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Global Hedging through Post-Decision State Variables

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Abstract: Unlike delta-hedging or similar methods based on Greeks, global hedging is an approach that optimizes some terminal criterion that depends on the difference between the value of a derivative security and that of its hedging portfolio at maturity or exercise. Global hedging methods in discrete time can be implemented using dynamic programming. They provide optimal strategies at all rebalancing dates for all possible states of the world, and can easily accommodate transaction fees and other frictions. However, considering transaction fees in the dynamic programming model requires the inclusion of an additional state variable, which translates into a significant increase of the computational burden. In this short note, we show how a decomposition technique based on the concept of post-decision state variables can be used to reduce the complexity of the computations to the level of a problem without transaction fees. The latter complexity reduction allows for substantial gains in terms of computing time and should therefore contribute to increasing the applicability of global hedging schemes in practice where the timely execution of portfolio rebalancing trades is crucial.

Keywords: hedging; transaction costs; dynamic programming; risk management; post-decision state variable

1. Introduction

In general, hedging designates a variety of trading strategies designed to reduce the risk related to the price movements of certain assets. Hedging strategies often involve trading securities that are closely related; in particular, derivative securities are predominantly used as hedges against their underlying assets.

In this note, we specifically consider the risk related to writing an option contract. It is interesting to recall that one of the basic models for the evaluation of option contracts is based on a hedging argument: in a complete market, the value of an option is equal to the capital required to set up a portfolio providing a perfect hedge, meaning one that produces the same payoff for all possible realizations of the price of the underlying asset. This is the rationale behind the so-called *delta-neutral* option hedging strategy. Assuming that trading can be done in continuous time and that there are no transaction costs, under the delta-neutral hedging strategy, the value of the hedging portfolio is equal to the value of the option at all times.

In practice, it is not possible to adjust hedging positions continuously. Under many market models where the number of traded securities is finite and the number of possible asset price transitions is infinite, it is not possible to eliminate risk completely, that is, to replicate the derivatives' payoff exactly, by using a discrete-time hedging strategy. The cost associated with a complete removal of risk through super-replication is often prohibitive (see Soner et al. 1995). Finally, when trading involves transaction costs, continuous-time hedging strategies may result in infinite costs. It then becomes necessary to design discrete-time hedging strategies that incorporate transaction costs. There are multiple papers studying the impact of transaction costs in hedging schemes, including the following: (Hodges and Neuberger 1989; Boyle and Vorst 1992; Grannan and Swindle 1996; Toft 1996; Clewlow and Hodges 1997; Martellini and Priaulet 2002; Zakamouline 2006; Zhao and Ziemba 2007).

In the context of an incomplete market, global hedging approaches aim at finding hedging strategies that optimize a criterion based on the terminal payoff at maturity. In other words, instead of trying to track the option value at all times, global hedging approaches consider only the end result of successive trades. Global hedging approaches are particularly interesting when risk cannot be completely eliminated or when transaction costs are present. The criterion considered may be a combination of risk and return considerations. For instance, assuming that the option is exercised at date T, and denoting the option payoff and the value of the hedging portfolio at T by C_T and V_T, respectively, the following criteria, among others, have been proposed in the literature (see for instance Schweizer 1995; Follmer and Leukert 2000; Francois et al. 2014; Hodges and Neuberger 1989; Ni et al. 2012):

- Expected shortfall: $E\left[(C_T - V_T)^+\right]$,
- Quadratic risk: $E\left[(C_T - V_T)^2\right]$,
- Semi-quadratic risk: $E\left[\left((C_T - V_T)^+\right)^2\right]$,
- Risk-return utility: $E\left[C_T - V_T + \lambda\,(C_T - V_T)^2\right], \quad \lambda \geq 0$,
- Exponential utility: $E\left[-e^{\lambda(C_T - V_T)}\right], \quad \lambda \geq 0$.

The use of global hedging procedures is currently hindered in practice by the heavy numerical burden associated with dynamic programs calculating the solution to the optimization problem. This issue is critical due to the importance of timely execution of trades when rebalancing hedging portfolios. The current work proposes a decomposition methodology based on post-decision state variables, which reduces the complexity of computations associated with the solution of a global hedging problem in the presence of transaction costs to the level of a problem without transaction fees. This approach reduces the dimensionality of the problem and leads to substantial gains in terms of computing time. The approach presented in the current paper should therefore contribute to increase the applicability of global hedging algorithms in practice by alleviating the issue of high computing time and thus allowing for a faster and more precise execution of trades during portfolio rebalancing operations.

2. Discrete-Time Global Hedging Model

Consider a set $\mathcal{T} := \{0, \ldots, T\}$ of discrete dates and a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ endowed with a filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ satisfying the usual conditions. Two assets are traded in an arbitrage-free market. Asset B is risk-free and its deterministic price at date t is denoted by B_t. Asset S is risky; its price process $\{S_t\}$ is $\mathcal{F}$-adapted and has the Markov property with respect to $\mathcal{F}$. An agent is hedging a short position on a simple contingent claim $C_T = C(S_T)$ with a self-financing portfolio.

In a hedging context, $\mathcal{T}$ is the set of rebalancing dates. For $t \in \mathcal{T}$, let $\theta_t := (x_t, y_t)$ represent the number of shares of assets B and S, respectively, within the portfolio at date t, before the rebalancing decision. A trading strategy $\theta = \{\theta_t\}_{t \in \mathcal{T}}$ is an $\mathcal{F}$-predictable process.[1] The portfolio value at date t associated with the trading strategy θ is defined as

$$V_t^\theta := B_t x_t + S_t y_t.$$

[1] θ being $\mathcal{F}$-predictable means that θ_0 is $\mathcal{F}_0$-measurable and that for all $t \in \{1, \ldots, T\}$, θ_t is $\mathcal{F}_{t-1}$-measurable.

When there are no transaction costs, a trading strategy is said to be self-financing if, for all $t \in \{0, 1, \ldots, T-1\}$,

$$B_t x_t + S_t y_t = B_t x_{t+1} + S_t y_{t+1}.$$

The global hedging problem that is solved by the agent is then

$$\min_{\theta \in \Theta} \mathbb{E}\left[g\left(C_T - V_T^{\theta}\right)\right] \tag{1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is called the penalty function and Θ is the set of admissible self-financing trading strategies. Under mild conditions on functions C and g and on the process $\{S_t\}$, it can be shown that Problem (1) has a solution.

This shows why the numerical burden associated with global hedging procedures is larger than with delta-hedging; the former relies on some optimization problem that must be solved, whereas delta-hedging only requires differentiating option prices with respect to the underlying asset value. Delta-hedging however has the disadvantage of being a myopic hedging strategy as it only considers risk for small variations of the underlying asset price. This becomes problematic when the hedge is performed in discrete time or when jumps occur within the underlying asset price path, e.g. as the result of occasional large price movements.

At date $t \in \mathcal{T}$, define the value function:

$w_t(s, v)$: expected minimal global hedging penalty at date t if the current price of the underlying asset is s and the current portfolio value is v.

The global hedging problem can then be characterized by the following discrete-time dynamic program.

$$w_T(s, v) = g\left(C(s) - v\right) \tag{2}$$

$$w_t(s, v) = \min_{d \in D_t(s,v)} \mathbb{E}_t\left[w_{t+1}\left(S_{t+1}, B_{t+1}\frac{(v - sd)}{B_t} + S_{t+1}d\right)\right], t = 0, \ldots, T-1 \tag{3}$$

$$\delta_t(s, v) \in \arg\min_{d \in D_t(s,v)} \mathbb{E}_t\left[w_{t+1}\left(S_{t+1}, B_{t+1}\frac{(v - sd)}{B_t} + S_{t+1}d\right)\right], t = 0, \ldots, T-1 \tag{4}$$

where, at each date $t < T$, a rebalancing decision $y_{t+1} = d$ is taken; $\mathbb{E}_t[\cdot]$ represents the expectation, conditional to $(S_t, V_t) = (s, v)$, which is information available at t; and $D_t(s, v)$ contains any constraint or limitation on the number of shares of asset S that can be held at t after the rebalancing, given that $(S_t, V_t) = (s, v)$.

The portfolio value at $t+1$ reflects the self-financing constraint

$$v = B_t x_{t+1} + sd$$

so that

$$\begin{aligned} V_{t+1} &= B_{t+1} x_{t+1} + S_{t+1} d \\ &= B_{t+1}\frac{(v - sd)}{B_t} + S_{t+1} d. \end{aligned}$$

Given any initial portfolio $\theta_0 = (x_0, y_0)$, setting

$$\theta_{t+1} = \left(\frac{V_t - S_t \delta_t\left(S_t, V_t\right)}{B_t}, \delta_t\left(S_t, V_t\right)\right)$$

for $t = 0, \ldots, T-1$ yields a hedging strategy solving Problem (1).

Notice that the solution of the dynamic program (2)–(4) characterizes the optimal hedging strategy at all rebalancing dates, for all possible instances of the underlying asset price and portfolio value.

The method is general and can be applied to any derivative characterized by a payoff function that depends on the price of the underlying asset at maturity or exercise,[2] to any penalty function g, and to any market model, in all cases provided that the conditional expectation in (3) can be computed or approximated.

Since, in many market models, asset prices are continuous and possibly unbounded, some form of approximation is needed for the computation of the value function (see for instance Breton and de Frutos 2012). Assume that M discrete possible values are considered for each of the three variables s, v and d. The complexity of the recursion (2)–(4) is $\mathcal{O}(M^4)$ for each time-step iteration of the dynamic program since it involves looping over all discretized values for state variables s and v, all decisions d and all transitions S_{t+1}, at each rebalancing date. Typically, finding the optimal hedging strategy for 12 time steps using a cubic splines interpolation scheme with 100 discretized values for each variable requires around one minute of computing time on a standard personal workstation.

Analogous continuous-time versions of the hedging problem (1) relying on the solution of the Hamilton-Jacobi-Bellman equation are presented in the literature; see for instance (Hodges and Neuberger 1989). However, as the partial differential equation often cannot be solved in closed-form in continuous-time for more complex models, the solution relies on the discretization of time, which showcases the relevance of the discrete-time version of the problem.

3. Transaction Fees and Post-Decision Variables

Under most market specifications, uncertainty about asset prices decreases with the frequency of observation dates. As a result, when there are no transaction costs, the expected hedging penalty can often be reduced by increasing the number of rebalancing dates. For instance, when trading is continuous, one can obtain a perfect hedge in the geometrical Brownian motion model by applying the delta-hedging strategy at an infinite number of rebalancing dates. This observation raises the issue of transaction costs, which should normally be considered when setting up a hedging strategy.

Where there are transaction costs, the self-financing constraint becomes, for all $t \in \{0, 1, \ldots, T-1\}$,

$$B_t x_t + S_t y_t = B_t x_{t+1} + S_t y_{t+1} + K_t$$

where $K_t = \kappa(S_t, y_t, y_{t+1})$ are transaction costs. Various types of costs can be considered, for instance,

$$K_t = \begin{cases} c\mathbb{I}_{\{y_{t+1} \neq y_t\}} & \text{fixed} \\ c\,|y_{t+1} - y_t| & \text{proportional to traded number of shares} \\ c\,|y_{t+1} - y_t|\,S_t & \text{proportional to traded market value} \end{cases}$$

for some positive constant c. The inclusion of transaction costs in the DP model requires an additional state variable because the composition of the portfolio is needed to compute the costs associated to a rebalancing decision.

At date $t \in \mathcal{T}$, define the value function $\omega_t(s, x, y)$: expected minimal global hedging penalty at date t if the current price of the underlying asset is s and the current portfolio composition is (x, y).

The global hedging problem can then be characterized by the dynamic program

$$\omega_T(s, x, y) = g\left(C(s) - (xB_T + ys)\right) \quad (5)$$

$$\omega_t(s, x, y) = \min_{d \in D_t(s,x,y)} \mathbb{E}_t\left[\omega_{t+1}\left(S_{t+1}, \frac{B_t x + s\,(y-d) - \kappa(s, y, d)}{B_t}, d\right)\right], t = 0, ..., T-1 \quad (6)$$

[2] The adaptation of the recursion for Bermudan instruments where early exercise is allowed at discrete dates is straightforward.

and the optimal hedging strategy by a function $\delta_t(s, x, y)$ achieving the minimum in (6). This approach to solving the global hedging problem is followed for instance in (Godin 2016). Note that when the length of time steps is small, the DP model endogenously decides on the optimal rebalancing frequency, as it would choose not to rebalance when it is not optimal to do so.

Assuming again that M discrete possible values are considered for each of the variables s, x, y and d, the complexity of the recursion becomes $\mathcal{O}(M^5)$ because of the additional state variable. Finding the optimal hedging strategy for 12 time steps using a cubic splines interpolation scheme with 100 discretized values for each variable, using the recursion (5)–(6), would now require roughly one hour of computing time on a standard personal workstation.

It is, however, possible to reduce the complexity of the algorithm to solve (1) to $\mathcal{O}(M^4)$, using a decomposition technique relying on the concept of post-decision state variables, as discussed in (Powell 2007).

For $t = 0, \ldots, T-1$, define an auxiliary post-decision value function through

$$\varphi_t(s, q, d) := \mathbb{E}_t\left[\omega_{t+1}\left(S_{t+1}, q, d\right)\right] \tag{7}$$

and set

$$h_t(s, x, y, d) = \frac{B_t x + s\,(y - d) - \kappa(s, y, d)}{B_t}.$$

The dynamic program becomes

$$\begin{aligned} \omega_T(s, x, y) &= g\left(C(s) - (xB_T + ys)\right) && (8)\\ \varphi_t(s, q, d) &= \mathbb{E}_t\left[\omega_{t+1}\left(S_{t+1}, q, d\right)\right], t = 0, ..., T-1 && (9)\\ \omega_t(s, x, y) &= \min_{d \in D_t(s,x,y)} \varphi_t\left(s, h(s, x, y, d), d\right), t = 0, ..., T-1. && (10) \end{aligned}$$

Equation (9) considers all possible transitions from the tri-dimensional state variable, while Equation (10) considers all possible rebalancing decisions at each state, so that the complexity is reduced to $\mathcal{O}(M^4)$ by performing the computations (9)–(10) sequentially instead of simultaneously.

The reason why using post-decision state variables makes it possible to reduce the complexity of the dynamic programming algorithm in the current framework is that the evolution of the asset price does not depend on the hedging decisions, and the post-decision state variable (the portfolio composition) does not depend on the transition of the exogenous asset price variable. Therefore, the dimension of the post-decision state variable is no greater than that of the state variable. This decomposition technique would work for any dynamic program where the evolution of endogenous and exogenous variable values at a given time step are independent. This is not the case for the global hedging problem without transaction fees presented in Section 2, where the endogenous state variable (the value of the portfolio) depends on the evolution of the underlying asset price.

4. Conclusions

An efficient method for obtaining the solution of a global hedging problem in the presence of transaction costs is presented. The method relies on a modification of the usual dynamic programming algorithm in order to incorporate post-decision state variables. This makes it possible to reduce the computational complexity from $\mathcal{O}(M^5)$ to $\mathcal{O}(M^4)$. This technique is shown to be applicable in the presented framework since endogenous variable transitions are not impacted by exogenous variable movements.

In finance, the ability to perform computations quickly is crucial for the timely execution of trades. Therefore, obtaining the hedging policy in less than a minute through the method presented in this paper, rather than in roughly an hour with the usual method, represents an appreciable improvement. The post-decision state variable approach should therefore increase the applicability of global hedging schemes in practice, which are currently hindered by their heavy associated numerical burden.

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