

Article

Stochastic Modelling of Expense-Augmented Insurance Portfolio Loss Process

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Abstract

The present research formulates, stochastically, the expense-augmented insurance portfolio loss process. This is achieved by developing a weighted mixed compound stochastic process which constitutes expenses driven by a Gamma process and claims driven by an exponential distribution. Their intensity process is driven by affine and known as the basic affine jump-diffusion process (BAJDP), which is tractable and integrable and aids in capturing the randomness inherent in insurance data. Monte Carlo simulations are conducted to estimate the intensity and the effectiveness of the simulated process by the moment matching method. The main aim of the present study is to formulate an expense-augmented insurance loss process that includes claims and other expenses as part of the loss to an insurance company as opposed to existing models such as the classical risk model, which only models the process with claims. The model is tested with data obtained from the IRA, Kenya website. The descriptive statistics generated show only Gross premium data are normal ($p > 0.05$) using the Shapiro–Wilks test of normality; the same is confirmed with Q-Q plots of the data. The model parameters are estimated through maximum likelihood estimates. The model is compared with three other existing models including a classical (HPP) loss model, classical (NHPP) loss, and a Gaussian loss model using a measure of probability of loss called the loss ratio, in which it stands out as a better loss model. Finally, the regression model that relates expenses and claims shows statistical significance at 5% significance level, and it can be used for respective prediction. The present model gives desired results since it takes into consideration other expenses in an insurance portfolio, and also loss is reduced significantly as compared with the existing models.

Keywords: affine intensity process; basic affine jump diffusion process; expenses; claims; mixed stochastic process



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1. Introduction

Expenses form a part of the money outflow in an insurance company, just like in any other business. Predicting and calculating expenses is an integral part of competitive financial management. When expenses are well understood, it enables cost control, improves profitability, and aligns the company's operations and financial plan. The result is more informed investment decisions. In a competitive industry like insurance, cost-efficient operations lead to sustainable growth. Therefore, studying expenses and investigating them helps evaluate the efficiency of firms and identify profitable insurance ventures.

Without proper investigation of an insurance company's expenses, accounting fraud cannot be detected. Regulatory authorities can also identify facts and establish industry benchmarks, bringing fairness and order to the insurance business and ensuring fair deals for policyholders. There are three major forms of expenses in an insurance company: expenses related to policy maintenance and acquisition (underwriting expenses), expenses associated with investment of the company's portfolio (investment expenses), and those arising from investigation and claim settlements (loss adjustment or claims expenses) as outlined in [Jang and Oh \(2021\)](#). In classical risk processes and other existing risk models, only the latter expense is captured, ignoring the other two (see [Cramér, 1930](#)). In the conventional risk model, it is outlined that an insurance company decreases its asset base through settling claims. That is, it is only through claims that insurance portfolio experiences money outgo. Therefore, the loss process in such a case is given by

$$S(t) = \sum_{i=0}^{\bar{N}(t)} X_i. \quad (1)$$

where, $S(t)$ is the only money outgo process in an insurance portfolio, according to the classical model as indicated in [Albrecher and Asmussen \(2006\)](#), [Bartlett \(1986\)](#), [Cox \(1955\)](#), [Debicki et al. \(2015\)](#), [Kyprianou \(2013\)](#), and [Liu \(2012\)](#).

$\bar{N}(t)$ is a Homogeneous Poisson Process (HPP) which models the total claims received, also known as the claims frequency, while X_i 's denote the i.i.d size of the claims.

This results in an inaccurate approximation of the financial position by such a risk model, highlighting the need to consider all expenses to create a more robust model.

Modeling of the loss process in an insurance setup has always been achieved by considering claims as the only source of money outgo; this is the case, for example, in the classical risk model in which the claims count process is a homogeneous Poisson process (HPP), the classical risk model in which the claim count is modeled as a non-homogeneous Poisson process (NHPP) whose intensity varies with time, and finally, the Gaussian risk model in which aggregate claims are modeled as a normal distribution. However, it is known that other expenses incurred are part of the money outgo. The other expenses are those due to underwriting or policy development and expenses due to investments. The present modeling framework considers claims and these two expenses combined to come up with an Expenses-Augmented Loss model. Hence, the present research aims to formulate the loss process, which is robust in the sense that it takes care of all the money outgo in an insurance portfolio, and this is carried out to enhance accurate approximation of loss in a risk process model.

The rest of this paper is organized as follows: Section 2 explores the literature that is relevant to this study. In Section 3, a mixed stochastic process is formulated and its statistical properties are investigated. Results, discussions, and numerical analysis are given in Sections 4 and 5. Finally, Section 6 gives conclusions and recommendations for future expansion of the concepts developed.

2. Literature Review

Expense modelling has always been crucial in determining how effective and efficient insurance companies need to be while tackling risk management and insurance literacy. The efficiency of a company is positive if the cost of inputs is less than the cost of outputs. Estimation of costs has been emphasized in [Gardner and Grace \(1993\)](#) for life insurance and the same has been looked into by [Cummins and Weiss \(1993\)](#) for property casualty insurance. [Berger et al. \(1997\)](#) modeled expenses to examine the effects of distribution systems on an insurer's efficiency. [Toivanen \(1997\)](#), [Hirao and Inoue \(2004\)](#), and [Segal](#)

(2002) also modeled expenses in an insurance company, investigating the economic capacity and its scope for some insurance portfolio in relation to other competitors. Choi and Weiss (2005) and Fenn et al. (2008) also investigated expenses to ascertain the link that exists between firm efficiency and the structure of the market.

Shi and Frees (2010) modeled insurance company expenses long-tail longitudinally using the property-casualty data from the National Association of Insurance Commissioners (NAIC) between the years 2001–2006. The authors established that the distribution of expenses before transformation is long-tailed and positively skewed, a property that is key in the present model. Quantile regression is used to analyze annual firm-level expenses and a distribution called the Asymptotic Laplace Distribution (ALD) with the CDF

$$F(Z; \sigma, \tau) = \begin{cases} \tau \exp(Z(1 - \tau)) & Z\sigma \leq 0 \\ 1 - (1 - \tau) \exp(-Z\tau) & Z\sigma > 0 \end{cases} \quad (2)$$

and PDF

$$f(Z; \sigma, \tau) = \frac{\tau(1 - \tau)}{\sigma} \exp(-Z[\tau - I(Z\sigma \leq 0)]) \quad (3)$$

where

- $\tau \in]0, 1[$ denotes the skew parameter which defines the level of skewness;
- $\sigma > 0$ denotes the scale parameter;
- $Z = \frac{y - \mu}{\sigma}$;
- $\mu \in \mathbb{R}$ is the location parameter.

It is very important to note that the ALD is defined on the whole real number line, allowing for negative expenses; also, its flexibility allows for a proper description of skewness and long trailed data. It is negatively (positively) skewed if $\tau > \frac{1}{2}$ ($\tau < \frac{1}{2}$) and the higher the σ , the longer the tail; however, when $\tau = \frac{1}{2}$, the density of the ALD(μ, σ, τ) becomes

$$f(Z; \tau = 1/2) = \frac{1}{4\sigma} \exp\left(-\frac{|Z|}{2}\right) \quad (4)$$

and according to Landsman and Valdez (2003), this translates to the Laplace or Double Exponential Distribution (DED).

In the present study, other expenses such as underwriting and investment are modeled alongside the usual loss adjustment, which normally takes into account the expenses as a result of claims to improve the accuracy of the approximate ruin probabilities in the risk model for an insurance portfolio. The expenses can also be modeled by Geometric Brownian Motion, whereas the claim count process is then assumed to follow a Doubly Stochastic Poisson Process (DSPP).

2.1. Wiener Process

Definition 1. A stochastic process $\{W(t), t \geq 0\}$ is known as a Wiener process or a standard Brownian motion if (i) $W(0) = 0$, the process is always zero rated at time $t = 0$; (ii) $W(t)$ has sample paths devoid of kinks; (iii) for any $t > s \geq 0$, $W(t) - W(s) \sim N(0, t - s)$, i.e., the statistical traits of the process revolve only around the interval $t - s$; and (iv) its increments, for any $0 \geq t_1 \cdots \geq t_n$, are independent of respective times—i.e., $W(t)_n - W(t)_{n-1} \cdots W(t)_2 - W(t)_1$ are random variables that are independent, and hence, the process is (Strong) Markovian. Markovian processes are independent of the past events.

From the definition, we can write $W(t) = W(t) - W(0) \sim N(0, t) \Rightarrow E[W(t)] = 0$. Thus, in general, Brownian motion is a process $\{B(t), t \geq 0\}$ such that $B(t) - B(s) \sim$

$N(\mu \times (t - s), \sigma^2(t - s))$. Consequently, if we set $\mu = 0, \sigma = 0$, and $B(0) = 0$, a Wiener process or the standard Brownian motion results.

2.2. Poisson Process

Definition 2. In a counting process, the number of events of interest and the even number process $\{N(t), t \geq 0\}$ satisfies the following conditions, as also highlighted in [Klugman et al. \(2012\)](#): (i) $N(0) = 0$, i.e., no number of events have occurred at time 0. (ii) For any time $t > 0$, $N(t)$ must be an integer valued—i.e., in terms of claims, there cannot be, say, 3.2 claims. (iii) When $s < t$, $N(s) \leq N(t)$, i.e., the number of events over a period of time is non-decreasing—for example, if we have counted, say, 10 deaths in 3 months, then the number of deaths encountered in a 4-month period includes the 3-month period and must be at least 10. (iv) When $s < t$, $N(t) - N(s)$ represents the number of events occurring in the time interval (s, t) , i.e., we counted $N(t)$ events up to time t , and $N(s)$ events up to time s , so there were $N(t) - N(s)$ events encountered between time s and t .

The Poisson process is an example of a counting process defined above [Klugman et al. \(2012\)](#). It is defined as follows:

Definition 3. It is a process $\{N(t), t \geq 0\}$; in addition to the counting process properties, we first have $P(N(t+s) = r | N(t) = r) = 1 - \lambda s + O(s)$, $P(N(t+s) = r+1 | N(t) = r) = \lambda s + O(s)$, and $P(N(t+s) > r | N(t) = r) = O(s)$. Secondly, when $s < t$, the number of events that occur does not overlap in that the events are in the $(s, t]$ interval.

2.3. Doubly Stochastic Poisson Process

The key motivation for the present research development of a stout framework for the approximation of ruin probability is developing a more pragmatic approach. This entails the formulation of a surplus process that takes into account economic factors and also ensures the insurance claims count captures the impacts of random atmospheric conditions and economic constraints on the claim severity evolution. A Cox process is realized when the claims severity of the Poisson process is allowed to be a stochastic process. Cox processes of an affine nature, which form a natural choice of processes to model insurance risk in mathematics, were first introduced by Ammeter in 1948 according to [Grandell and Grandell \(1991\)](#). It is a very good process for modelling the frequency and size of risk fluctuations in an insurance portfolio since it accounts for both overdispersion, which is associated with count data, and environmental factors like seasonal variation. The idea was conceptualized by [Cox \(1955\)](#). [Diggle and Milne \(1983\)](#) applied the idea in the physical sciences; [Lando \(1998\)](#) applied the idea in credit risk modelling. The model has also been applied in many physical sciences and particularly in [Biffis \(2005\)](#), it was used for modelling mortality. In the present study, BAJDP, which is simply a Cox–Ingersoll–Ross (CIR) process with jump, has been used to model the claims plus the expenses intensity.

It has been observed that when stochasticity is allowed in the claims intensity, the randomness of variation results in a stout and pragmatic model for claim modelling and consequently, a robust model for approximating ruin probabilities, ultimately allowing for accurate estimation of the reserves and capital required for evaluating the financial position of an insurance portfolio. Empirical applications of the Cox process in financial modelling have been achieved—for example, in credit risk modelling using Vasicek models and interest rate modelling using CIR models, mortality rates modelling as indicated in [Biffis \(2005\)](#), and life insurance as outlined in [Cox \(1955\)](#). On the same financial and insurance application, it has also been used in pricing catastrophic reinsurance derivatives where the claims evolutions conform to the Cox process as indicated in [Dassios and Jang \(2003\)](#). Theoretical developments such as [Albrecher and Asmussen \(2006\)](#) and [Björk and Grandell](#)

(1988) established ruin probabilities in the case where the counting aspect of the claims was modelled as a doubly stochastic Poisson process. The idea has been applied in non-life insurance scenarios empirically and with the classical risk model in order to model claim count data as indicated in Liu (2012). The authors of Liu (2012) developed a DSPP whose intensity was modelled by a class of affine intensity processes known as the shot noise Cox process, in which the stochastic aspect of the claims was always assumed to be a positive jump. This assumption ensured non-negativity in the claims realized.

The Cox process was introduced, for example, in Liu (2012) to address some of these issues. The researcher filtered the Cox process with a short noise intensity where the stochastic aspect of the claim intensity arrived as strictly positive jumps; this clearly showed the reduced flexibility of the Cox process. To overcome this, a scheme known as expense-augmented loss is explored in the next section. In an insurance set up, it has always been assumed that the only source of money outflow is purely through claim settlement, which is not true. Money outgo is also realized through expenses that may include licenses and fees, sales and marketing expenses, and other general expenses like risk classification. Therefore, stochastic modelling for claims and expenses can be achieved by using a Cox process together with expenses as part of the loss process; such will enable a better selection of the intensity process that reflects the nature of the claims arrival and its intensity for each of the business activities in an ideal economic environment. In fact, the claims process is herein referred to as a loss process in the sense that it captures the actual and pragmatic nature of liabilities incurred in an insurance portfolio. This results in improvement of the accuracy of the insurance portfolio's money outgo modelling process. Hence, the net effect is the accurate and reliable approximation of the probability of ruin, whose value reflects the appropriate financial position of an insurance enterprise.

Definition 4. Consider a stochastic process $\Lambda(t)$. If $\Lambda(t) \equiv (\Lambda(t))_{t < \infty}$, $\Lambda(0) = 0$ with a non-decreasing sample realization, then it is referred to as diffuse if it has continuous sample paths.

Definition 5. Since $M(t)$ is a random measure, let $\tilde{N}$ be an HPP ($\Lambda = 1$) that does not depend on $\Lambda(t)$. We refer to $N(t) \equiv \tilde{N} \circ \Lambda(t)$ as a point process called a Cox process.

Lemma 1. If $M(t) = \int_0^t \Lambda(s)ds$, then $(\Lambda(t))_{t < \infty}$ is the mean value process. The Cox process is well defined if the intensity has Riemann integrable and continuous sample paths.

Proof. See Grandell (2006). $\square$

Cox processes exhibit interesting statistical properties, which are defined according to Cox and Isham (1980). Both the expected value and variance of a DSPP for a sample difference between two Poisson processes $N(s) \leq N(t)$ conditioned on $\Lambda(u) \leq \Lambda(v)$, with $s \leq u \leq v < t$, are given as $\int_s^t \Lambda(v)dv$. However, by the law of iterated expectation, unconditional moments are defined as follows:

$$E[N(t) - N(s)] = \int_s^t E[\Lambda(v)]dv \quad (5)$$

$$Var[N(t) - N(s)] - \int_s^t E[\Lambda(v)]dv = Var\left[\int_s^t [\Lambda(v)]dv\right] \quad (6)$$

The right-hand-side (RHS) term in Equation (6) arises from uncertainty, which comes with variability, and it takes care of overdispersion in count data more so than in claim frequency, as outlined in Liu (2012). Overdispersion is a phenomenon that occurs when the variance exceeds the mean. It is not accounted for in both homogenous and non-

homogeneous Poisson processes since in both cases, it is assumed that the mean equals the variance. The aspect of overdispersion is well captured in the current modelling framework.

Figure 1 gives the results of a Monte Carlo simulation of a CIR process. The parameter were chosen as follows: the number of steps $N = 100$; total time $T = 50$; time step size is given by $dt = T/N$; the rate of mean reversion $k = 0.05$; $\theta = 0.05$; the rate of volatility is given by $\sigma = 0.03$; and finally, the initial rate of intensity $\Lambda(0) = 0.1$. From the simulated model, the mean intensity rate for this CIR process is given by $\Lambda(t) = 0.0573$.

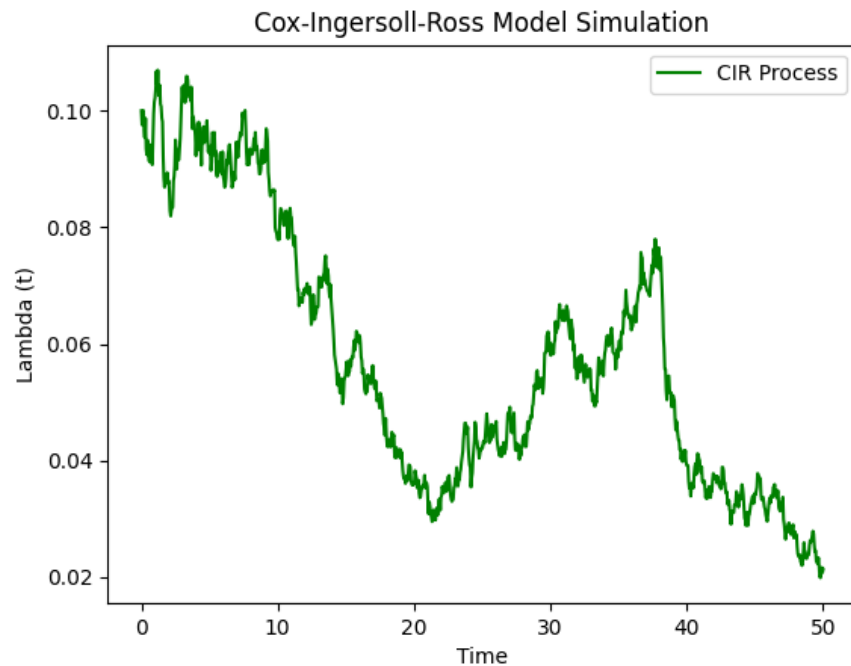


Figure 1. Realization of CIR process with $N = 1000$, $T = 50$, $dt = T/N$, $\kappa = 0.05$, $\theta = 0.05$, $\sigma = 0.03$, and $x_0 = 0.1$.

Another interesting process is the shot-noise intensity process defined as in Liu (2012) by

$$\Lambda(t) = \Lambda(0) \exp(-kt) + \sum_{i=1}^{N(t)} X_i \exp(-k(t - \tau_i)) \quad (7)$$

where τ_i gives the time from HPP(λ) and X_i gives i.i.d positive jump sizes with density μ . It is vital to note that $k > 0$ and stands for a completely deterministic exponential rate of process decay as a result of jumps. The jumps brings about randomness. To obtain Figure 2, the following assumptions are necessary for the shot noise DSPP: $T = 50$; $\Lambda(0) = 0.55$; the intensity of the shot noise = 1.11; the decay rate $k = 1.15$; and the number of shots = 100.

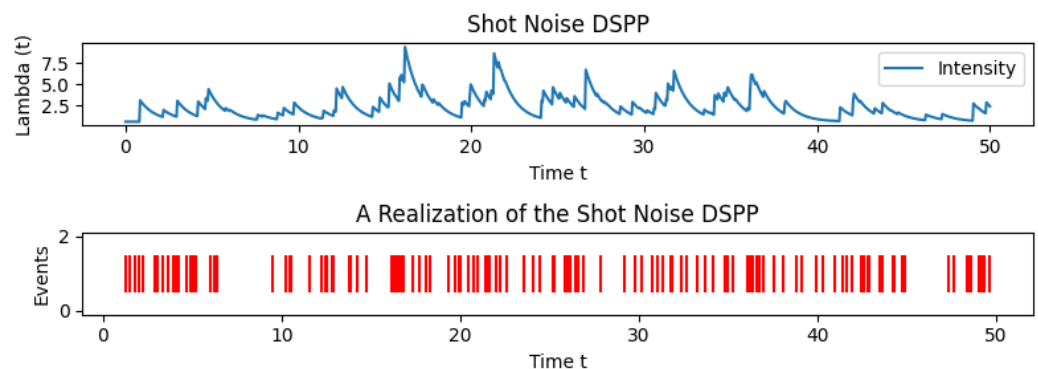


Figure 2. Realization of shot noise DSPP and its intensity with $N = 1000$, $\Lambda(0) = 0.55$, $\lambda_{short} = 1.11$, decay rate = 1.15, and Numebr of shots = 100.

These processes emanate from the affine intensity process, which is defined by the following Stochastic differential equation as per Liu (2012):

$$d\Lambda(t) = \rho(t, \Lambda(t))dt + \sigma(t, \Lambda(t))dW(t) + d\Xi(t) \quad t \geq 0 \quad (8)$$

where $W(t)$ is the Wiener process and $d\Xi(t)$ is a point process—i.e., a jump intensity process $\eta(t, \Lambda(t))$, $t \geq 0$.

Another process that is worth mentioning is the Hawkes process, which is defined according to Jang and Oh (2021) as

$$\Lambda(t) = \Lambda(0) + \sum_{\tau_i < t} \sum_{j=1}^P X_j \exp(-\beta_j(t - \tau_j)). \quad (9)$$

If we set $P = 1$, the process becomes the classical shot noise process. Another one is DCP. It is also defined as follows:

Definition 6. DCP is a point process $N(t) = \sum_{j \geq 1} (\tau_{2,j} \leq t)_{j=1,2,\dots}$ with $\Lambda(t) > 0$ —i.e.,

$$\begin{aligned} \Lambda(t) = & \rho + (\Lambda(0) - \rho) \exp(-kt) + \sum_{i \geq 1} X_i \exp(-k(t - \tau_{1,i})) \mathbb{I}(\tau_{1,i} \leq t) \\ & + \sum_{j \geq 1} Y_j \exp(-k(t - \tau_{2,j})) \mathbb{I}(\tau_{2,j} \leq t). \end{aligned} \quad (10)$$

In a shot noise self-exciting point process, $\Lambda(0)$ denotes the zero intensity, $\rho \geq 0$ represents the mean reversion rate, and $k > 0$ stands for the rate of exponential decay. The X_i 's are independently and identically externally excited jumps with distribution function $F(x)$, $x > 0$, occurring at times $\tau_{1,i}$ following a Poisson process with parameter $\lambda > 0$. The indicator function is denoted by $\mathbb{I}$. Additionally, the Y_i 's represent a sequence of independently and identically distributed positive self-excited jumps with distribution function $F(y)$, $y > 0$, occurring at times $\tau_{2,j}$ and generated by the intensity function $\lambda(t)$. Generally, it is assumed that X_i 's, X_j 's, τ_i 's, and τ_j 's are independent. For further details, please refer to Jang and Oh (2021).

3. Mixed Stochastic Process

In statistical theory, a mixture is a representation of marginal distributions of a random variable as conditional distributions or as a weighted average of components. A mixed stochastic process according to Li et al. (2020) and Bartlett (1986) constitutes weighted averages of stochastic processes. Therefore, an ideal insurance portfolio loss process is denoted by $L(t)$ and is defined as follows:

$$L(t) = w_1 X_1(t) + w_2 X_2(t) \quad (11)$$

where $\{X_1(t), t \geq 0\}$ is a stochastic process that models other expenses and $\{X_2(t), t \geq 0\}$ is another stochastic process modelling claims themselves and w is some weight.

$L(t)$, a mixed stochastic process, can be formulated as follows:

$$L(t) = w_1 X_1(t) + w_2 X_2(t) + w_3 X_3(t) + \dots + w_n X_n(t) = \sum_{i=1}^n w_i X_i(t)$$

where $w_i \in \{w_1, w_2, \dots, w_n\}$ are continuous non-negative weights with respect to various money outgoes in the portfolio. The stochastic processes $\{X_i(t) \mid t \geq 0, i = 1, 2, \dots, n\}$ are random variables for individual money outflows.

Also, $X_i(t) \equiv \{i = 1, 2, \dots, n\}$ is constrained to $\sum_{i=1}^n w_i = 1$.

If one of the weights $w_i = 1$, then the mixed stochastic process degenerates to a single stochastic process, and as in the case of the conventional risk process, since other expenses are not considered, only claims are looked into.

When two stochastic processes, i.e., claims and other expenses, are considered, as in the present study, a mixed stochastic process $L(t)$ is realized as indicated in Li et al. (2020).

Assumptions of the loss process $L(t)$ include the following: The loss in an insurance company takes place as a result of claims and other expenses. The claims process is assumed to conform to a doubly stochastic Poisson process or the Cox process.

3.1. Distribution of Mixed Stochastic Process $L(t)$

Let $X_1(t), t \geq 0$ be a stochastic process which defines other expenses in an insurance portfolio. It is assumed that due to the fact that expenses are positively skewed and long tailed, $X_1(t) \sim \text{Gamma}(\alpha\Lambda(t), \beta)$ where $\alpha\Lambda(t) > 0$ and $\beta > 0$ are shape and scale parameters, respectively. The PDF of $X_1(t)$ is given by

$$f(x_1)_{\text{other Expenses}} = \frac{\beta^{\alpha\Lambda(t)}}{\Gamma(\alpha\Lambda(t))} x_1^{\alpha\Lambda(t)-1} \exp(-\beta x_1), \quad x_1 > 0 \quad (12)$$

Its expected value and variance are given by $E(X) = \frac{\alpha\Lambda(t)}{\beta}$ and $\text{Var}(X) = \frac{\alpha\Lambda(t)}{\beta^2}$. Its moment generating function and coefficient of skewness are also given by $M(t) = \left(1 - \frac{t}{\beta}\right)^{-\alpha\Lambda(t)}$, $t < \beta$ and $\frac{2}{\sqrt{\alpha\Lambda(t)}}$, respectively. Also, $E(x^r) = \frac{\Gamma(\alpha\Lambda(t)+r)}{\Gamma(\alpha\Lambda(t))\beta^r}$, $r = 1, 2, 3, \dots$, where $\Lambda(t)$ is the intensity of the Cox process, specifically BAJDP. Refer to Hogg (1986) for more on the probability distribution.

Let a Poisson process $\{X_2(t), t \geq 0\}$ with mean $\Lambda(t) > 0$ be another stochastic process which models claims in the present set up. The PDF is given by

$$f(x_2)_{\text{claim}} = \frac{(\Lambda(t))^{x_2} \exp(-\Lambda(t)x_2)}{x_2!}, \quad \text{for } x_2 = 0, 1, 2, \dots \quad (13)$$

Now, the PDF of a mixed stochastic process $L(t)$ is given by

$$f(x) = \begin{cases} w_1 \frac{\beta^{\alpha\Lambda(t)}}{\Gamma(\alpha\Lambda(t))} x_1^{\alpha\Lambda(t)-1} \exp(-\beta x_1) + w_2 \frac{(\Lambda(t))^{x_2} \exp(-\Lambda(t)x_2)}{x_2!} & \text{for } x_1 > 0, x_2 \geq 0 \\ 0 & \text{elsewhere} \end{cases} \quad (14)$$

Properties of Mixed Stochastic Process $L(t)$

We investigate the statistical properties of the mixed distribution. We derive the expected value and the variance of $L(t)$.

Proposition 1. The expected value of the mixed stochastic process $L(t)$ is given by

$$EL(t) = \left[w \frac{\alpha\Lambda(t)}{\beta} + (1-w)\mu\Lambda(t) \right]$$

- (i) $\Lambda(t)$ is the intensity process of the basic affine jump diffusion process.
- (ii) w is the weight used to enhance the mixing of the stochastic processes.

Proof.

$$EL(t) = wEX_i(t) + (1-w)E(X_2(t))$$

$$\begin{aligned}
&= w \int_0^\infty \frac{x_1 \beta^{\alpha \Lambda(t)}}{\Gamma(\alpha \Lambda(t))} x_1^{\alpha \Lambda(t)-1} \exp(-\beta x_1) dx_1 + (1-w) \sum_{n=0}^\infty E[X_2(t) | N(t) = n] P(N(t) = n) \\
&= w \int_0^\infty \frac{\beta^{\alpha \Lambda(t)}}{\Gamma(\alpha \Lambda(t))} x_1^{(\alpha \Lambda(t)+1)-1} \exp(-\beta x_1) dx_1 + (1-w) \sum_{n=0}^\infty n \mu \frac{(\Lambda(t))^n}{n!} \exp(-\Lambda(t)) \\
&= w \left(\int_0^\infty \frac{1}{\beta} \frac{\beta^{\alpha \Lambda(t)+1}}{\Gamma(\alpha \Lambda(t)+1)} x_1^{(\alpha \Lambda(t)+1)-1} \exp(-\beta x_1) dx_1 \right) + (1-w) \left(\mu \exp(-\Lambda(t)) \Lambda(t) \times \sum_{n=1}^\infty \frac{(\Lambda(t))^{n-1}}{(n-1)!} \right)
\end{aligned}$$

Recall that $\Gamma(\alpha + 1) = \alpha \times \Gamma(\alpha)$ and also let $x_2 = n - 1$.

$$\begin{aligned}
&= w \left(\int_0^\infty \frac{\alpha \Lambda(t)}{\beta} \frac{\beta^{\alpha \Lambda(t)+1}}{\Gamma(\alpha \Lambda(t)+1)} x_1^{(\alpha \Lambda(t)+1)-1} \exp(-\beta x_1) dx_1 \right) \\
&\quad + (1-w) \left(\mu \exp(-\Lambda(t)) \Lambda(t) \times \sum_{x_2=0}^\infty \frac{(\Lambda(t))^{x_2}}{x_2!} \right) \\
&= w \frac{\alpha \Lambda(t)}{\beta} \left(\int_0^\infty \text{Gamma}(x_1; \alpha \Lambda(t) + 1, \beta) dx_1 \right) + (1-w) \left(\mu \exp(-\Lambda(t)) \Lambda(t) \times \sum_{x_2=0}^\infty \frac{(\Lambda(t))^{x_2}}{x_2!} \right)
\end{aligned}$$

Now, using the densities of the Gamma and that of Poisson stochastic processes, we obtain

$$EL(t) = w \frac{\alpha \Lambda(t)}{\beta} (1) + (1-w) [\mu \exp(-\Lambda(t)) \Lambda(t) \times \exp(\Lambda(t))]$$

$$EL(t) = \left[w \frac{\alpha \Lambda(t)}{\beta} + (1-w) \mu \Lambda(t) \right]$$

□

Proposition 2. The moment generating function of the process $L(t)$ is given by

$$M_L(t) = \exp \left[\Lambda(t) \left(\left(1 - \frac{tw}{\beta} \right)^{-\alpha} + (1 - \mu t(1-w))^{-1} - 2 \right) \right] \quad (15)$$

Proof. The MGF of any arbitrary random variable X is usually given by

$$M_X(t) = E(e^{tX})$$

For further theory, see [Hogg \(1986\)](#). We obtain the MGF of $L(t)$ as follows:

$$M_L(t) = E[e^{tL}] = E[e^{t(wX_i + (1-w)X_j)}]$$

via the independent assumption

$$M_L(t) = E[e^{twX_i}] \cdot E[e^{t(1-w)X_j}] \iff M_L(t) = M_{X_i}(t) \cdot M_{X_j}(t)$$

Recall that

$$L(t) = w \sum_{i=1}^{N(t)} X_i + (1-w) \sum_{i=1}^{N(t)} X_i$$

where $X_i \sim \text{Gamma}(\alpha, \beta)$ and $X_j \sim \text{Exp}(\mu^{-1})$ and $N(t) \sim \text{Poi}(\Lambda(t))$, which is simply a DSPP. We can also write

$$M_{X_i} = [M_{X_i}]^{N(t)} = E[\exp(N(t) \ln M_{X_i}(t))] = M_{N(t)}[\ln M_{X_i}(t)]$$

For $N(t) \sim \text{Poi}(\Lambda(t))$, we have

$$M_N(t) = \exp(\Lambda(t)(e^t - 1))$$

Thus, for any compound distribution

$$M_X(t) = \exp[\Lambda(t)(M_{X_i}(t) - 1)] \quad (16)$$

in the present case,

- (i) $\Lambda(t)$ is the compound intensity process.
- (ii) $M_X(t)$ is the MGF of intensities being counted.

Next, we can now write down the individual MGFs for each the two stochastic random processes.

$$M_{X_i}(t) = \exp\left\{\Lambda(t)\left[\left(1 - \frac{tw}{\beta}\right)^{-\alpha} - 1\right]\right\} \quad (17)$$

$$M_{X_j}(t) = \exp\left\{\Lambda(t)\left[(1 - (1 - w)\mu t)^{-1} - 1\right]\right\} \quad (18)$$

From $M_L(t) = M_{X_i}(t) \cdot M_{X_j}(t)$, we obtain

$$M_L(t) = \exp\left\{\Lambda(t)\left[\left(1 - \frac{tw}{\beta}\right)^{-\alpha} - 1\right]\right\} \cdot \exp\left\{\Lambda(t)\left[(1 - (1 - w)\mu t)^{-1} - 1\right]\right\}$$

This simplifies to

$$M_L(t) = \exp\left[\Lambda(t)\left(1 - \frac{tw}{\beta}\right)^{-\alpha} + \Lambda(t)(1 - \mu t(1 - w))^{-1} - 2\Lambda(t)\right] \quad (19)$$

The final result follows upon factoring $\Lambda(t)$ out. $\square$

Next, we confirm the first moment by differentiating and substituting $t = 0$, after which we obtain the expected value of $L(t)$:

$$EL(t) = \frac{dM_L(t)}{dt}\bigg|_{t=0} = \Lambda(t)\left[w\frac{\alpha}{\beta} + (1 - w)\mu\right]$$

The second moment is given as follows:

$$E^2L(t) = \frac{d^2M_L(t)}{dt^2}\bigg|_{t=0} = \Lambda(t)\left[(\alpha + 1)w^2\frac{\alpha}{\beta^2} + 2\mu^2(1 - w)^2\right]$$

Therefore,

$$\begin{aligned} \text{Var}(L(t)) &= E^2L(t) - EL(t)^2 = M_L^2(0)'' - (M_L(0)')^2 \\ \text{Var}(EL(t)) &= \Lambda(t)\left[\frac{\alpha w^2}{\beta^2} + \mu^2(1 - w)^2 - 2w\mu\frac{\alpha}{\beta}w(1 - w)\right] \end{aligned} \quad (20)$$

To determine the weights, we state that the sum of all wights is equal to one.

$$w_1 + w_2 + \dots + w_n = \sum_{i=1}^n w_i = 1 \quad (21)$$

We assume that the weight of each expense denotes the magnitude of the money outgo—i.e., policy underwriting expenses, investment expenses, and claims expenses are

equal for each. For w_i , $i = 1, 2, \dots, n$, each weight is given by $w = (i)^{-1}$, for $i = 1, 2, 3, \dots, n$. In the present study, we have developed a loss process in an insurance portfolio as a mixed stochastic process from two processes—i.e., $i = 2$. Thus, the equal weight is given by

$$w = (i)^{-1} = \frac{1}{2}$$

This is because

$$w_i = \frac{1}{i} \text{ for } i = 1, 2, \dots, n. \quad (22)$$

Hence,

$$EL(t) = \left(w \frac{\alpha \Lambda(t)}{\beta} + (1 - w) \mu \Lambda(t) \right)$$

Simplifying further, the following equation is obtained:

$$EL(t) = \Lambda(t) \left[w \frac{\alpha}{\beta} + (1 - w) \mu \right] \quad (23)$$

4. Results and Discussions

To determine the intensity process $\Lambda(t) \geq 0$ for all $t > 0$ in Equation (23) above, the expression must be analytically tractable since the claims have to be non-negative at all costs. The process also ensures that the mean value $M(t) = \int_0^t \Lambda(s) ds$ is monotonically increasing as time goes on. This is a reasonable and realistic phenomenon from the insurer's point of view. Therefore, $\Lambda(t)$ has to be estimated using the basic affine jump-diffusion process. One such process is the Cox–Ingersoll–Ross process. The only shortcoming of the CIR process is the absence of the jump component, leaving it as a continuous process that cannot capture the dynamics of the general expenses of an insurance portfolio. Although the classical shot noise process has a closed-form expression and is analytically tractable, the exponential decay that excites the process may be limited but not for positively skewed and long-tailed processes. The stochasticity of the shot noise process rests purely with the jumps, in which their exponential decay is completely deterministic Liu (2012). Hence, we concentrate on BAJDP.

4.1. Basic Affine Jump Diffusion Process (BAJDP)

The general affine intensity process with jumps is defined by the following stochastic differential equation as per Liu (2012):

$$d\Lambda(t) = \rho(t, \Lambda(t))dt + \sigma(t, \Lambda(t))dW(t) + dJ(t) \quad t \geq 0 \quad (24)$$

where $W(t)$ is the Weiner process, and $J(t)$ is a point process with jump intensity process $\nu(t, \Lambda(t))$, $t \geq 0$.

However, the following Stochastic Differential Equation satisfies the basic affine intensity process with κ being the rate of mean reversion according to Bluhm et al. (2003):

$$d\Lambda(t) = \kappa(\theta - \Lambda(t))dt + \sigma\sqrt{\Lambda(t)}dW(t) + dJ(t) \quad t \geq 0 \quad (25)$$

4.2. Discretization of BAJDP

The Euler–Maruyama scheme is a numerical method that makes it easy to transform the SDE into a discrete equation. The scheme involves the approximation of each difference term as a finite difference. It is important to note that the accuracy of the methodology improves with a reduction in the time steps. Equation (25) can be discretized as follows:

$$\Lambda(t + \delta t) - \Lambda(t) = \kappa(\theta - \Lambda(t))\delta t + \sigma\sqrt{\Lambda(t)}N(0, \delta t) + J(t)\delta t \quad (26)$$

where $J(t) = \sum_{i=1}^{N(t)} X_i$ is a point process that brings the jump aspect; the process reverts to the mean θ at rate k . For $\Lambda(t) \geq 0$ for every $t \geq 0$, the condition $\sigma^2 \leq 2\kappa\theta$ must hold.

It can be rewritten as

$$\frac{\Lambda(t + \delta t) - \Lambda(t)}{\sqrt{\Lambda(t)}} = \frac{\kappa\theta\delta t}{\sqrt{\Lambda(t)}} - \kappa\sqrt{\Lambda(t)}\delta t + \sigma\sqrt{\delta t}N(0,1) + \frac{\left(\sum_{i=1}^{N(t)} X_i\right)\delta t}{\sqrt{\Lambda(t)}} \quad (27)$$

This can be interpreted as

$$y_i = \beta_1 V_{1,i} + \beta_2 V_{2,i} + \epsilon_i \quad (28)$$

where

$$y_i = \frac{\Lambda(t + \delta t) - \Lambda(t)}{\sqrt{\Lambda(t)}}; \beta_1 = \left(\kappa\theta + \sum_{i=1}^{N(t)} X_i\right); \beta_2 = -\kappa; V_{1,i} = \frac{\delta t}{\sqrt{\Lambda(t)}};$$

$$V_{2,i} = \sqrt{\Lambda(t)}\delta t \text{ and } \epsilon_i = \sigma\sqrt{\delta t}N(0,1)$$

The parameters can be estimated as follows:

$$\hat{\kappa} = -\hat{\beta}_2 \quad (29)$$

$$\hat{\theta} = \frac{\left(\hat{\beta}_1 - \sum_{i=1}^{N(t)} X_i\right)}{\hat{\kappa}} \quad (30)$$

$$\hat{\sigma} = \sqrt{\frac{\text{Var}(\epsilon_i)}{\delta t}} \quad (31)$$

Monte Carlo simulations are applied in this case to scrutinize how the estimates perform. The model, at the outset, compares a deterministic and a stochastic approach. In the former model, we calibrated the model using a vector of Gaussian samples, resulting in a generated path. After this, the later process is redone with several different Gaussian vector values, resulting in multiple paths while making real parameters. For reasonably accurate estimators, a small Mean Absolute Percentage Error (MAPE) should be realized and, consequently, the convergence of the Monte Carlo estimates should tend towards the real parameters, in principle indicating unbiased data. Refer to Table 1 for estimates and Figure 3. It can be observed that there is consistency with the MAPE values indicated in the table.

Due to discretization errors, there exists a possibility of generating negative intensity paths. This was dealt with before calibration by shifting all series by a 99th percentile as suggested by Orlando et al. (2020). This allows for a positive series of data; also, some negative quantities can be removed from the prediction. This is further necessary to ensure the intensity process remains positive as expected in any insurance portfolio.

Table 1. Assumed values vs. Monte Carlo simulated mean values for parameters.

	κ	θ	σ
Assumed Means	5	0.5	0.03
Monte Carlo Simulated Means	5.0065	0.4893	0.0294
MAPE	0.13%	2.14%	2.00%

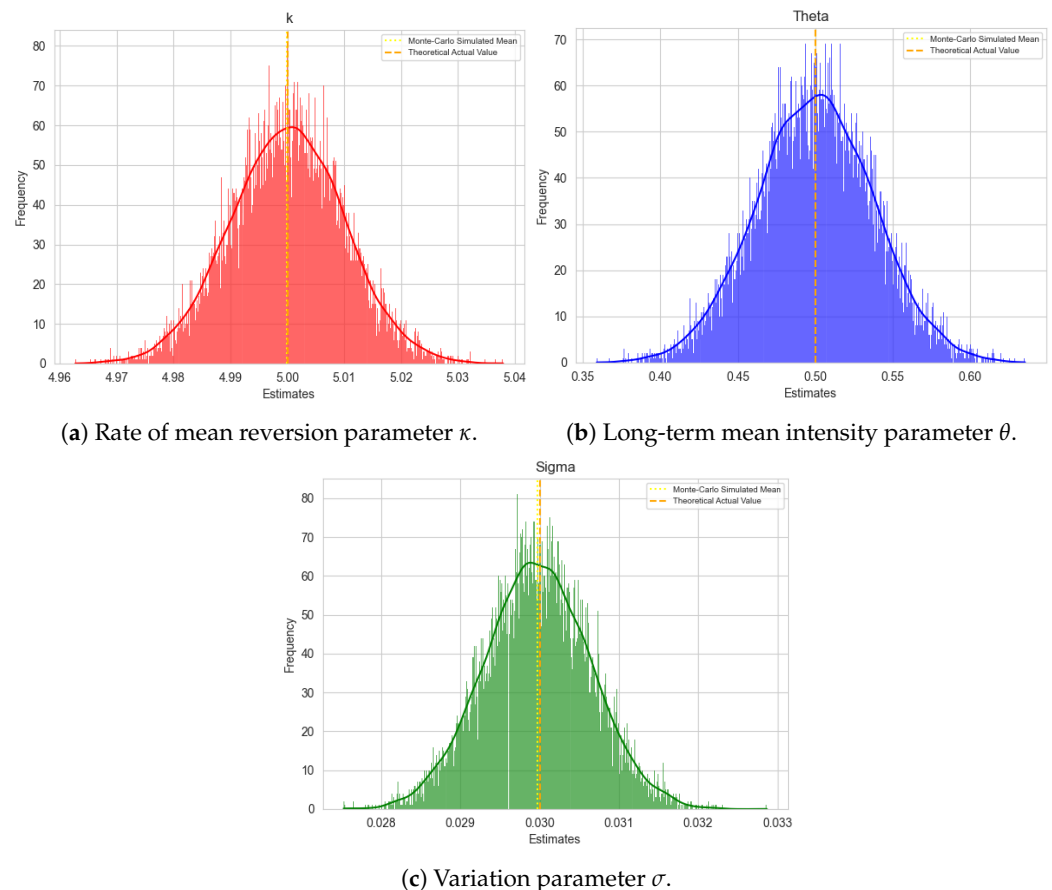


Figure 3. Absolute value plots between assumed and simulated BADJP parameters.

Simulating Basic Affine Jump Diffusion Process: We are going to assume $N = 1000$, $T = 1.0$, $dt = \frac{T}{N}$, $\kappa = 0.05$, $\theta = 0.05$, and $\sigma = 0.03$. We begin at $\Lambda(0) = 0.3$. The intensity of the Poisson process for the jump is given by $\lambda_{jump} = 0.1$ and the mean by $\mu_{jump} = 0.05$. The Monte Carlo simulated mean of the intensity process equal to $\Lambda(t) = 0.3018$.

Thereafter, we fixed $N = 1000$ and varied T to obtain the corresponding means of intensities for $T = 10$, $\Lambda(t) = 0.21132$, $T = 100$, $\Lambda(t) = 0.048935$, $T = 1000$, $\Lambda(t) = 0.044300$, and (finally) $T = 10,000$. For the latter, the mean of the intensity process does not exist. The corresponding plots can be seen in Figures 4 and 5.

To determine the effectiveness of the simulated value of the BAJDP, we compare the value of the simulated process output to the theoretical or empirical benchmarks. Such comparison is carried out to assess whether properties such as moments, distributions, and jump characteristics resonate as indicated in Table 2. Theoretical properties of the BAJDP include

$$E[\Lambda(t)] = (\theta + (\Lambda(0) - \theta) \exp(-\kappa t)) \quad (32)$$

and the variance, which is derived from the Feller condition, ensuring non-negativity, is given by

$$\text{Var}[\Lambda(t)] = \frac{\sigma^2}{2\kappa} (1 - \exp(-2\kappa t)) + \text{Jump Variance} \quad (33)$$

Furthermore, the jump component of the CIR process also has as its properties

$$E[\text{Jump}] = \lambda_{jump} \quad (34)$$

$$\text{Var}[\text{Jump}] = \sigma_{jump}^2 \quad (35)$$

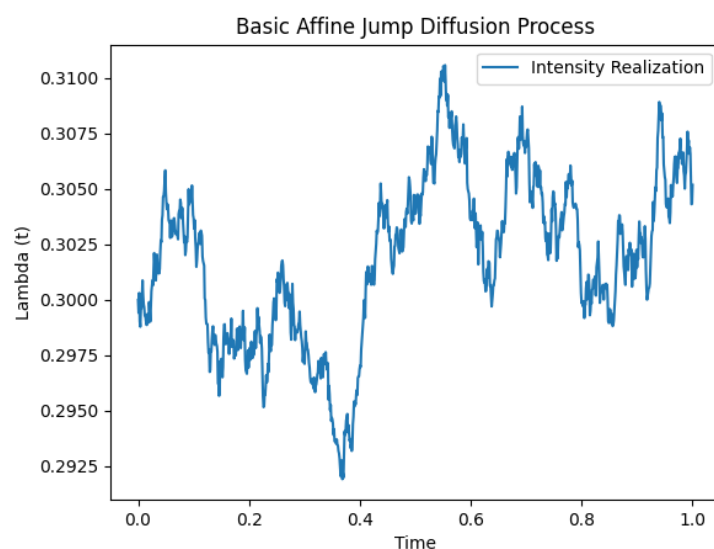


Figure 4. Simulated BAJDP when $T = 1$, $\kappa = 0.05$, $\theta = 0.05$, $\sigma = 0.03$, and $\Lambda(0) = 0.3$.

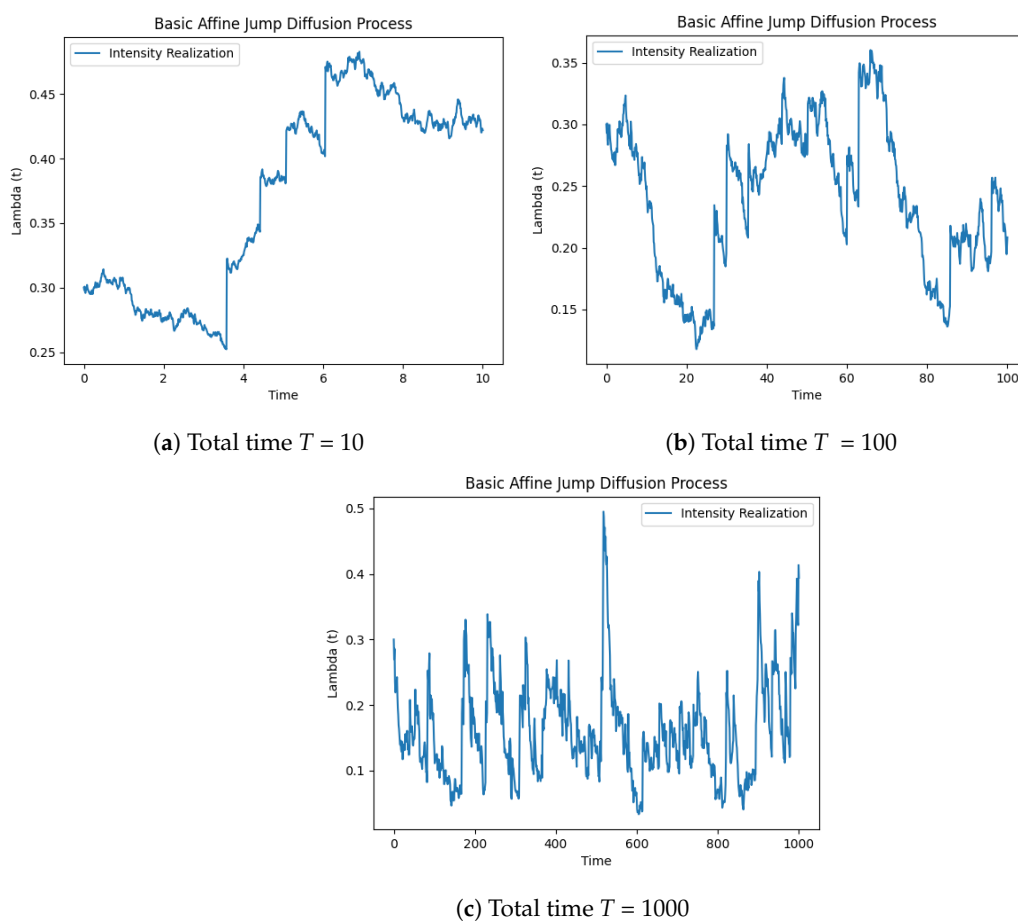


Figure 5. Plots of trajectories of intensity process when $\kappa = 0.05$, $\theta = 0.05$, $\sigma = 0.03$, and $\Lambda(0) = 0.3$.

Table 2. Effectiveness of simulated values of intensity process.

Time (T)	Simulated Mean	Theoretical Mean	MAPE
1	0.30185	0.28787	4.65%
10	0.21132	0.201633	4.80%
100	0.048935	0.051684	5.32%
1000	0.0443	0.05000	11.40%

We are now in a position to estimate the expected value of a mixed stochastic process $EL(t)$; see Table 3. To approximate the process, we make some other assumptions with respect to the parameters used. The process is given by Equation (23).

Table 3. Approximate expected value of $EL(t)$ where $\alpha = 3, \beta = 2, \Lambda(t) = 0.30185$, and $\mu = 10$.

Total Time	Expected Value of Mixed Process
10	17.356

Taking $\beta = 2$ and $\alpha = 3$, the approximated value of the intensity process $\Lambda(t) = 0.30185$ and the expected value of the individual claims size $\mu = 10$.

The results of Figures 4 and 5 show that the intensity of the BAJDP is inversely proportional to the time; i.e., as time goes on, the expected rate of intensity decreases. Such results are consistent with those of Liu (2012) in which there is exponential decay in the spikes of the intensity process.

Finally, in Figure 6, the expected value of the loss process increases as time goes on. This suggests that the rate of expenses of any nature increases as time goes on. More claims and other expenses are anticipated as the business grows positively. This is consistent within an insurance portfolio setup. These results are consistent with those obtained by Liu (2012) in which the intensity process is modelled as a shot noise Cox process, in which there is exponential decay in the spikes of the intensity process. The intensity process is non-deterministic based on the graphical output, which indicates that it is stochastic. This stochastic nature of the intensity process helps to effectively capture the inherent randomness in the insurance expense/claims data. It has been demonstrated that this variability can also stem from environmental factors. Based on the comparison between the aggregate loss of the present model and the previous model in which only claims were taken as sources of loss, there is a significant reduction in the expected value of the loss process, as indicated in Figure 6.

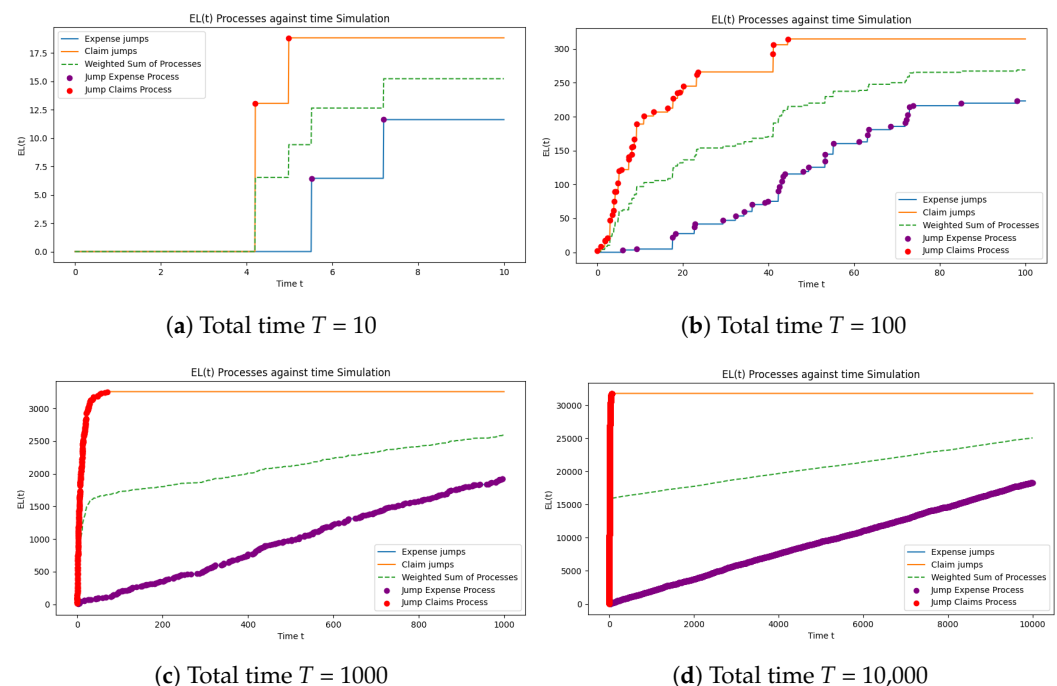


Figure 6. Plots of time taken vs. expected expense-augmented loss process T : $\Lambda(t) = 0.30185$, $\beta = 2$, $\alpha = 3$, $\lambda = 0.1$, and $w = 0.5$.

5. Data Analysis and Results

5.1. Descriptive Statistics of the Data

Table 4 presents the measures of central tendency of the data given. This table gives the minimum observed value, first quartile, media, mean, third quartile and the maximum observed value for the aggregated gross premium, claims and expenses quarterly data of the identified general insurance companies in Kenya. Clearly the gross premium presents the highest quarterly mean of 8.197×10^{10} followed at a distance by claims with 3.480×10^{10} . The correlation coefficient between claims and expenses is 0.6154442, as indicated in Table 5.

Table 4. Descriptive summary statistics of data.

	Gross Premium	Claims	Expenses
Min	2.056×10^{10}	7.193×10^9	3.005×10^9
1st Qu	4.768×10^{10}	1.801×10^{10}	9.555×10^{10}
Median	7.311×10^{10}	3.036×10^{10}	1.739×10^{10}
Mean	8.197×10^{10}	3.480×10^{10}	2.029×10^{10}
3rd Qu	1.051×10^{11}	4.598×10^{10}	2.674×10^{10}
Max	1.913×11^{10}	8.609×10^{10}	8.502×10^{10}

Table 5. Correlation coefficient matrix.

	Gross Premium	Claims	Expenses
Gross Premium	1.0000000		
Claims	0.9312012	1.0000000	
Expenses	0.6154442	0.6405205	1.0000000

5.2. Testing the Normality of the Data

The general insurance data used to test the model were obtained from the Insurance Regulatory Authority (IRA) website [Source: <https://ira.go.ke/annual-reports/>] for all the general insurance products offered in Kenya. The data extracted featured gross premiums, claims, and expenses for each of the four quarterly reports spanning from 1 January 2012 to 31 December 2023, inclusive. The data used were sourced from the IRA website under annual and quarterly reports of the general insurance section. These data were sourced from the following insurance companies in Kenya: AAR Insurance, AIG Kenya, AMACO, APA, British American, Cannon, Concord, CIC General, Corporate, Directline, Fidelity Shield, First Assurance, Gateway, Geminia, GA, Heritage Insurance, ICEA Lion General, Intra Africa, Invesco, Jubilee, Kenindia, Kenya Orient, Kenya Alliance, Madison, Mayfair, Mercantile, Occidental, Pacis, Phoenix, Real, Resolution Health, Tausi, The Monarch, Trident, UAP Insurance, Takaful, and Xplico. Out of all these, the general insurance business in Kenya is dominated by CIC, Jubilee, Britam, ICEA Lion General, and APA. It is also important to note that general insurance dominates the insurance industry, accounting for 60% of the industry's gross written premiums as indicated in [<https://ira.go.ke/annual-reports/>].

Before testing the data, we conducted a normality test to ascertain the nature of the data used. Various methods such as graphical representation using graphs such as frequency distribution graphs, distribution density graphs of the data, and quantile-quantile (Q-Q) plots were used. The representations given in Figures 7–9 indicate that the data are slightly positively skewed for both gross premiums and claims as opposed to the expenses data, which are always positively skewed.

However, statistical tests of normality, such as the D'Augusto–Pearson test, for the null hypothesis indicate that the data do not significantly differ from the normal compared

to the alternative hypothesis. This test is usually used in cases where the dataset contains repeated entries; thus, it is not of importance in the present study. The other important test of normality is the Kolmogorov–Sminorv test, which is important but its application is limited to sample data that are greater or equal to 50. Our data are slightly below this; hence, the only test for normality we employ is the Shapiro–Wilks test. The output and verdict of the hypothesis at a significance level of 5% is presented in Table 6.

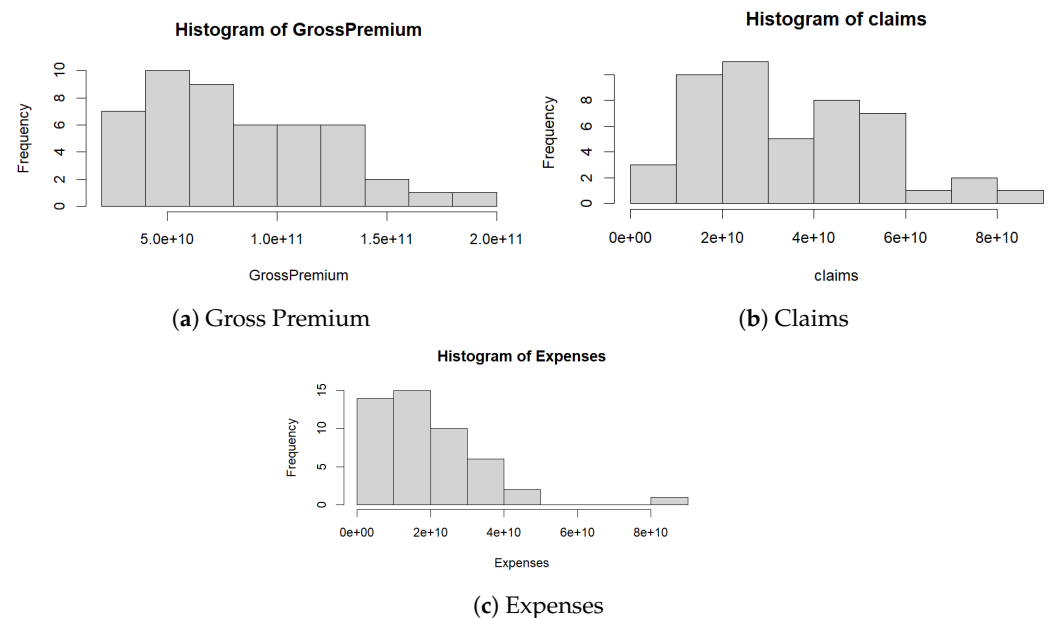


Figure 7. Histogram graphs of gross premium, claims, and expenses.

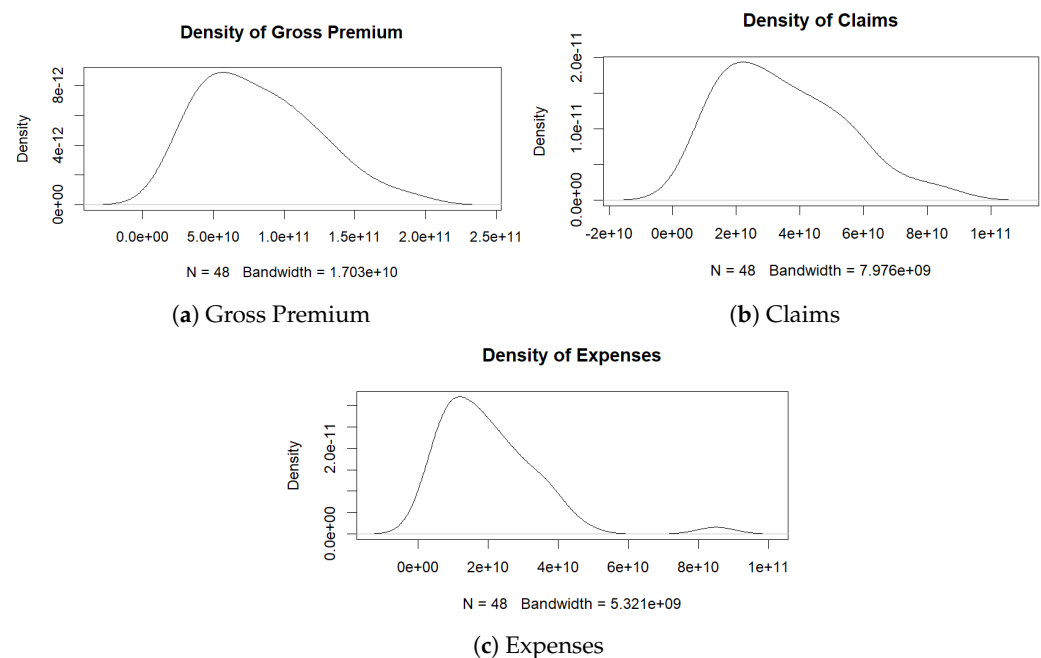
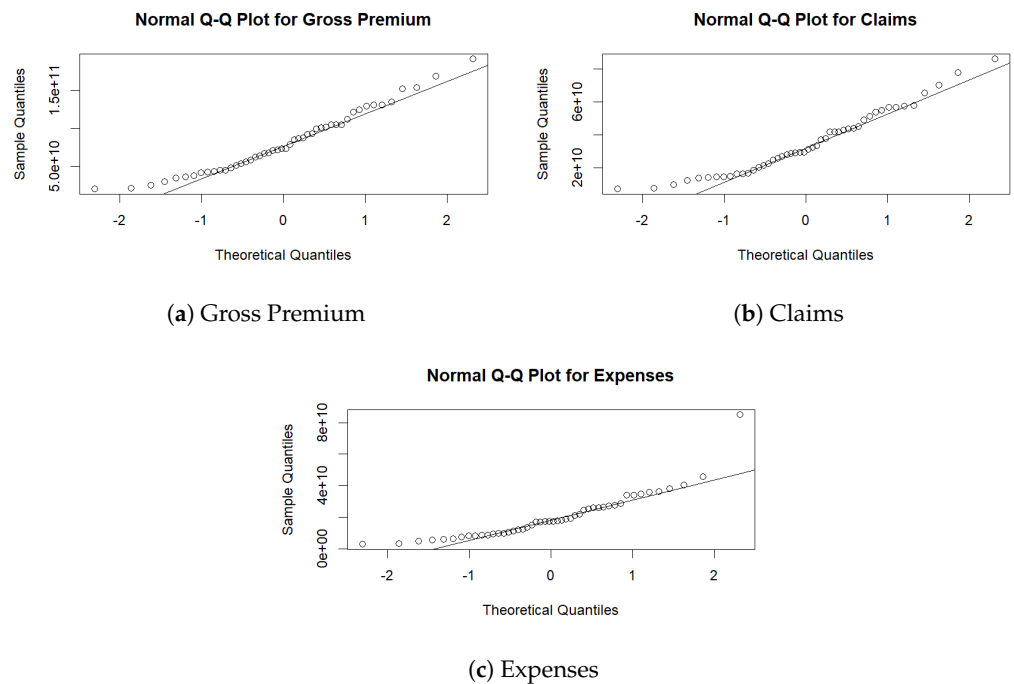


Figure 8. Density representation of gross premium, claims, and expenses.

Table 6 shows that the gross premium data are normal ($p > 0.05$). However, the graphical representations show there is a slight positive skewness. In the same vein, the claims data are also slightly positively skewed, clearly indicating the data are not normally distributed. This may be a result of a few extreme values, which is a characteristic of claims data.

Table 6. Shapiro–Wilks test of normality (H_0 : pop is normal vs. H_1 : pop is not normal).

	Gross Premium	Claims	Expenses
w	0.9590	0.94878	0.83672
p-value	0.09278	0.03576	9.833×11^{-6}
Verdict	($p > 0.05$) Fail to Reject H_0 (Normal)	($p \leq 0.05$) Reject H_0 (Not Normal)	($p \leq 0.05$) Reject H_0 (Not Normal)

**Figure 9.** Q-Q plots of gross premium, claims, and expenses.

5.3. Parameter Estimation and Model Fitting

The expected value of the classical loss process (Classical Loss Model (HPP)) is given by $E[S(t)] = \mu\lambda t$. Under the present modeling framework, the loss process is modeled as $EL(t) = \Lambda(t) \left[w \frac{\alpha}{\beta} + (1 - w)\mu \right]$. We use the method of maximum likelihood estimation to obtain the parameters from the dataset [Source: <https://ira.go.ke/annual-reports/> (Accessed on 8 January 2025)].

Table 7 gives a comparison of the means of fitted parameters for the loss processes. The present model is referred to as the Expense-Augmented model since it takes into consideration other expenses as part of the money outgo, making its expected value the highest. This makes the model more robust. The classical loss model (HPP) was developed in Cramér (1930). Though mostly used for its simplicity, it does not reflect the actual loss in an insurance company. This is because it does not capture the randomness inherent in the claims data, for example. The NHPP loss model was developed in Grandell (2006), Schmidli (2017), and Kyprianou (2013). This model makes the intensity process vary with time but it does not capture overdispersion in claims data like the HPP loss model. For its approximation and fitting, we made the necessary assumptions for its intensity process $\lambda(t)$ simulation. It was then fitted using a generalized linear model to obtain the intensity value.

$$\lambda(t) = \lambda_0 + \lambda_1 \sin(2\pi t) \quad (36)$$

Table 7. Comparison between existing expected value of losses and $EL(t)$.

Loss Model	Expected (Parameter) Values	Expected Value of Loss
Expense-Augmented Loss Process	$EL(t) = \Lambda(t) \left[w \frac{\alpha}{\beta} + (1 - w)\mu \right]$	27,546,013,380
Classical Loss Model (HPP)	$ES(t) = \mu\lambda t$	11,655,405,820
Classical Loss Model (NHPP)	$ES(t) = \mu\lambda(t)$	19,560,050,000
Gaussian Model $S(t) \sim N(\mu t, \sigma^2 t)$	$ES(t) = \mu t$	27,453,528,920

The model also does not take expenses into account when modeling loss in an insurance portfolio. [Debicki et al. \(2015\)](#) developed a loss model in which the claims process conforms to a Gaussian model. Though the model mean result is closer to those of the present model, it does not take other expenses into account, making it not accurate in the approximation of the aggregate loss. The parameters for the loss processes in Table 7 were estimated by use of the maximum likelihood estimation technique in R.

Table 8 gives a comparison of the performance of the present expense-augmented loss model and the other three existing models. Loss ratio (LR), according to [Gill and Hewitt \(1964\)](#), [Van Kampen \(2003\)](#), and [Maggioni and Turchetti \(2024\)](#), is the ratio between the actual loss and the total gross premiums in an insurance portfolio over a given period, such as from January 2012 to December 2023 as per the data used. It is used to show the rate of expenditure on claims and other expenses, giving the profitability index. Furthermore, the loss ratio distribution describes the probability of the loss ratio being realized. In the present data, the expense-augmented loss ratio is highest at 67.20%, indicating insurers are settling claims as anticipated. Due to the failure to include other expenses in the loss modelling framework for classical loss models and Gaussian loss models, the loss ratio produced is relatively lower at 42.48%. This measure is not only applied by players such as rating agencies, lenders, customers, and regulatory agencies such as the IRA and Central Bank of Kenya (CBK), but also by the insurers themselves to assess their financial position. Customers may use the data to monitor and assess the performance of companies in order to choose from different companies and services. Regulatory agencies may set minimum levels on certain products to protect both the insurer and the company, as indicated in [Maggioni and Turchetti \(2024\)](#). The statistical properties of the loss ratio distribution are given for the present model by

$$LR = \frac{\text{Claims (S)} + \text{Expenses (E)}}{\text{Gross Premium (P)}}, \quad E[LR] = \frac{E[S] + E[E]}{P}, \quad \text{Var}[LR] = \frac{\text{Var}[S] + \text{Var}[E]}{P^2}$$

Table 8. Comparison of models in terms of loss ratio and expected loss ratio.

Loss Model	Loss Ratio	Expected Loss Ratio
Expense-Augmented Loss Process	0.6720	1.40000%
Classical Loss Model (HPP)	0.4248	0.02962%
Classical Loss Model (NHPP)	0.4248	0.04960%
Gaussian Model $S(t) \sim N(\mu t, \sigma^2 t)$	0.4248	0.06900%

To obtain the same statistical properties for the other existing three loss models, the expenses are set equal to zero. This is because expenses have been ignored for the simplicity of the modelling framework. Further, the scatter plot of the two variables is given in Figure 10. There is a relationship between the variables. Because there is a close relationship between the two variables, we also investigate the linear regression.

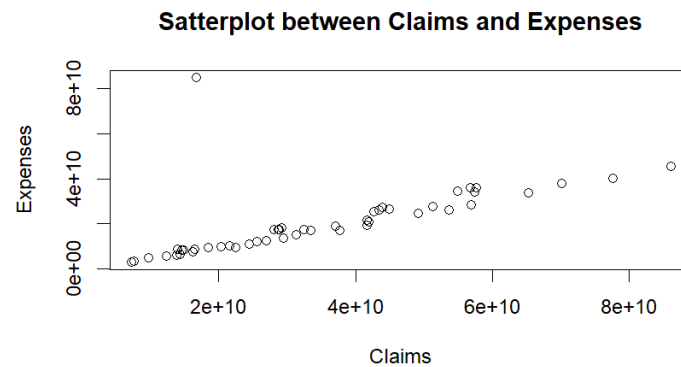


Figure 10. Scatter plot of claims and expenses.

Table 9 gives the output of the regression residuals from the regression output between claims and expenses. Its analysis plays a crucial role in diagnosing and validating the assumptions of the model. It is the difference between the observed and predicted values.

Table 9. Regression residuals between claims and expenses.

Min	Q1	Median	Q3	Max
-4.660×10^9	-3.622×10^9	-2.44504×10^9	4.269×10^8	7.348×10^{10}

Residual analysis is very important as it may be used to estimate linearity, independence, normality, and homoscedasticity, i.e., the ability of the residual variance to be constant across all levels of the predicted values.

From Table 10, the predicted value when a claim is equal to zero is 3.461×10^9 . It has a standard error of 3.390×10^9 and its p -value ($p > 0.05$), hence its insignificance according to the model. Expenses on the other side have a coefficient estimate of 4.835×10^{-1} with a standard error of 8.547×10^{-2} , with a p -value of 9.43×10^{-7} , which is below 0.05, hence indicating significance—i.e., the coefficient is significantly different from zero in the test statistics. This coefficient is the slope, and it shows a change in expenses for a unit change in claims.

Table 10. Coefficients between claims and expenses.

	Estimate	Std. Error	t Value	Pr(> t)
Intercept	3.461×10^9	3.390×10^9	1.021	0.313
Expenses	4.835×10^{-1}	8.547×10^{-2}	5.657	9.43×10^{-7}

Table 11 shows other components of the summary of the regression model of claims and expenses. The residual standard error (1.126×10^{10}) on 46 degrees of freedom gives the average deviation of the observed values from the fitted values, and a better fit is achieved by a smaller RSE. A total of 41.03% of the variability in expenses is explained by the claims variable. This level of variability can also be confirmed by the low correlation coefficient between the two variables. Finally, the F-statistic and its p -value test the overall significance of the model. The p -value is 9.431×10^{-7} , which is statistically significant (given the established 5% significance level).

The linear regression model between claims and expenses is thus given by the equation below.

$$\text{Expenses} = 0.4835\text{Claims} + 3461000000 \quad (37)$$

Table 11. RSE, multiple R^2 , adjusted R^2 , F-statistic, and p -value on regression model.

Residual Std. Error (RSE)	Multiple R^2	Adjusted R^2	F-Statistic	p -Value
1.126×10^{10} on 46 d.f	0.4103	0.3974	32 on 1 & 46 d.f	9.431×10^{-7}

6. Conclusions

Conventional risk processes and many other existing models have always assumed that the only source of money outgo in an insurance portfolio is claims, as indicated in Albrecher and Asmussen (2006), Bartlett (1986), Cox (1955), Debicki et al. (2015), Kyprianou (2013), and Liu (2012). In an ideal insurance setup, money outflows result from other factors such as expenses related to policy maintenance and acquisition—herein referred to as underwriting expenses—and costs relating to the investment of an insurance company's assets. The present study examines and includes other expenses to ensure that a robust money outgo process is realized. It is worth noting that the intensity of both the claims and other costs has been modeled by an affine intensity process known as BAJDP or, simply, a CIR process with a jump component. The net effect, as depicted in Figure 6, is that there is a reduction in the money outflow from the insurance portfolio; this consequently increases retention, and thus financial stability is guaranteed. Investment expenses can be included in the model and also expenses can be modeled by other distributions that are also heavy-tailed, such as Pareto and log-normal distributions. Runge–Kutta methods normally produce more accurate approximations as compared to Euler–Maruyama methods; hence, these can be used for numerical analysis of BAJDP SDE to improve our results. Finally, the expense-augmented loss process has been compared to other existing models through the measurement of the LR and expected LR. Other methodologies such as premium-adjusted claims ratios, a technique that assesses the proportion of premiums consumed by aggregate losses, were also explored. This method is applicable when it comes to the identification of the probability of claims exceeding premiums in reinsurance arrangements.

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Abbreviations

The following abbreviations are used in this manuscript:

HPP	Homogeneous Poisson Process
NHPP	Non-Homogeneous Poisson Process
DSPP	Doubly Stochastic Poisson Process
LR	Loss Ratio
ALD	Asymptotic Laplace Distribution
DED	Double Exponential Distribution

CIR	Cox–Ingersoll–Ross
BAJDP	Basic Affine Jump Diffusion Process
SDE	Stochastic Differential Equation
DCP	Dynamic Contagion Process
CDF	Cumulative Distribution Function
PDF	Probability Distribution Function
NAIC	National Association of Insurance Commissioners
MGF	Moment Generating Function.
IRA	Insurance Regulatory Authority
CBK	Central Bank of Kenya

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