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Option Pricing with the Logistic Return Distribution

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Abstract: The Black–Scholes model and many of its extensions imply a log-normal distribution of stock total returns over any finite holding period. However, for a holding period of up to one year, empirical stock return distributions (both conditional and unconditional) are not log-normal, but rather much closer to the logistic distribution. This paper derives analytic option pricing formulas for an underlying asset with a logistic return distribution. These formulas are simple and elegant and employ exactly the same parameters as B&S. The logistic option pricing formula fits empirical option prices much better than B&S, providing explanatory power comparable to much more complex models with a larger number of parameters.

Keywords: option pricing; distribution of returns; logistic distribution; holding period

JEL Classification: G13

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1. Introduction

The seminal works of Black and Scholes (1973) and Merton (1973) are the foundation for a wide spectrum of option pricing models, incorporating stochastic volatility (see for example, Hull and White (1987); Scott (1987); Wiggins (1987); Melino and Turnbull (1990); Stein and Stein (1991); Heston (1993); Naik (1993); Duffie et al. (2000); and Barone-Adesi et al. (2008)), jumps (Merton (1976); Bates (1991) and Madan et al. (1998); Kou (2002); Eraker (2004); Cremers et al. (2008); Ackerer and Filipović (2020)), stochastic interest rates (Merton (1973) and Amin and Jarrow (1992)), leverage effects (Geske (1979)), and various combinations of these factors (Bailey and Stulz (1989); Amin and Ng (1993); Bakshi et al. (1997); Pan (2002); Willems (2019); Kirkby and Nguyen (2020); Bartl et al. (2020)). In the basic B&S model, the volatility is constant, and the underlying asset price follows a geometric Brownian motion, implying that the total return distribution (1 + rate of return) is log-normal for any finite holding period. In the stochastic volatility models, the unconditional return distribution is a mixture of the conditional distributions across states and may thus have fat tails. However, the conditional return distribution for a given volatility level is still log-normal. Jumps may induce deviations from the log-normal distribution; however, as these jumps are typically infrequent, the bulk of the distribution remains approximately log-normal.[1]

This feature of the main option pricing models is in sharp contrast to the empirical evidence: statistical tests clearly reject the log-normality of returns. For horizons of up to a year, which is the relevant horizon for most options, the empirical total return distribution is very different from the log-normal. Rather, it is much closer to the logistic distribution (Mantegna and Stanley (1995); Levy and Duchin (2004)). This is true for both the unconditional and the conditional distributions. Figures 1 and 2 illustrate this point. Figure 1 shows the empirical distribution of one-day total returns on the S&P 500 index, the best log-normal fit and the best logistic fit to this distribution. It is evident that the logistic distribution fits the data much better, and this is confirmed by formal tests. In fact, out of 10 theoretical distributions widely used in the literature, the logistic distribution provides the best fit (we expand on these points in the next section). Figure 2 shows the same result for the return distribution conditioned on the previous day's VIX. This empirical observation of the rejection of log-normality in favor of the logistic distribution is the motivation for the present paper.

This paper analytically derives option pricing formulas for the case of the logistic return distribution. Our approach is different than the standard option pricing approach in the following sense: rather than starting from an instantaneous return process, our starting point is the distribution of the underlying asset's value at expiration. We do not make any assumptions about the instantaneous return process (such as a Brownian motion). Instead, we directly assume a logistic distribution of stock prices at expiration, and we employ the no-arbitrage argument to derive the call option price via risk-neutral valuation. From this price we also derive the hedge ratio facilitating continuous hedging. What is the origin of the empirically observed logistic return distribution? We suggest a possible explanation, but this is rather speculative, and alternative explanations are certainly possible. The important point is that regardless of the reason for the logistic stock price distribution at expiration, this distribution allows us to derive a simple analytical formula for option prices.

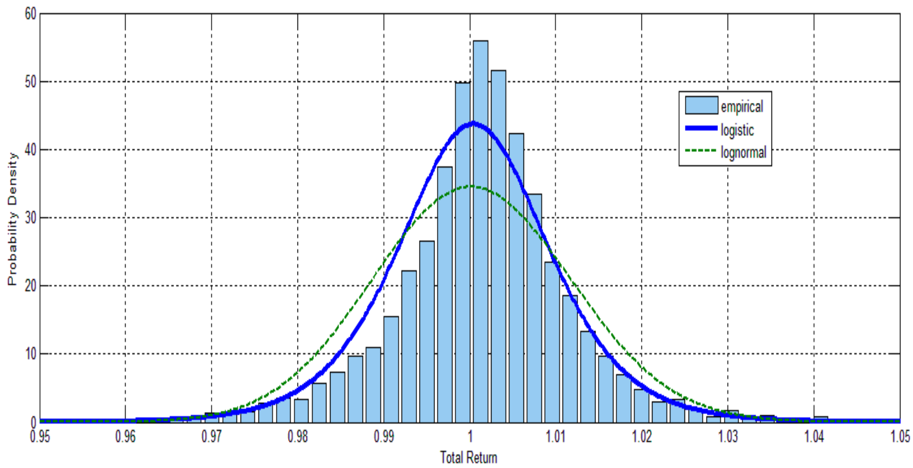

Figure 1. The unconditional daily return distribution for the S&P 500 index. The dashed line is the best log-normal fit. The bold line is the best logistic fit. The logistic distribution provides the best fit of all 10 theoretical distributions examined. The description of the data and more details are provided in Section 2.

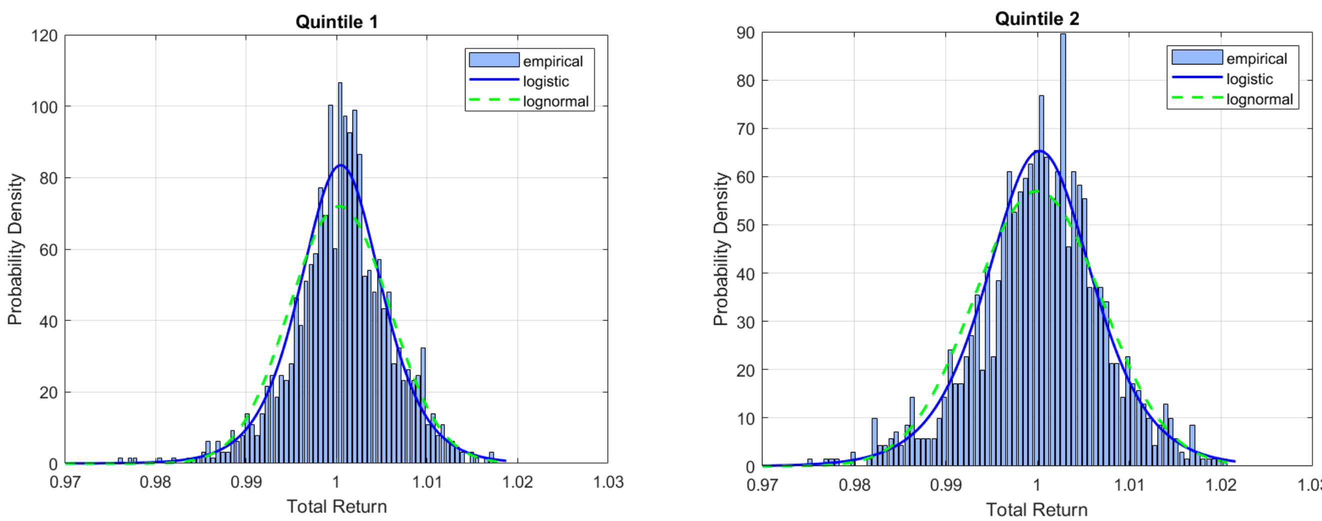

Figure 2. *Cont.*

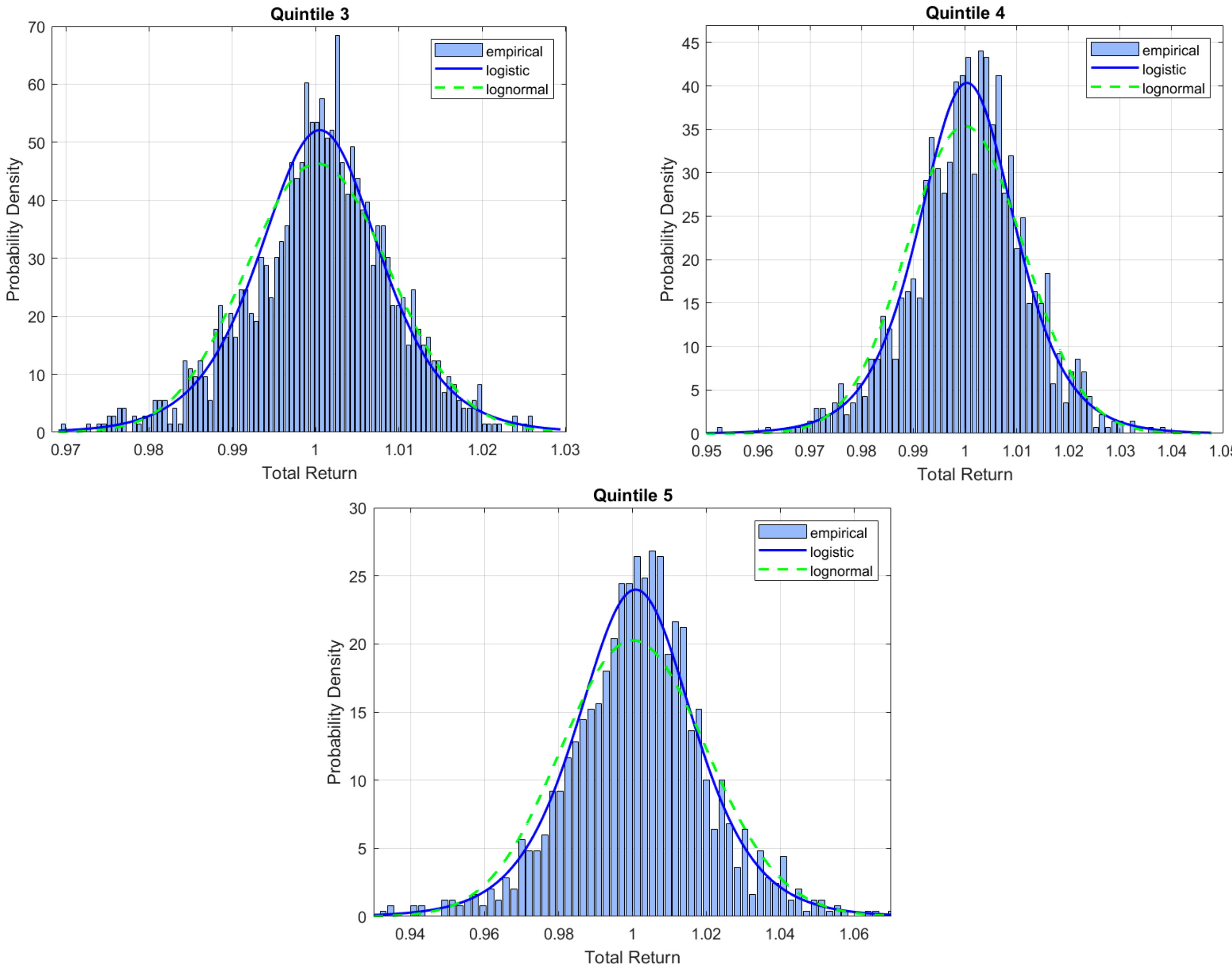

Figure 2. The distributions of daily total returns conditioned on the pervious-day's VIX. Quintile 1 is the one with the lowest VIX. For all five quintiles, the logistic distribution fits better than the log-normal (see also Tables 1 and 3).

Table 1. Goodness of fit for the conditional and unconditional daily return distribution.

	Unconditional	Quintile 1 (Lowest VIX)	Quintile 2	Quintile 3	Quintile 4	Quintile 5 (Highest VIX)
Logistic	−18,958	−4566	−4274	−4010	−3701	−3063
Log-logistic	−18,955	−4565	−4273	−4009	−3700	−3062
Rice	−18,399	−4534	−4253	−4005	−3683	−3012
Normal	−18,399	−4534	−4253	−4005	−3683	−3011
Gamma	−18,396	−4533	−4252	−4004	−3681	−3011
Log-normal	−18,394	−4532	−4251	−4004	−3680	−3010

This unorthodox approach has pros and cons relative to the traditional approach of starting with an instantaneous return process. The disadvantages are that (1) the model does not explain *why* the distribution is logistic and (2) the logistic distribution can only serve as an approximation because it is not limited to the positive domain; in addition, the sum of two logistic random variables is not exactly logistically distributed (thus, if the one-day return distribution is exactly logistic, the two-day return distribution can only be approximately logistic). We advocate that these limitations are not severe, because even if the logistic distribution only approximates the true distribution, for the relevant holding periods this approximation is very good (and much better than the log-normal), and it is thus very useful. The advantage of our approach is that it yields a simple analytic option pricing formula that is based directly on the empirically observed return distribution. We believe that the non-standard approach suggested here compliments the traditional approach and yields a new and useful perspective for option pricing.

The option pricing formulas derived for the logistic case are surprisingly simple and elegant. They employ the same parameters as B&S, with a single variable that needs to be estimated: volatility. They are even easier to implement than B&S. Thus, the challenging econometric issues involved with the sophisticated option pricing models are circumvented. Empirically, the logistic option pricing formulas perform much better than B&S and are comparable to more complex option pricing models with a large number of parameters. We view this model as a first step. One could potentially extend the logistic option pricing model presented here to incorporate stochastic volatility, stochastic interest rates, etc., which may further improve the performance of the model. Of course, this would come at a price of increased complexity. For the sake of clarity, in this paper we present the simplest logistic option pricing model, which can serve as a springboard for further extensions.

The structure of this paper is as follows. The next section shows that the empirical return distribution is best fitted by the logistic distribution, which is the motivation of the paper. Section 3, which is the main contribution of the paper, derives option prices under the logistic return distribution. Section 4 examines the empirical performance of the logistic option pricing formula. In Section 5, we suggest a possible explanation for the empirically observed logistic return distribution. Section 6 concludes.

2. The Empirical Return Distribution

It has been long known that the log-normal distribution does not fit the empirical return distribution well. Focusing on short-horizon returns, Mandelbrot (1963) and Fama (1965) suggest that the stable Pareto (Lévy) distribution provides a better fit. Officer (1972) suggests that the distribution should have fat tails but a finite second moment. These ideas are supported by Mantegna and Stanley (1995), who find that a truncated stable Paretian distribution fits the empirical return distribution well for holding periods of up to about a month (more on the relation between the truncated stable Paretian distribution and the logistic distribution in Section 5). One of the most comprehensive studies comparing the empirical return distribution with a large number of theoretical distributions is that of Levy and Duchin (2004). They examine the fit of many different distributions as a function of the holding period and conclude that:

> "*For investment horizons shorter than one year the logistic distribution best describes the empirical data*" (p. 61).

They find this result for stock indices, individual stocks, and even bonds.[2]

In examining the shape of the return distribution, we confirm and extend the results of Levy and Duchin (2004) in the following ways. First, we employ a different and longer sample period. Second, we include in our analysis several additional theoretical candidate distributions not considered in the Levy and Duchin analysis. Third, we employ several different criteria for goodness-of-fit. Most importantly, we examine not only the unconditional return distribution, but also the return distribution conditional on the anticipated volatility, as captured by the VIX index.

We examine the returns of the S&P 500 index for holding periods of 1 day up to 1 year, which is the range of horizons relevant for most options. For the one-day horizon we employ the daily returns on the S&P 500 index over the period from January 1990–December 2013, as reported in the CRSP files. We test the goodness-of-fit to the empirical return distribution by 10 theoretical distributions employed in the literature: log-normal, normal, logistic, gamma, inverse-Gaussian, log-logistic, Weibull, Rice, generalized extreme value, and beta distributions. We employ two criteria for goodness-of-fit: negative log-likelihood and the Bayesian information criterion.

For a general impression, Figure 1 plots the empirical daily return histogram, the best log-normal fit, and the best logistic fit. The logistic distribution clearly fits better than the log-normal. This is confirmed by the formal tests. When comparing the 10 potential distributions with the empirical data, we find that the distribution that provides the best fit is the logistic distribution, followed by the log-logistic distribution, the Rice distribution, the normal distribution, and the gamma distribution. The log-normal distribution comes in sixth place (see Table 1). We obtain the exact same ranking with the three different goodness-of-fit criteria.

As the market is characterized by periods of high daily volatility and other periods of low daily volatility, one could argue that the unconditional return distribution is a mixture of several different distributions corresponding to different market conditions. In order to examine the conditional return distribution, we partition the return sample into five quintiles, sorted by the previous day's level of the VIX index. For each quintile, the conditional return distribution is the set of next-day returns. Hence, we have five different empirical conditional return distributions. For each of these distributions, we examine which theoretical distribution fits best. Table 2 provides some summary statistics on the VIX quintiles and the five conditional return distributions. As expected, the variance in the return distribution increases with the VIX quintile, implying that there is a positive relation between the VIX and the variance in the next-day return. Figure 2 shows the conditional daily return distributions for the five quintiles, as well as the best log-normal fit and best logistic fit for each distribution. The figure reveals that the logistic distribution provides a better fit than the log-normal for each of the five conditional distributions. This is confirmed by the formal goodness-of-fit test: the logistic distribution provides the best fit for each of the five conditional empirical distributions, followed by the log-logistic and the Rice distributions. We obtain the same ranking for all five conditional distributions, and by all three goodness-of-fit criteria. Table 1 reports the negative log-likelihood for all five conditional return distributions, as well as for the unconditional return distribution (recall that the more negative the statistic, the better the fit).

Table 2. Summary statistics.

Quintile	1	2	3	4	5
VIX range (%)	9.3–13.6	13.6–16.9	16.9–20.5	20.5–25.1	25.1–80.9
Mean of next-day return: (%)	0.04	0.01	0.01	0.02	0.07
Standard deviation of next-day return: (%)	0.56	0.70	0.86	1.14	1.97

The table reports the negative log-likelihood for various candidate distributions to fit the empirical daily return distribution. The first column corresponds to the unconditional return distribution, as shown in Figure 1. The next columns correspond to the daily return distribution conditional on the previous day's VIX, as reported in Figure 2. In all cases the logistic distribution provides the best fit, while the lognormal comes in sixth place (recall that the more negative the negative log-likelihood, the better the fit). The exact same ranking is obtained with the BIC goodness-of-fit criterion.

How does the length of holding period affect the return distribution? In order to examine this issue, we analyze the one-month, three-month, six-month, nine-month, and twelve-month return distributions of the S&P 500 index. For each holding period we report the six distributions that provide the best fit, in order of their performance, by the two goodness-of-fit criteria. The results are shown in Table 3. As the table reveals, the logistic distribution provides the best fit for all horizons shorter than a year. This holds for both of the criteria employed. Raftery (1995) interprets a difference of 0–2 in the BIC as "weak" evidence, a difference of 2–6 as "positive" evidence, 6–10 as "strong" evidence, and a BIC difference exceeding 10 as "very strong" evidence. Table 3 reveals that for monthly returns, the evidence in favor of the logistic distribution is "very strong" in comparison to all other alternatives. As the horizon increases, the difference between the logistic and the log-logistic, which provides the next best fit, decreases. For annual horizons the distribution that fits best is either the Weibull distribution or the generalized extreme value distribution, depending on the criterion employed. This is consistent with the findings of Levy and Duchin, who find that the logistic distribution dominates for horizons of up to a year but the Weibull distribution provides the best fit for annual returns (see Levy and Duchin 2004, Exhibit 3 on p. 53).[3]

The 1990–2013 daily sample is partitioned into five quintiles, sorted by the VIX. We create five conditional distributions of the next-day returns. For example, the return distribution of quintile 1 is the set of returns on the days following the days with the lowest VIX, etc. The second row shows the VIX cut-off points for each quintile, and the third and fourth rows provide the mean return and standard deviation of returns for the associated conditional return distributions. As expected, higher VIX values are associated with more volatile next-day returns. The entire distributions are shown in Figure 2.

For each holding-period, the table ranks the six distributions that provide the best fit to the empirical returns, in order of their performance. The logistic distribution provides the best fit for all horizons shorter than a year, by both log-likelihood and the BIC criterion.

Table 3. The return horizon and the best-fitting distributions.

Goodness-of-Fit Criterion	1 Month	3 Months	6 Month	9 Months	12 Months
Negative log-likelihood	**logistic—1667** log-logistic—1660 normal—1568 Rice—1568 Gamma—1566 log-normal—1562	**logistic—324** log-logistic-322 log-normal—286 inverse Guassian—285 gamma—285 Rice—274	**logistic—101** log-logistic-97 normal—94 Rice—94 gamma—90 log-normal—86	**logistic—42** Weibull—41 normal—40 Rice—40 general extreme value—39 log-logistic—37	**general extreme value—22** Weibull—21 normal—19 Rice—19 logistic—19 gamma—16
Bayesian information criterion	**logistic—3319** log-logistic—3306 normal—3122 Rice—3122 gamma—3118 log-normal—3110	**logistic—635** log-logistic—632 log-normal—561 inverse Guassian—559 gamma—558 Rice—537	**logistic—192** log-logistic—185 normal—177 Rice—177 gamma—170 log-normal—162	**logistic—73** Weibull—72 normal—70 Rice—70 general extreme value—64 log-logistic—64	**Weibull—34** normal—30 Rice—30 general extreme value—30 logistic—28 gamma—23

Most options have maturities less than a year. The relevant return distribution for pricing these options is the logistic distribution. For horizons of up to 6 months, the distribution which provides the second-best fit is the log-logistic. This distribution has the theoretical advantage of total returns being restricted to the positive domain. However, even for annual returns, a negative total return represents an event that is about 5.5 standard deviations to the left of the mean, so this is not a major consideration. The logistic return distribution not only provides the best fit to the empirical distribution, but it also implies very elegant option pricing formulas, as shown in the next section.

3. Logistic Option Pricing

Consider an underlying asset with a value at maturity distributed according to some probability density function $f(x)$. The absence of arbitrage implies that one can employ risk-neutral valuation to price options on this asset. In a risk-neutral world, the value of a European Call option with strike price K and maturity T is given by:

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \int_K^{\infty} (x-K) f(x) dx, \tag{1}$$

where C_0 is the Call price and r_f is the risk-free rate. In the risk-neutral world, the current price of the underlying asset is its expected value at maturity, discounted at the risk-free rate: $S_0 = \frac{1}{\left(1+r_f\right)^T} E[x]$. One way to obtain the B&S formula is to plug the log-normal distribution of terminal asset values implied by the B&S geometric Brownian motion as $f(x)$ in Equation (1) (see, for example, Cox et al. (1979); and Hull (2009))[4]. In what follows, we employ Equation (1) with the logistic distribution for $f(x)$ to obtain logistic option pricing.

The logistic distribution p.d.f. is given by[5]:

$$f(x;\mu,s) = \frac{e^{-\frac{x-\mu}{s}}}{s\left(1+e^{-\frac{x-\mu}{s}}\right)^2}, \tag{2}$$

where μ is the mean of the distribution and the standard deviation is $\sigma = \frac{\pi s}{\sqrt{3}}$. It is convenient to express the logistic distribution and its integrals using the hyperbolic functions defined as (see, for example, Becker 1931):

$$\sinh(x) \equiv \frac{e^x - e^{-x}}{2} \cdot \cosh(x) \equiv \frac{e^x + e^{-x}}{2} \cdot \tanh(x) \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \sec h(x) \equiv \frac{2}{e^x + e^{-x}}. \tag{3}$$

The logistic distribution (2) can be equivalently written as:

$$f(x;\mu,s) = \frac{1}{4s} \sec h^2\left(\frac{x-\mu}{2s}\right). \tag{4}$$

Plugging this distribution as $f(x)$ in Equation (1), we obtain:

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \int_K^{\infty} (x-K) \frac{1}{4s} \sec h^2\left(\frac{x-\mu}{2s}\right) dx. \tag{5}$$

Employing the following change in variables: $z \equiv \frac{x-\mu}{2s}$ $dz = \frac{1}{2s}dx$ yields:

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \int_{\frac{K-\mu}{2s}}^{\infty} (2sz + \mu - K) \frac{1}{4s} \sec h^2(z) 2s dz,$$

or:

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \int_{\frac{K-\mu}{2s}}^{\infty} \left(sz + \frac{\mu-K}{2}\right) \sec h^2(z) dz \tag{6}$$

Making use of the relations:
$(\tanh(x))\prime = \sec h^2(x)$ and $\int x \sec h^2(x) dx = x \cdot \tanh(x) - \ln(\cosh(x))$ leads to:

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \left[\frac{\mu-K}{2}\tanh(z) + sz\tanh(z) - s \cdot \ln(\cosh(z))\right] \Big|_{\frac{K-\mu}{2s}}^{\infty} =$$

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \left\{ \left[\frac{\mu-K}{2} + sz - sz + s \cdot \ln(2)\right] - \left[\frac{\mu-K}{2}\tanh\left(\frac{K-\mu}{2s}\right) + \frac{K-\mu}{2}\tanh\left(\frac{K-\mu}{2s}\right) - s \cdot \ln\left(\cosh\left(\frac{K-\mu}{2s}\right)\right)\right] \right\} \tag{7}$$

$$C_0 = \frac{1}{\left(1+r_f\right)^T} \left[\frac{\mu-K}{2} + s \cdot \ln(2) + s \cdot \ln\left(\cosh\left(\frac{K-\mu}{2s}\right)\right)\right],$$

or:

$$C_0 = \frac{1}{\left(1+r_f\right)^T}\left[\frac{\mu - K}{2} + \ s \cdot \ln\left(2\cosh\left(\frac{K-\mu}{2s}\right)\right)\right]. \tag{8}$$

As in B&S, under risk-neutrality the underlying asset's expected value grows at the risk-free rate. Thus, if we denote the underlying asset's current price with S_0, we obtain: $\mu = S_0\left(1+r_f\right)^T$(recall that μ is the expected value of the end-of-period asset value). $\frac{\pi s}{\sqrt{3}}$ is the standard deviation of the end-of-period underlying asset value. We can express this instead as the standard deviation of returns from date 0 to maturity: $\sigma = \frac{1}{S_0}\frac{\pi s}{\sqrt{3}}$, or: $s = \sigma S_0 \frac{\sqrt{3}}{\pi}$. Plugging these values of μ and s in Equation (8), we finally obtain:

$$C_0 = \frac{S_0}{2} - \frac{K}{2\left(1+r_f\right)^T} + \frac{1}{\left(1+r_f\right)^T}\frac{\sqrt{3}}{\pi}S_0\sigma \cdot \ln\left(2\cosh\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right)\right). \tag{9}$$

This is the option pricing formula under the logistic return distribution. It is straightforward to verify that $C_0 \to S_0$ as $K \to 0$ and that $C_0 \to 0$ as $K \to \infty$, as in B&S.

The logistic option pricing formula is based on exactly the same parameters as the B&S formula, and it is very easy to implement. Just as in B&S, there is a single parameter that needs to be estimated—the standard deviation. Note, however, that the notation for the standard deviation is slightly different here and in B&S. In B&S, σ denotes the annual standard deviation of log-returns; thus, the standard deviation of returns from date 0 until expiration is $\sigma\sqrt{T}$. Here, σ simply denotes the standard deviation of returns from date 0 until expiration.[6]

To get a feeling for the logistic option pricing relative to the B&S pricing, consider an underlying asset with $S_0 = \$100$ and $\sigma = 20\%$, a risk-free rate of 1%, and maturity in 1 year. For an in-the-money option with $K = \$80$, the B&S Call price is USD 26.22. The logistic option price for this option, given by Equation (9), is USD 26.73. Thus, in this case the logistic price is 1.95% higher than the B&S price. For the at-the-money option with $K = \$100$, the B&S price is USD 8.28 while the logistic price is USD 8.07. In this case, the logistic price is 2.5% lower than the B&S price. Finally, if the option is out-of-the-money with $K = \$120$, the B&S price is USD 2.22 while the logistic price is USD 1.79. In this case, the relative pricing difference is quite large, with the logistic option priced 19.4% below the B&S price.

We can derive the price of a European Put by employing Put–Call parity. This yields:

$$\begin{aligned} &P_0 = C_0 - S_0 + \tfrac{K}{\left(1+r_f\right)^T} \Rightarrow \\ &P_0 = -\tfrac{S_0}{2} + \tfrac{K}{2\left(1+r_f\right)^T} + \tfrac{1}{\left(1+r_f\right)^T}\tfrac{\sqrt{3}}{\pi}S_0\sigma \cdot \ln\left(2\cosh\left(\tfrac{(K/S_0)-\left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right)\right). \end{aligned} \tag{10}$$

The formula for the price of the Put is exactly the same as the one for the Call, only with the signs of the first two terms reversed.

It is straightforward to calculate the hedge ratio Δ from Equation (9):

$$\Delta \equiv \frac{\partial C_0}{\partial S_0} = \frac{1}{2}\left[1 - \tanh\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right)\right]. \tag{11}$$

Γ, the rate of change in Δ with respect to the underlying asset's price, is given by:

$$\Gamma \equiv \frac{\partial^2 C_0}{\partial S_0^2} = \frac{\pi\left(1+r_f\right)^T}{4\sqrt{3}\sigma S_0} \cdot \sec h^2\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right). \tag{12}$$

V, the rate of change in the option value with respect to the underlying asset's volatility is given by:

$$V = \frac{1}{\left(1+r_f\right)^T}\frac{\sqrt{3}}{\pi}\left[\ln\left(2\cosh\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right)\right) - \frac{1}{\sigma S_0}\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}}\right)\tanh\left(\frac{(K/S_0) - \left(1+r_f\right)^T}{2\frac{\sqrt{3}}{\pi}\sigma}\right)\cdot\right] \tag{13}$$

Just as in B&S, pricing and hedging require the estimation of only the volatility.

4. Empirical Performance of the Logistic Option Pricing Formula

In this section, we compare the empirical performance of the logistic option pricing formula with B&S and more complex option pricing models along two dimensions: the implied volatility smile and dynamic hedging errors.

4.1. Data

We employ daily prices of all call options on the S&P 500 index in the OptionMetrics file in the period 1996–2013. To exclude low-liquidity options, with notoriously unreliable price records, we impose the following three requirements:

a. The bid-ask spread is less than 2.5% of (bid + ask)/2.
b. Bid > USD 10
c. Open interest > 500

With this screening, we have a total of 61,820 option price observations. The risk-free rate is taken as the corresponding T-Bill rate. To match the rate until expiration, we extrapolate linearly between the two T-Bills with the closest maturities. Following Hull (2009), the underlying asset price is taken as the level of the S&P 500 index minus the present value of all dividends until expiration.

4.2. Volatility Smile

For each option price, we numerically calculate the implied B&S volatility and the volatility implied by the logistic option price formula, Equation (9). Figure 3 shows the implied volatilities as a function of moneyness for the two models. To get more detailed results, we partition the sample according to days to expiration. Panel A corresponds to options with 25–50 days to expiration, Panel B to options with 50–75 days to expiration, and Panel C to options with 75–100 days to expiration.

Note that if the model fits the data well, no smile is expected. While the smile is not completely eliminated with the logistic formula, it is greatly attenuated relative to the one obtained with B&S. Notice that moneyness range in Figure 3 is much wider than the range usually reported in the literature. In the more typically employed narrower moneyness range, the variability in the implied volatility under the logistic model is comparable to that obtained with more complex models with a larger number of parameters. For example, Bakshi et al. (1997, Figure 1, p. 2022) report that the implied volatility for various stochastic volatility models (with and without jumps) and 60–180 days to expiration varies by about 3%–4% in the $0.93 < S/K < 1.08$ range. We find similar variability in the implied volatility for the logistic pricing formula over the same moneyness range and similar maturities. This similar performance is obtained with the very simple logistic option pricing model, which has only a single free parameter.

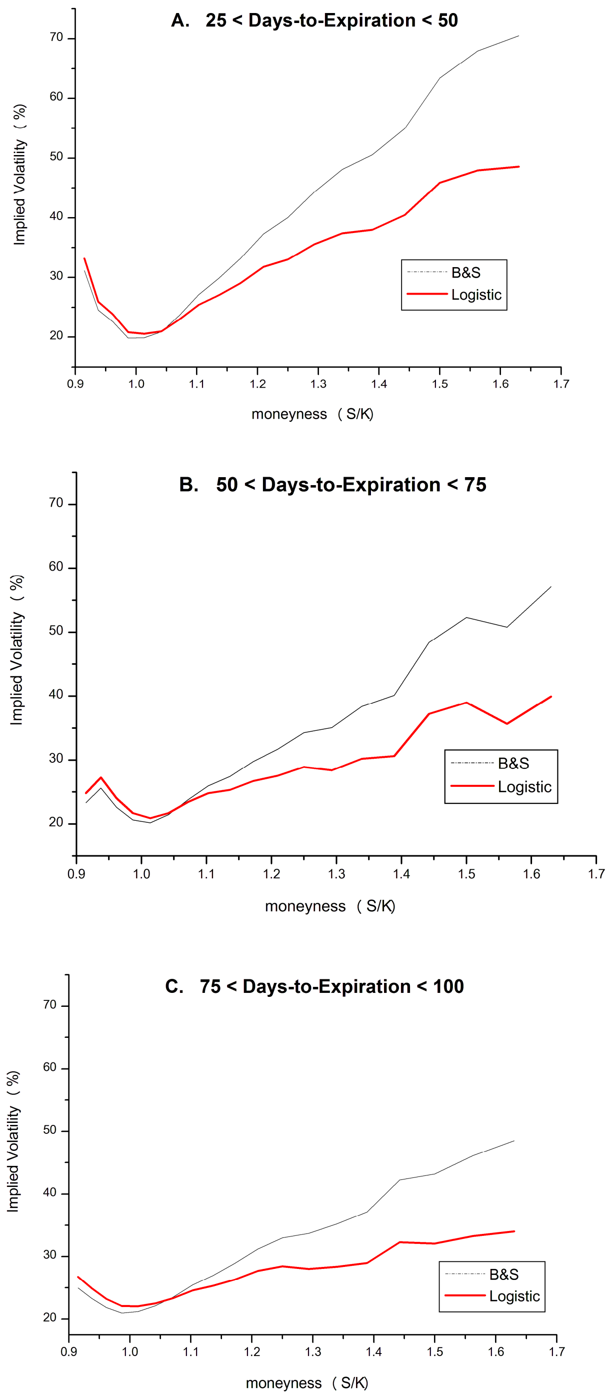

Figure 3. The implied volatility as a function of moneyness for B&S (light) and the logistic option pricing formula (bold). The volatility smile still exists with logistic option pricing, but it is much less pronounced than for B&S. Note that the moneyness range is much wider than ranges typically reported in the literature.

4.3. Hedging Errors

The logistic pricing formula implies a delta hedge given by Equation (11). We test the performance of this hedge ratio relative to the standard B&S delta hedging by shorting the call and hedging the position with Δ shares and the risk-free asset. We record the value of this zero-cost portfolio on the next day:

$$H = -C(t+1) + \Delta(t)S(t+1) + cash(t)e^{rdt}, \tag{14}$$

where $cash(t) = C(t) - \Delta(t)S(t)$. According to the theoretical model with perfect hedging, H should be zero. Thus, H measures the empirical hedging error, i.e., the difference between the next-day value of the option and the replicating portfolio (see, for example, Bakshi et al. (1997)). As we are hedging many options with very different values, we look at the normalized error, which is the error H divided by the absolute size of the position (i.e., $C(t) - cash(t) = \Delta(t)S(t)$; note that *cash* is negative). In other words, we measure the error for a position of USD 1. We analyze the error both for the B&S delta and for the logistic delta above. In both cases, we use the *VIX* as an estimate of the volatility.[7] In the B&S formula, σ is the annual volatility; thus, we simply take $\sigma_{B\&S}$ = *VIX*. In the logistic option pricing formulas, $\sigma_{Logistic}$ is the volatility of returns for the period until expiration (not annualized). We approximate this value by $\sigma_{Logistic} = VIX \cdot \sqrt{T}$, where T is the time to expiration in years. While the scaling by $\sqrt{T}$ is mathematically precise only for i.i.d. log-returns, it is a good approximation for returns (rather than log-returns) as well, at least for horizons exceeding one month—see Figure A1 in the Appendix A.

Table 4 reports the mean absolute normalized hedging error, as well as the mean square normalized hedging error for three different ranges of time to expiration. In all cases the logistic formula yields smaller hedging errors than B&S. The difference is significant at the 1% level in all cases. While the hedging errors, and hence the differences reported in the table, are small in absolute values, the relative improvement is rather large. The mean square normalized error of the logistic model is 15.5% lower than B&S for 25–50 days to expiration, 20% lower for 50–75 days to expiration, and 10.9% lower for 75–100 days to expiration. Bakshi et al. (1997) report that the hedging errors for the various stochastic volatility models are similar to those of B&S when a single instrument is used. Thus, it seems that the logistic model outperforms these models in single-instrument hedging. Extending the model and adding instruments may further reduce hedging errors.

The hedging error is given by $H = -C(t+1) + \Delta(t)S(t+1) + cash(t)e^{rdt}$. It is the difference between the next-day price of the option and the replicating portfolio. The normalized hedging error is given by the error, normalized by the absolute size of the position $H/\Delta(t)S(t)$, i.e., the error per USD 1 position. Both the absolute and the squared normalized hedging error are smaller for the logistic option pricing model than for B&S. This holds for all three time-to-expiration categories. In all cases the difference is significant at the 1% level.

Table 4. Hedging errors.

	25–50 Days to Expiration		**50–75 Days to Expiration**		**75–100 Days to Expiration**	
	B&S	**Logistic**	**B&S**	**Logistic**	**B&S**	**Logistic**
Mean absolute normalized hedging error $(\cdot 10^{-3})$	2.79	2.64	2.94	2.76	2.47	2.41
Mean square normalized hedging error $(\cdot 10^{-5})$	2.65	2.24	3.57	2.86	1.92	1.71

5. Possible Origin of the Logistic Return Distribution

Section 2 shows that the empirical return distribution is closer to the logistic distribution than to any of the other distributions examined for horizons of up to one year. This is the motivation for deriving an option pricing formula based on the logistic return distribution. For this purpose, which is the main contribution of the present paper, one does not necessarily need to know *why* the return distribution is approximately logistic. Still, it is an intriguing question. While we have no pretense of offering a definitive answer to this question, we would like to sketch a possible explanation.

The central limit theorem asserts that the sum of i.i.d. random variables with finite variance converges to the normal distribution. Thus, if we denote the one-period total return (1 + rate of return) with $\widetilde{r}$, the T-period total return is given by $\widetilde{R} = \widetilde{r}_1 \cdot \widetilde{r}_2 \cdot \cdots \cdot \widetilde{r}_T$ and $\log\left(\widetilde{R}\right) = \sum_{t=1}^{T} \log(\widetilde{r}_t)$. If the $\widetilde{r}$ values are i.i.d., then so are their logs; therefore, taking the central limit theorem as $T \to \infty$, $\log\left(\widetilde{R}\right)$ converges to the normal distribution, and thus $\widetilde{R}$ converges to the log-normal distribution. In B&S's continuous time model, the $\widetilde{r}$ values are instantaneous; therefore, the distribution of $\widetilde{R}$ is log-normal for any finite time interval, no matter how short.

Empirically, this is clearly not the case, as shown in Section 2. One possible explanation is that the one-period returns are not independent. This may certainly be true. However, the central limit theorem also holds for many cases with serial correlation (Billingsley (2008); Bradley (2005); and Durrett (2010)). More importantly, it is not clear why serial correlation would lead specifically to the logistic distribution.

We would like to suggest an explanation which is not based on the violation of the i.i.d assumption, but rather of the continuous-time concept of an "instantaneous" return. It seems that there is some natural limit of a maximal frequency at which new information arrives, and that it is meaningless to talk about returns over shorter periods. The minimal time period can be a minute, a second, or even a millisecond, but it is not instantaneous.[8] Therefore, even if returns are i.i.d. and with finite variance, it will take some time for the return distribution to converge to the log-normal. This may seem like a minor technical point, but in fact, we believe that it may be the basis for understanding the origin of the empirical logistic return distribution. This understanding is based on the following three facts:

- The empirical return distribution over very short periods (15 s to a minute) is very close to a truncated Lévy (stable Paretian) distribution with an exponent α of approximately 1.4 (Mantegna and Stanley 1995).
- This distribution is very similar to the logistic distribution. Figure 4 demonstrates this point by comparing these two distributions.
- It is well-known that the Lévy distribution is stable. When the Lévy distribution is truncated, it is pseudo-stable. This means that it eventually converges to the normal distribution because it has finite variance and the conditions of the central limit theorem hold; however, this convergence can take a *very long* time (Mantegna and Stanley 1994). For holding periods shorter than this convergence time, the distribution remains an approximately truncated Lévy distribution, and hence approximately logistic.

This is one possible way to think about the origin of the logistic return distribution. However, it only pushes the mystery one level deeper: why is the very short horizon return distribution a truncated Lévy distribution? Here the explanation becomes even more speculative. Levy (2005) suggests that the truncated Lévy distribution of returns is related to the power law tail distribution of wealth.[9] He compares the empirical wealth distribution power law exponent with the Lévy return distribution exponent across different countries and finds close agreement between these values in each country (but a difference in these parameters across countries). Of course, other explanations for the short-horizon truncated Lévy return distribution are possible. We leave this line of inquiry for future work.

The truncated Lévy and logistic distributions are very similar. However, working with the truncated Lévy distribution is much more cumbersome, as there is no analytical expression for the p.d.f of this distribution. In contrast, the logistic distribution proves very easy to work with, as shown in Section 3.

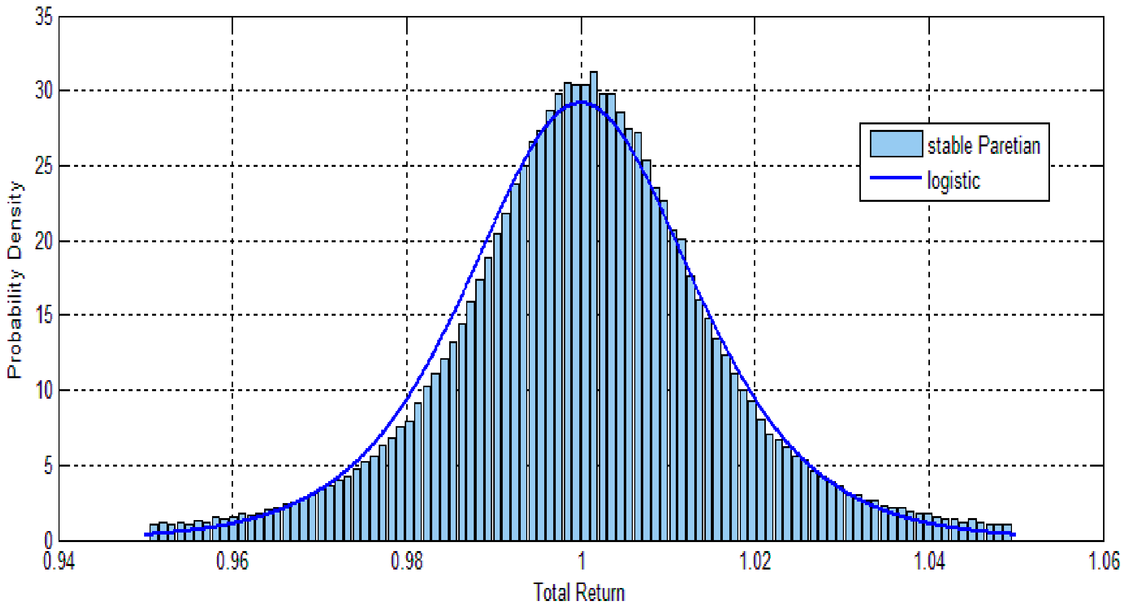

Figure 4. The logistic distribution and the truncated symmetric stable Paretian distributions are very close. The histogram is created by drawing 200,000 random observations from a truncated stable Paretian distribution with $\alpha = 1.4$, $\beta = 0$ (asymmetry parameter), $\gamma = 0.01$ (scale parameter), and $\delta = 1$ (location parameter). The solid line is the best logistic fit. The fit depends on the stable Paretian distribution being symmetric ($\beta = 0$), but it is not sensitive to the values of γ and δ. The excellent fit holds for a wide range of the exponent α.

6. Concluding Remarks

The standard approach to option pricing is to start with a continuous-time process for the evolution of the underlying asset's price and to derive option prices under this process. Great progress has been made with this approach. More and more sophisticated models incorporating time-varying volatility, jumps, and stochastic interest rates have been solved. These models typically improve empirical pricing and hedging errors relative to the cornerstone B&S model, but this comes at a price of introducing more parameters that are usually econometrically challenging to estimate. This approach typically leads to a conditional return distribution of the underlying asset that is either log-normal or approximately log-normal with small deviations due to infrequent jumps. However, for horizons of up to a year, the empirical return distribution is very different from the log-normal. Rather, it is much closer to the logistic distribution, which provides the best fit to the empirical distribution out of a wide range of theoretical distributions examined.

This observation is the motivation for the present paper, which takes a different approach to option pricing. Rather than starting with a continuous-time return process, our starting point is the assumption of a logistic distribution of the underlying asset value at expiration (i.e., a logistic return distribution). By employing risk-neutral valuation, we analytically derive option prices under the logistic distribution. From this valuation formula we also derive the corresponding hedge ratio.

The logistic option pricing formulas turn out to be very elegant and easy to implement. While the logistic distribution is similar to the stable Paretian distribution employed by Carr and Wu (2003), it is much easier to work with and yields a simple analytic pricing formula. The logistic option pricing formula involves exactly the same parameters as the B&S formula, with only one free parameter that needs to be estimated—the volatility. Empirically, the logistic option pricing formulas yield prices and hedge ratios which are much better than B&S and are roughly comparable to more complex models, involving a larger number parameters that are typically econometrically challenging to estimate.

We see this paper as a first step. Combining the logistic pricing approach with elements that have been documented to be important for option pricing, such as time-varying volatility, jumps, and possibly stochastic interest rates could be a promising next step. This would require explaining the logistic distribution as the result of an underlying short-term (but not continuous) return process with these elements. Clearly, there is much to do along these lines. Adding these elements should hopefully even further improve the pricing and hedging performance of the logistic option pricing model.

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Appendix A

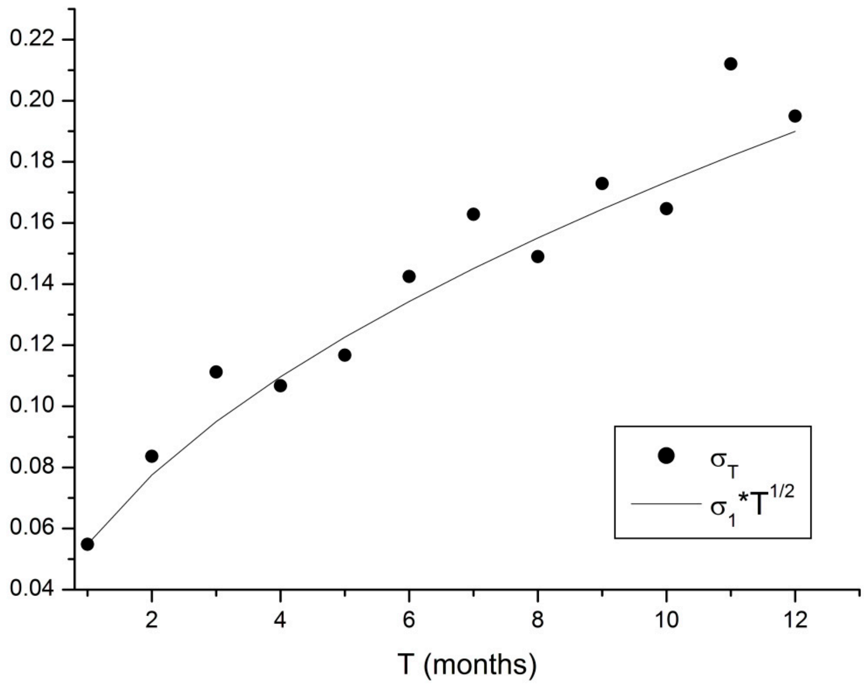

Figure A1. The sample is the set of S&P 500 monthly returns over 1926–2013. σ_1 is the empirical one-month standard deviation of returns. σ_T is the empirical standard deviation of returns for the T-month holding period. While the theoretical relation $\sigma_T = \sigma_1 \cdot \sqrt{T}$ holds mathematically only for i.i.d log-returns, empirically it provides a very good approximation for returns as well, at least for horizons exceeding one month.

Notes

1 An important exception that does not imply a log-normal return distribution is the model of Carr and Wu (2003), who employ the stable Pareto return distribution to explain the observation that the volatility smirk does not flatten out as maturity increases. The logistic distribution employed in the present study is very similar to a truncated stable Paretian distribution (see Section 5 and Figure 4). The advantage of using the logistic distribution is that it yields a very simple analytic option pricing formula.

2 Raja (2009) reaches the same conclusion for individual U.K. stocks. He finds that the best fit for log-returns is provided by the log-logistic distribution, implying that returns are distributed logistically.

[3] For the longer holding periods the conditional distribution analysis is both less relevant and less feasible. The typical length of volatility regimes is probably not very long. Practically, partitioning the 88 year sample into quintiles leads to less than 18 annual observations per quintile, which makes estimating the shape of the conditional distribution impractical. Finney (1978) shows that for a special choice of parameters, the logistic distribution is close to the normal distribution. This is employed by Hofstetter and Selby (2001) to obtain an analytic approximation for the B&S implied volatility. The results reported in this section show that the empirical return distribution is significantly different than the log-normal and normal distributions and is much closer to the logistic distribution. In other words, for the empirical distribution, the special parametrization under which the logistic and the normal are similar does not hold, and the two distributions are quite different.

[4] If the asset's end-of-period value at maturity is distributed log-normally, the return distribution is also log-normal. This is true both for the physical return distribution and the risk-neutral return distribution, although these two distributions have different means. Similarly, a logistic end-of-period asset value implies a logistic risk-neutral return distribution.

[5] The logistic distribution is a part of the family of elliptical distributions, of which the normal distribution is also a member. The CAPM holds for all distributions in this family (with the other standard assumptions; see Chamberlain (1983), Owen and Rabinovitch (1983), and Berk (1997)).

[6] For practical purposes, it is convenient to have a simple way to translate annual volatility to the volatility for different horizons. While the scaling of the standard deviation with time by $\sqrt{T}$ is mathematically precise only for log-returns and under the assumption of no serial correlation, Figure A1 in the Appendix A shows that this scaling is a very good approximation for empirical returns (rather than log-returns) as well.

[7] Similar results, although more noisy, are obtained when the option's implied volatility is used instead of the VIX.

[8] For example, suppose that new information/liquidity shocks arrive every second, and this determines some 1 s return distribution. One could technically define a 1 ms return distribution for this process, but this would be meaningless—it would not change the 1 s return distribution dictated by Nature, nor induce it to be log-normal. For the advantages of the discrete-time approach see Brennan (1979).

[9] The idea is based on the fact that the sum of i.i.d. random variables drawn from any distribution with a power law tail and exponent $\alpha < 2$ converges to the Lévy distribution with an exponent α. Thus, if an individual's influence on the stock price is proportional to her wealth, and the wealth distribution has a power law tail, then the return, which is the sum of the actions of many individuals, will be distributed according to the Lévy distribution with the same exponent as that of the wealth distribution. The truncation is due to the finiteness of the economy.

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