thermocline layer. The buoyancy frequency reaches a maximum of 10^{−3} rad/s, representing the strongest stratification and active internal wave activity. The relatively high surface in situ density value conforms to the real hydrological characteristics of the northern Shenhui Sea continental slope. The higher the buoyancy frequency, the easier the water column returns to its original state. Here, the water column tends to maintain a stratified state and the vertical turbulence exchange is suppressed. In

the slow-fluctuating deep layer (600–1800 m), temperature drops slowly to 5.7 °C and density increases again to approximately 1033.5 kg/m³. Buoyancy frequency maintains a low magnitude of approximately 10^{−5} rad/s, with many peaks fluctuating above 2–3 orders of magnitude. Furthermore, peaks of buoyancy frequency indicate the presence of internal waves breaking and double-diffusive convection, which would trigger turbulence mixing.

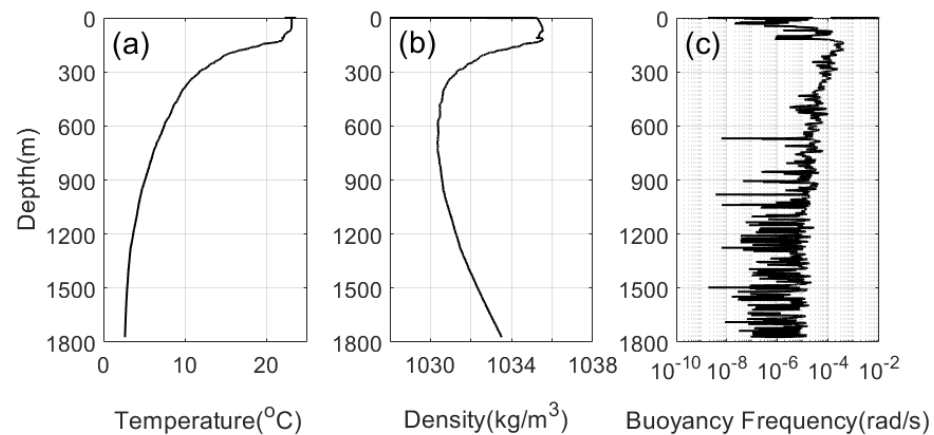


Figure 5. (a) Temperature profile; (b) density profile; (c) buoyancy frequency profile.

A representative shear segment spanning depths 518–520 m (Figure 6) was selected for detailed ECMDN construction. Applying EMD decomposition to this interval yields six IMF components and one residual (Figure 7), capturing oscillatory features across a broad range of scales.

The marginal Hilbert spectrum (Figure 8) shows that the IMF components obtained from EMD decomposition exhibit distinct frequency scales, effectively capturing multi-scale vortex structure information within the observing fluid field. Subsequently, wavelet transform, LIM, and Hilbert transform methods were applied to quantify the vortex scale, energy, and phase parameters at each sampling point and depth.

Thus, based on the multi-parameter fusion node identification and multi-scale energy interaction determination, the hidden physical process of cross-scale turbulent energy transfer within the shear signal was mapped into a topological structure. Core network topological metrics further realize the quantitative characterization of turbulence intermittency and spatial inhomogeneity. The node degree distribution exhibits obvious localized aggregation characteristics, with most nodes maintaining sparse connection states and a small number of nodes forming high-degree connection clusters, which quantitatively corresponds to the bursty intermittent energy burst phenomenon of thermocline turbulence. The inter-layer edge weight distribution is significantly asymmetric: edges between small-scale vortex layers have higher weight values, indicating stronger local energy coupling intensity, while large-scale layer edges have lower and more uniform weights. The uneven layer-to-layer connectivity density directly reflects the spatial inhomogeneity of cross-scale energy transfer in stratified turbulence, realizing the quantitative description of turbulent cascade heterogeneity and intermittency. The ECMDN results are shown in Figure 9. Topological nodes, which are the fusion of four physical statistics, are mapped into a three-dimensional coordinate space, and each individual axis represents specific physical information. In this coordinate space, the x -axis represents the depth sequence, the y -axis represents the scale, and the z -axis represents the fluctuation in the local intermittency value. According to their specific physical statistics, these nodes are distributed in different and non-overlapping coordinate-space positions, and nodes with the same scale are marked with the same color. Furthermore, connecting edges are depicted by black lines. If a pair of nodes v_i and v_j at the same depth, located on layers L_α and L_β , respectively, satisfy the

connection criteria, an edge e_{ij} exists between node v_i and node v_j . This edge e_{ij} indicates a cross-scale energy transfer between two vortices of scale $_{\alpha}$ and scale $_{\beta}$ at depth $_i$.

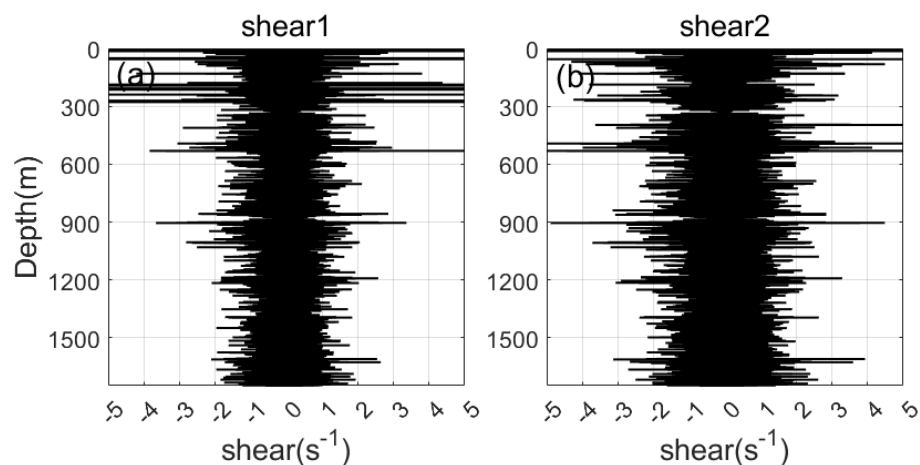


Figure 6. (a) Shear profile 1; (b) shear profile 2.

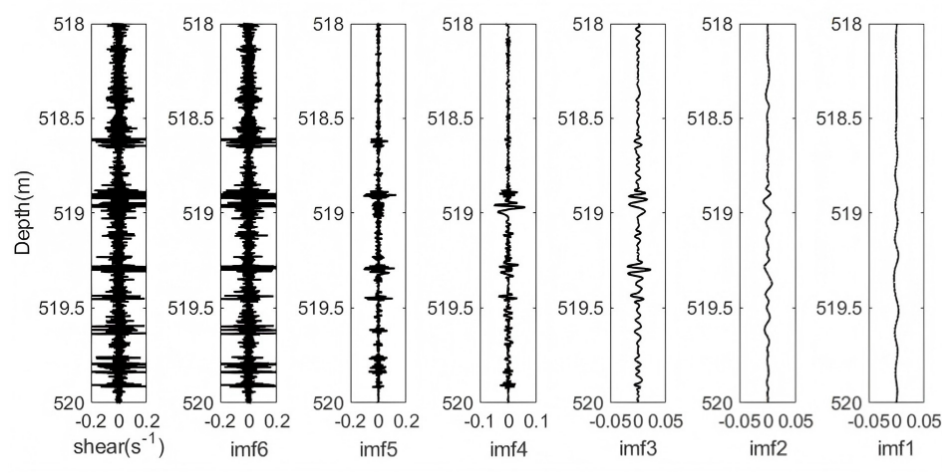


Figure 7. Raw shear signal and corresponding IMF components.

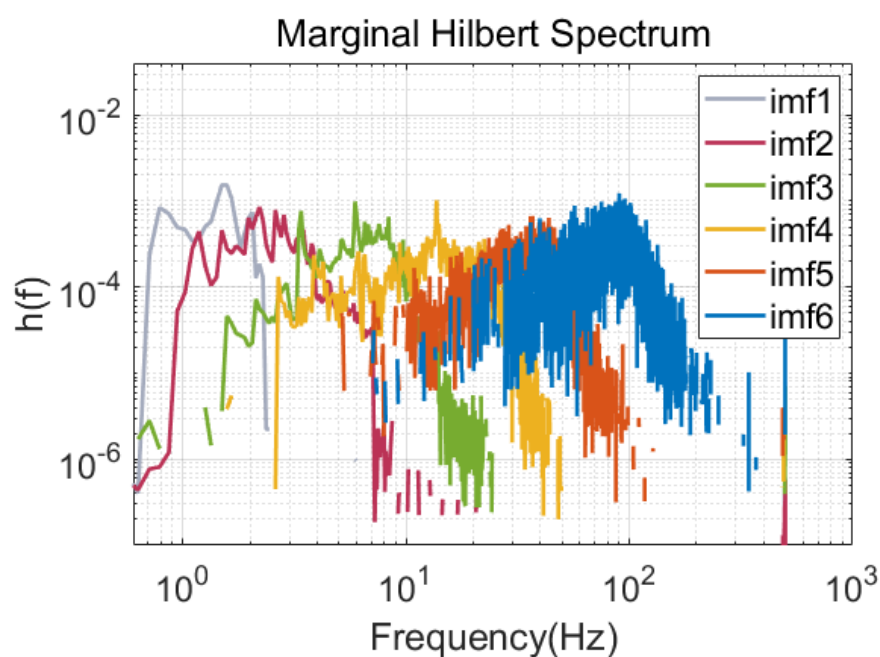


Figure 8. Marginal Hilbert spectrum showing distinct frequency bands of IMF components.

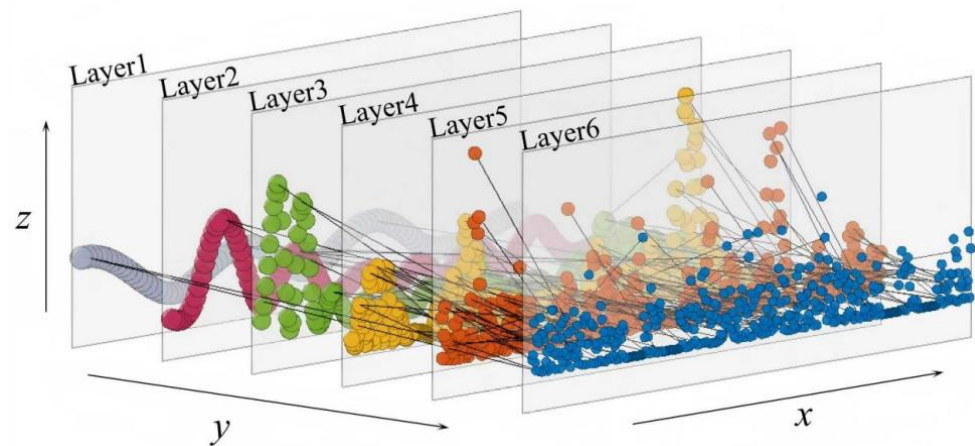


Figure 9. ECMDN constructed from the 518–520 m shear segment. The x-axis represents vertical sampling depth (unit: m), the y-axis represents EMD-decomposed vortex scale layers corresponding to different turbulent spatial scales, and the z-axis represents local intermittency value (LIM, dimensionless). The unified color bar quantifies the strength of inter-layer cross-scale energy coupling edges, with black lines marking all valid energy transfer edges. Forward energy cascade edges (large-to-small scale) and inverse energy cascade edges (small-to-large scale) are explicitly annotated, enabling intuitive visual verification of the coexisting dual-cascade pattern and its depth-scale-dependent distribution characteristics.

The pathways of energy cascade are characterized by the network edges. Considering the overlapping connections, a subgraph connected to the Layer2 network layer was extracted from the multi-layer network results for detailed analysis. As illustrated in Figure 10, nodes (v_{21} , Layer2) located in Layer2 are connected to nodes (v_{31} , Layer3), (v_{41} , Layer4), and (v_{61} , Layer6) in Layer3, Layer4, and Layer6, respectively, and these edges are marked with solid blue lines. The presence of these edges indicates cross-scale energy transfer between vortex structures of scale₂ corresponding to Layer2 and those of scale₃, scale₄, and scale₆ corresponding to Layer3, Layer4, and Layer6, respectively. Similarly, edges are also observed between node (v_{22} , Layer2) in Layer2 and nodes (v_{42} , Layer4) and (v_{62} , Layer6) in Layer4 and Layer6, respectively, marked with solid red lines, suggesting that at this depth, turbulent energy within the vortex structure of scale₂ is transferred only to the vortex structures of scale₄ and scale₆.

Considering the identical energy transfer direction of all inter-layer edges between Layer α and Layer β , the connection between layers is determined in advance before the direction judgment, which represent the global existence of energy transfer path. Given the inter-layer edges in the multi-layer network, the nodes and edges in Figure 9 are aggregated into a single-layer weighted network for connection determination. Each network layer $\text{Layer}\alpha$ in the multi-layer model was aggregated into a single layer-node $\overline{\text{Layer}\alpha}$ in the single-layer network. Furthermore, new weighted edges were formed according to the number of original inter-layer edges. Here, the global connection between $\overline{\text{Layer}\alpha}$ and $\overline{\text{Layer}\beta}$ is given as $c_{\alpha\beta}$. Based on the topological characteristics, the global connection has been summarized, e.g., c_{14} , c_{15} , c_{16} , c_{23} , c_{24} , c_{25} , c_{26} , c_{35} , c_{36} , c_{45} , c_{46} , c_{56} .

Then, the direction of turbulent energy transfer was quantitatively determined using the wavelet bispectrum method. Based on the marginal Hilbert spectrum (Figure 8) and standardized time-space scale conversion, the peak energy frequency and corresponding physical spatial scale (cpm) of each IMF component are extracted, yielding the respective parameters for Layer1–Layer6: $f = [1.3, 2.1, 6, 12, 35, 90]$. As shown in Figure 4, the turbulence inertial subrange cutoff wavenumber is 30 cpm (cycles per meter). Combined with the stable descending velocity $U = 0.45$ m/s of the profiler, the standard wavenumber-frequency conversion formula $f = U \cdot k$ is adopted, and the corresponding cutoff frequency

is calculated as 13.5 Hz. The IMF6 component with a characteristic frequency of 90 Hz is far higher than the inertial subrange cutoff frequency, falling completely within the turbulence dissipation range. Therefore, the vortex structure corresponding to IMF6 is unstable and belongs to the dissipative scale, only receiving energy input from large- and medium-scale vortices without participating in reversible cross-scale energy coupling. Therefore, this study concludes that vortex structure at 90 Hz only receive energy input from larger-scale vortices, indicating a forward transfer process. It means that in the aggregated single-layer network, edges are directed from large-scale layer-nodes $\overline{\text{Layer1}}$, $\overline{\text{Layer2}}$, $\overline{\text{Layer3}}$, $\overline{\text{Layer4}}$, $\overline{\text{Layer5}}$ to dissipative-scale layer-node $\overline{\text{Layer6}}$. $\overline{\text{Layer6}}$, and the global connections c_{16} , c_{26} , c_{36} , c_{46} , c_{56} are ultimately expressed as $\bar{e}_{16}$, $\bar{e}_{26}$, $\bar{e}_{36}$, $\bar{e}_{46}$, $\bar{e}_{56}$ for the directional purpose, as indicated by the red edge annotation in Figure 12.

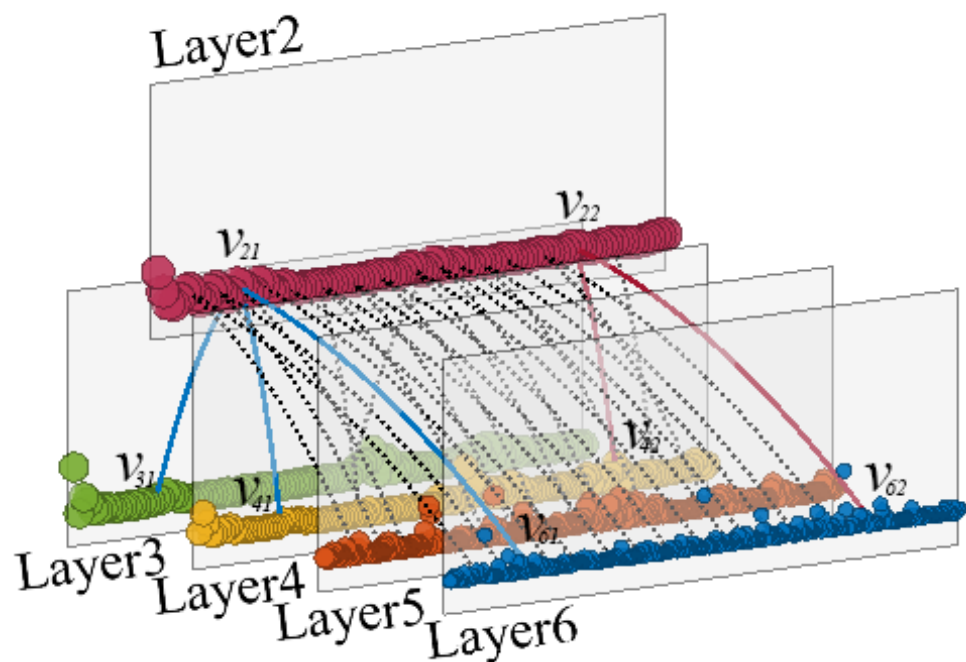


Figure 10. Subgraph illustrating connections originating from Layer 2.

Regarding the remaining directed edges, the imaginary part of the bispectrum results is obtained in Figure 11. The values of $b_{\text{im}}(f_1, \delta, f_4)$, $b_{\text{im}}(f_1, \delta, f_5)$, $b_{\text{im}}(f_2, \delta, f_3)$, $b_{\text{im}}(f_2, \delta, f_4)$, $b_{\text{im}}(f_2, \delta, f_5)$, $b_{\text{im}}(f_3, \delta, f_5)$, and $b_{\text{im}}(f_4, \delta, f_5)$ need to be calculated and used to determine the corresponding directions of c_{14} , c_{15} , c_{23} , c_{24} , c_{25} , c_{35} , and c_{45} . As can be obtained from the imaginary part of the bispectrum results, $b_{\text{im}}(f_1, \delta, f_4)$, $b_{\text{im}}(f_1, \delta, f_5)$, $b_{\text{im}}(f_2, \delta, f_3)$, $b_{\text{im}}(f_2, \delta, f_4)$, $b_{\text{im}}(f_2, \delta, f_5)$, and $b_{\text{im}}(f_4, \delta, f_5)$ are all positive, denoted by upward-pointing red triangles, indicating that turbulent energy is transferred from low-frequency, large-scale vortex structures to high-frequency, small-scale vortex structures.

$b_{\text{im}}(f_1, \delta, f_4) > 0$ and $b_{\text{im}}(f_1, \delta, f_5) > 0$, that is, energy stored in the large-scale scale₁ vortex structure at frequency f_1 is transferred to small-scale vortices with frequencies f_4 and f_5 . $b_{\text{im}}(f_2, \delta, f_3) > 0$, $b_{\text{im}}(f_2, \delta, f_4) > 0$, and $b_{\text{im}}(f_2, \delta, f_5) > 0$, that is, the turbulent energy contained in the vortex structure at scale₂ splits and transfers to small-scale vortex structures corresponding to frequencies f_3 , f_4 , and f_5 . Similarly, $b_{\text{im}}(f_4, \delta, f_5) > 0$ means energy stored in the vortex structure at scale₄ is also transferred in the forward direction to the vortex structure at scale₅. In other words, edges adjacent with layer-node $\overline{\text{Layer1}}$ are directed to layer-nodes $\overline{\text{Layer4}}$ and $\overline{\text{Layer5}}$, and the global connections c_{14} and c_{15} are ultimately expressed as $\bar{e}_{14}$ and $\bar{e}_{15}$, as indicated by the red edge annotation in Figure 12. Edges adjacent to layer-node $\overline{\text{Layer2}}$ are pointed to layer-nodes $\overline{\text{Layer3}}$, $\overline{\text{Layer4}}$, and $\overline{\text{Layer5}}$, and the global connections c_{23} , c_{24} , and c_{25} are ultimately expressed as $\bar{e}_{23}$, $\bar{e}_{24}$, and $\bar{e}_{25}$,

as indicated by the red edge annotation in Figure 12. Moreover, connection c_{45} is translated as $\bar{e}_{45}$, as indicated by the red edge annotation in Figure 12, indicating energy in layer-node $\overline{\text{Layer4}}$ transfer to layer-node $\overline{\text{Layer5}}$.

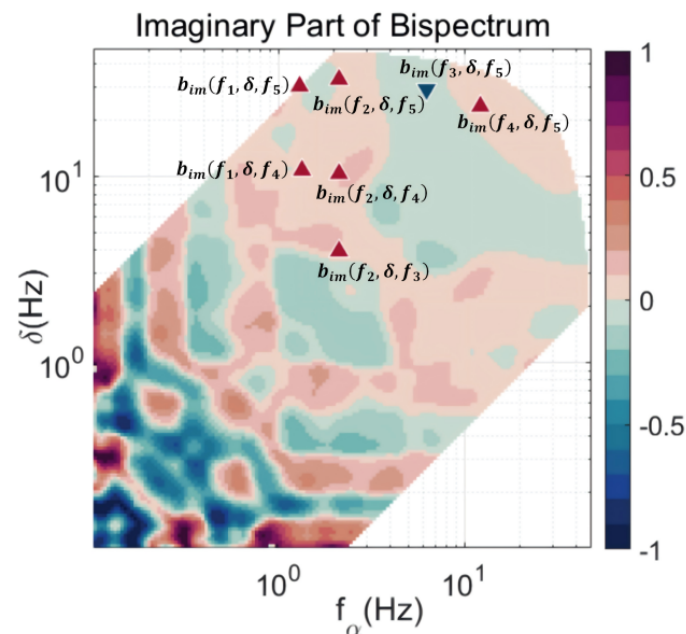


Figure 11. Imaginary part of the normalized bispectrum indicating energy transfer direction.

Conversely, $b_{im}(f_3, \delta, f_5)$ is negative, denoted by a downward-pointing blue triangle, indicating that during the interaction between vortex structures at scale₃ and scale₅, turbulent energy is transferred from the high-frequency f_5 , small-scale scale₅ vortex structure to the low-frequency f_3 , large-scale scale₃ vortex structure. The global connection c_{35} is ultimately expressed as $\bar{e}_{53}$, as indicated by the blue edge annotation in Figure 12. This process represents an inverse energy transfer.

In the single-layer directed weighted network, except for the edge from layer-node $\overline{\text{Layer5}}$ to $\overline{\text{Layer3}}$, all other edges are directed from large-scale layer-nodes to small-scale layer-nodes. This indicates that within the 518–520 m depth segment, the majority of turbulent energy participates in the forward energy cascade, while only a small fraction is involved in the inverse cascade. Compared with traditional spectral flux and third-order structure function diagnostics, the ECMDN results show consistent macroscopic cascade characteristics: traditional integral methods also conclude that ocean thermocline turbulence is dominated by downscale forward energy cascade. However, conventional methods rely on global statistical averaging, discard spatial depth coordinate information, and cannot separate weak inverse energy transfer signals submerged in dominant forward cascade trends. In contrast, the proposed ECMDN framework retains point-wise spatial localization information, effectively captures fine-scale bidirectional energy transfer features at specific depths and scales, and realizes quantitative differentiation of dual cascade behaviors, which is the unique technical advantage of the multilayer network method over traditional turbulence diagnostic tools. Furthermore, analysis based on the single-layer directed weighted network reveals obvious scale-dependent dominant characteristics of dual energy cascades in the 518–520 m strong thermocline. Medium- and large-scale vortex structures (Layer1–Layer4) in this depth segment are completely dominated by forward downscale energy transfer, which is consistent with the classic dynamic characteristics of submesoscale stratified turbulence in the South China Sea Shenhu Sea, where large-scale vortices dissipate energy through gradual downscale splitting. Only the small-scale vortex group corresponding to Layer3–Layer5 presents sporadic inverse upscale energy transfer, forming an asynchronous

dual-cascade pattern of dominant forward cascade and occasional inverse cascade. Spatially, this bidirectional energy transfer phenomenon is stably confined to the strong stratification thermocline layer, with no dual-cascade signal observed in uniform mixed layers. This scale-depth differentiated dual cascade feature is highly consistent with the intermittent nonlinear interaction mechanism of stratified shear turbulence, supplementing fine-scale dynamic evidence for submesoscale ocean turbulence energy evolution.

3.2. Limitations and Future Prospects

This study's ECMDN framework and analytical results have specific conditional limitations based on the current dataset and modeling settings. First, the network edge identification relies on empirical fixed thresholds of LIM and phase synchronization, making the network topology and dual-cascade detection results threshold-dependent to a certain extent. Second, all conclusions in this work are summarized based on a single 518–520 m thermocline segment, lacking multi-depth and multi-region ensemble statistical verification, and the universality of the dual-cascade phenomenon needs further confirmation. Third, in weakly stratified, fully mixed ocean turbulence with low intermittency, traditional spectral analysis and third-order structure function methods have higher computational efficiency and simpler operational processes, showing better applicability. In contrast, the ECMDN method has unique advantages in localized fine-scale dual-cascade identification for strongly stratified intermittent turbulence. Future research will carry out threshold sensitivity analysis and multi-segment ensemble verification to optimize the stability and universality of the proposed framework.

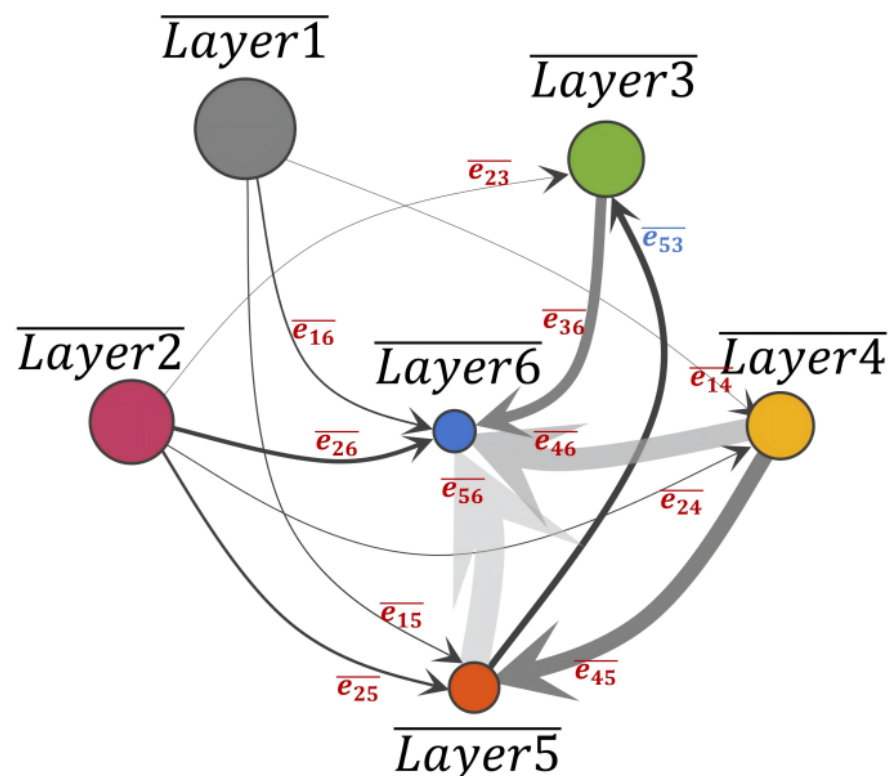


Figure 12. Single-layer directed weighted network summarizing cross-scale fluxes of turbulent energy. Nodes represent six vortex scale layers derived from EMD decomposition, and directed edges represent cross-scale energy transfer directions. Red edges denote forward downscale energy cascade, blue edges denote inverse upscale energy cascade, and edge weights represent the relative intensity of energy coupling. All axes and annotation labels are standardized with physical scale correspondence, clearly showing the scale-dependent dominant law of bidirectional energy transfer in thermocline turbulence.

4. Conclusions

Vortex structures serve as the primary carriers of energy in turbulent flows, governing the energy cascade and dominating turbulent mixing and ocean circulation. In this study, we propose an Energy Cascade Multi-layer Directed Network (ECMDN) model based on complex network theory to characterize cross-scale energy transfer pathways and directions in oceanic turbulence. By integrating multi-parameter fusion for node definition, identifying multi-scale energy interactions, and determining energy transfer directions through wavelet bispectrum analysis, the ECMDN framework successfully maps the nonlinear, intermittent, and inhomogeneous nature of turbulent energy cascades into a topologically interpretable structure. Applied to in situ shear measurements from the Shenhu Sea, the model reveals the coexistence of forward and inverse energy transfer processes within the same spatiotemporal domain—a dual energy cascade—indicating that vortex structures at a given scale can simultaneously receive energy from and transfer energy to vortices at other scales. The results demonstrate that while the majority of turbulent energy undergoes forward cascading toward smaller scales, a notable fraction participates in inverse transfer, highlighting the bidirectional and dynamically coupled nature of energy exchange in real marine environments. This work provides a novel theoretical framework and quantitative tool for investigating the evolutionary mechanisms of turbulent mixing, offering new insights into the complexity of oceanic energy budgets.

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Data Availability Statement: The raw turbulence shear observation data and ECMDN analysis scripts generated and used in this study are available from the corresponding author upon reasonable formal research request.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Richardson, L.F. *Weather Prediction by Numerical Process*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1922.
2. Zhao, S. Synchrocascade Pattern in the Atmospheric Turbulence. *J. Geophys. Res.* **2003**, *108*, 4238. [\[CrossRef\]](#)
3. Kolmogorov, A.N. A Refinement of Previous Hypotheses Concerning the Local Structure of Turbulence in a Viscous Incompressible Fluid at High Reynolds Number. *J. Fluid Mech.* **1962**, *13*, 82–85. [\[CrossRef\]](#)
4. Batchelor, G.K.; Townsend, A.A. The Nature of Turbulent Motion at Large Wave-Numbers. *Proc. R. Soc. Lond. Ser. A. Math. Phys. Sci.* **1949**, *199*, 238–255. [\[CrossRef\]](#)
5. McMillan, J.M.; Hay, A.E.; Lueck, R.G.; Wolk, F. Rates of Dissipation of Turbulent Kinetic Energy in a High Reynolds Number Tidal Channel. *J. Atmos. Ocean. Technol.* **2016**, *33*, 817–837. [\[CrossRef\]](#)
6. Biferale, L. Shell Models of Energy Cascade in Turbulence. *Annu. Rev. Fluid Mech.* **2003**, *35*, 441–468. [\[CrossRef\]](#)
7. Chakraborty, S.; Jensen, M.H.; Sarkar, A. On Two-Dimensionalization of Three-Dimensional Turbulence in Shell Models. *Eur. Phys. J. B* **2010**, *73*, 447–453. [\[CrossRef\]](#)
8. Frisch, U.; Sulem, P.L.; Nelkin, M. A Simple Dynamical Model of Intermittent Fully Developed Turbulence. *J. Fluid Mech.* **1978**, *87*, 719–736. [\[CrossRef\]](#)
9. Parisi, G.; Frisch, U. On the Singularity Structure of Fully Developed Turbulence. In *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*; North-Holland: Amsterdam, The Netherlands, 1985; pp. 84–87.
10. Sreenivasan, K.R. Fractals and Multifractals in Fluid Turbulence. *Annu. Rev. Fluid Mech.* **1991**, *23*, 539–604. [\[CrossRef\]](#)
11. Yamada, M.; Ohkitani, K. Lyapunov Spectrum of a Chaotic Model of Three-Dimensional Turbulence. *J. Phys. Soc. Jpn.* **1987**, *56*, 4210–4213. [\[CrossRef\]](#)

12. Ohkitani, K.; Yamada, M. Temporal Intermittency in the Energy Cascade Process and Local Lyapunov Analysis in Fully-Developed Model Turbulence. *Prog. Theor. Phys.* **1989**, *81*, 329–341. [\[CrossRef\]](#)
13. She, Z.S.; Leveque, E. Universal Scaling Laws in Fully Developed Turbulence. *Phys. Rev. Lett.* **1994**, *72*, 336–339. [\[CrossRef\]](#) [\[PubMed\]](#)
14. Arneodo, A.; Audit, B.; Bacry, E.; Manneville, S.; Muzy, J.F.; Roux, S.G. Thermodynamics of Fractal Signals Based on Wavelet Analysis: Application to Fully Developed Turbulence Data and DNA Sequences. *Phys. A Stat. Mech. Its Appl.* **1998**, *254*, 24–45. [\[CrossRef\]](#)
15. Squire, J.; Meyrand, R.; Kunz, M.W.; Arzamasskiy, L.; Schekochihin, A.A.; Quataert, E. High-Frequency Heating of the Solar Wind Triggered by Low-Frequency Turbulence. *Nat. Astron.* **2022**, *6*, 715–723. [\[CrossRef\]](#)
16. Meyrand, R.; Squire, J.; Schekochihin, A.A.; Dorland, W. On the Violation of the Zeroth Law of Turbulence in Space Plasmas. *J. Plasma Phys.* **2021**, *87*, 535870301. [\[CrossRef\]](#)
17. Miloshevich, G.; Laveder, D.; Passot, T.; Sulem, P.L. Inverse Cascade and Magnetic Vortices in Kinetic Alfvén-Wave Turbulence. *J. Plasma Phys.* **2021**, *87*, 905870201. [\[CrossRef\]](#)
18. Valente, P.C.; Dallas, V. Spectral Imbalance in the Inertial Range Dynamics of Decaying Rotating Turbulence. *Phys. Rev. E* **2017**, *95*, 023114. [\[CrossRef\]](#) [\[PubMed\]](#)
19. Stumpo, M.; Benella, S.; Alberti, T.; Pezzi, O.; Papini, E.; Consolini, G. Relating Intermittency and Inverse Cascade to Stochastic Entropy in Solar Wind Turbulence. *Astrophys. J. Lett.* **2023**, *959*, L20. [\[CrossRef\]](#)
20. Zhu, Y.; Semisalov, B.; Krstulovic, G.; Nazarenko, S. Direct and Inverse Cascades in Turbulent Bose–Einstein Condensates. *Phys. Rev. Lett.* **2023**, *130*, 133001. [\[CrossRef\]](#) [\[PubMed\]](#)
21. Malkin, V.M.; Fisch, N.J. Transition Between Inverse and Direct Energy Cascades in Multiscale Optical Turbulence. *Phys. Rev. E* **2018**, *97*, 032202. [\[CrossRef\]](#) [\[PubMed\]](#)
22. Sahoo, G.; Alexakis, A.; Biferale, L. Discontinuous Transition from Direct to Inverse Cascade in Three-Dimensional Turbulence. *Phys. Rev. Lett.* **2017**, *118*, 164501. [\[CrossRef\]](#) [\[PubMed\]](#)
23. Apostolidis, A.; Laval, J.P.; Vassilicos, J.C. Turbulent Cascade in Fully Developed Turbulent Channel Flow. *J. Fluid Mech.* **2023**, *967*, A22. [\[CrossRef\]](#)
24. de Wit, X.M.; Fruchart, M.; Khain, T.; Toschi, F.; Vitelli, V. Pattern Formation by Turbulent Cascades. *Nature* **2024**, *627*, 515–521. [\[CrossRef\]](#) [\[PubMed\]](#)
25. Balwada, D.; Xie, J.H.; Marino, R.; Feraco, F. Direct Observational Evidence of an Oceanic Dual Kinetic Energy Cascade and Its Seasonality. *Sci. Adv.* **2022**, *8*, eabq2566. [\[CrossRef\]](#) [\[PubMed\]](#)
26. Zhang, Z.; Liu, Y.; Qiu, B.; Luo, Y.; Cai, W.; Yuan, Q.; Liu, Y.; Zhang, H.; Liu, H.; Miao, M.; et al. Submesoscale Inverse Energy Cascade Enhances Southern Ocean Eddy Heat Transport. *Nat. Commun.* **2023**, *14*, 1335. [\[CrossRef\]](#) [\[PubMed\]](#)
27. Pouquet, A.; Marino, R.; Mininni, P.D.; Rosenberg, D. Dual Constant-Flux Energy Cascades to Both Large Scales and Small Scales. *Phys. Fluids* **2017**, *29*, 111108. [\[CrossRef\]](#)
28. Tulloch, R.; Marshall, J.; Hill, C.; Smith, K.S. Scales, Growth Rates, and Spectral Fluxes of Baroclinic Instability in the Ocean. *J. Phys. Oceanogr.* **2011**, *41*, 1057–1076. [\[CrossRef\]](#)
29. Arbic, B.K.; Polzin, K.L.; Scott, R.B.; Richman, J.G.; Shriver, J.F. On Eddy Viscosity, Energy Cascades, and the Horizontal Resolution of Gridded Satellite Altimeter Products. *J. Phys. Oceanogr.* **2013**, *43*, 283–300. [\[CrossRef\]](#)
30. Cheikh, M.I.; Chen, J.; Wei, M. Small-Scale Energy Cascade in Homogeneous Isotropic Turbulence. *Phys. Rev. Fluids* **2019**, *4*, 104610. [\[CrossRef\]](#)
31. Cardesa, J.I.; Vela-Martín, A.; Dong, S.; Jiménez, J. The Temporal Evolution of the Energy Flux Across Scales in Homogeneous Turbulence. *Phys. Fluids* **2015**, *27*, 111702. [\[CrossRef\]](#)
32. Boffetta, G. Energy and Enstrophy Fluxes in the Double Cascade of Two-Dimensional Turbulence. *J. Fluid Mech.* **2007**, *589*, 253–260. [\[CrossRef\]](#)
33. Boffetta, G.; Musacchio, S. Evidence for the Double Cascade Scenario in Two-Dimensional Turbulence. *Phys. Rev. E* **2010**, *82*, 016307. [\[CrossRef\]](#) [\[PubMed\]](#)
34. Balwada, D.; LaCasce, J.H.; Speer, K.G. Scale-Dependent Distribution of Kinetic Energy from Surface Drifters in the Gulf of Mexico. *Geophys. Res. Lett.* **2016**, *43*, 10856–10863. [\[CrossRef\]](#)
35. Poje, A.C.; Özgökmen, T.M.; Bogucki, D.J.; Kirwan, A.D. Evidence of a Forward Energy Cascade and Kolmogorov Self-Similarity in Submesoscale Ocean Surface Drifter Observations. *Phys. Fluids* **2017**, *29*, 020701. [\[CrossRef\]](#)
36. Berta, M.; Griffa, A.; Haza, A.C.; Horstmann, J.; Huntley, H.S.; Ibrahim, R.; Lund, B.; Özgökmen, T.M.; Poje, A.C. Submesoscale Kinematic Properties in Summer and Winter Surface Flows in the Northern Gulf of Mexico. *J. Geophys. Res. Ocean.* **2020**, *125*, e2020JC016085. [\[CrossRef\]](#)
37. Xie, J.H.; Bühler, O. Third-Order Structure Functions for Isotropic Turbulence with Bidirectional Energy Transfer. *J. Fluid Mech.* **2019**, *877*, R3. [\[CrossRef\]](#)

38. Isern-Fontanet, J.; Turiel, A. On the Connection between Intermittency and Dissipation in Ocean Turbulence: A Multifractal Approach. *J. Phys. Oceanogr.* **2021**, *51*, 2639–2653. [[CrossRef](#)]
39. Lee, W.; Kovačič, G.; Cai, D. Cascade Model of Wave Turbulence. *Phys. Rev. E* **2018**, *97*, 062140. [[CrossRef](#)] [[PubMed](#)]
40. McKeown, R.; Ostilla-Mónico, R.; Pumir, A.; Brenner, M.P.; Rubinstein, S.M. Turbulence Generation through an Iterative Cascade of the Elliptical Instability. *Sci. Adv.* **2020**, *6*, eaaz2717. [[CrossRef](#)] [[PubMed](#)]
41. Shirazi, A.H.; Reza Jafari, G.; Davoudi, J.; Peinke, J.; Reza Rahimi Tabar, M.; Sahimi, M. Mapping Stochastic Processes onto Complex Networks. *J. Stat. Mech. Theory Exp.* **2009**, 2009, P07046. [[CrossRef](#)]
42. Iacobello, G.; Ridolfi, L.; Scarsoglio, S. A Review on Turbulent and Vortical Flow Analyses via Complex Networks. *Phys. A Stat. Mech. Its Appl.* **2021**, *563*, 125476. [[CrossRef](#)]
43. Shen, B.; Xiao, S.; Zong, X.; Zhang, C.; Xu, Z.; Liang, X.; Zhou, J.; Fu, W. Functional Adaptation of Brain Network Topology in Individuals with Chronic Ankle Instability: A Graph-Theoretical Study. *Med. Sci. Sports Exerc.* **2025**, *58*, 410–421. [[CrossRef](#)] [[PubMed](#)]
44. Iacobello, G.; Scarsoglio, S.; Ridolfi, L. Visibility Graph Analysis of Wall Turbulence Time-Series. *Phys. Lett. A* **2018**, *382*, 1–11. [[CrossRef](#)]
45. Scarsoglio, S.; Iacobello, G.; Ridolfi, L. Complex Networks Unveiling Spatial Patterns in Turbulence. *Int. J. Bifurc. Chaos* **2016**, *26*, 1650223. [[CrossRef](#)]
46. Liu, C.; Zhou, W.X.; Yuan, W.K. Statistical Properties of Visibility Graph of Energy Dissipation Rates in Three-Dimensional Fully Developed Turbulence. *Phys. A Stat. Mech. Its Appl.* **2010**, *389*, 2675–2681. [[CrossRef](#)]
47. Gürçan, Ö.D. Dynamical Network Models of the Turbulent Cascade. *Phys. D Nonlinear Phenom.* **2021**, *426*, 132983. [[CrossRef](#)]
48. Gao, Z.K.; Lv, D.M.; Dang, W.D.; Liu, M.X.; Hong, X.L. Multilayer Network from Multiple Entropies for Characterizing Gas-Liquid Nonlinear Flow Behavior. *Int. J. Bifurc. Chaos* **2020**, *30*, 2050014. [[CrossRef](#)]
49. Zheng, D.; Zhang, T.; Xing, J.; Mei, J. Improvement of the HHT Method and Application in Weak Vortex Signal Detection. *Meas. Sci. Technol.* **2007**, *18*, 2769–2776. [[CrossRef](#)]
50. Gopinath, S.; Prince, P.R. Investigations into Solar Flare Effects Using Wavelet-Based Local Intermittency Measure. *Acta Geophys.* **2019**, *67*, 687–701. [[CrossRef](#)]
51. Perri, S.; Carbone, V.; Vecchio, A.; Bruno, R.; Korth, H.; Zurbuchen, T.H.; Sorriso-Valvo, L. Phase-Synchronization, Energy Cascade, and Intermittency in Solar-Wind Turbulence. *Phys. Rev. Lett.* **2012**, *109*, 245004. [[CrossRef](#)] [[PubMed](#)]
52. Boccaletti, S.; Bianconi, G.; Criado, R.; del Genio, C.I.; Gómez-Gardeñes, J.; Romance, M.; Sendiña-Nadal, I.; Wang, Z.; Zanin, M. The Structure and Dynamics of Multilayer Networks. *Phys. Rep.* **2014**, *544*, 1–122. [[CrossRef](#)] [[PubMed](#)]

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