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Modeling Placement-Based Student Pathways in Higher Education: A Compartmental Framework of Flow from Placement to Assessment

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Abstract

Student progression through placement-based course pathways can be represented as a continuous-time dynamical system. We utilize a theoretical three-compartment framework to model institutional student flow in placement-exam-determined courses, with movement through placement (P : students placed but not yet engaged), active engagement (E : students participating in coursework), and assessment (A : students reaching the assessment stage). The model consists of ordinary differential equations describing recruitment, stage-to-stage transitions, stage-specific attrition, and remediation that returns students who failed assessments to re-enrollment in the course. We establish the existence and uniqueness of solutions, characterize the equilibria, and derive conditions for local stability. The system is solved numerically using an adaptive hybrid block method, and parameters are examined under five simulated scenarios: high recruitment, low placement, high attrition, slow progression, and low pass rates. The result suggests that high placement recruitment increases enrollment and may not be proportional to success at the assessment stage; low placement-to-engagement transitions can limit access, slow progression constrains the conversion of enrolled students into successful completers, low pass rates generate remediation cycles that may create additional resource demands, and high attrition produces substantial losses across the pathway. Future work should focus on empirical validation using real-world educational datasets to calibrate the model.

Keywords: mathematical modeling; compartmental model; student flow; institutional progression; assessment; student success



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1. Introduction

Understanding how students advance through academic programs requires models that capture the dynamic nature of their educational trajectories. As such, it is crucial to distinguish between cognitive learning progressions, the micro-level evolution of a student's conceptual understanding within a specific knowledge domain [1], and institutional progression, which describes the macro-level movement of students across administrative academic states. The former, learning progressions, describe the ordered sequence of conceptual developments that learners experience as they move from intuitive understanding

to advanced competency [2]. It allows educators and researchers to model not only what students know at a given moment but also learning pathways through which deeper understanding develops. Therefore, modeling learning progressions requires identifying where learners begin, defining intermediate stages of understanding, and empirically validating the developmental pathways connecting them [3–5]. Recent advances in learning progression theory argue that learning progressions are most effective when embedded within coordinated instructional systems [6]. Learning progressions incorporate disciplinary core ideas, cognitive structures, measurable progress variables, and assessment tools that situate student thinking along a developmental continuum [7]. Hence, these perspectives on learning progression suggest that learning is a dynamic, developmental process shaped by the interaction of cognitive resources, instructional supports, and prior knowledge [8].

Institutional progression, on the other hand, dictates how those developmental stages translate into formalized academic advancement through placement, course enrollment, and institutional assessment. Unlike cognitive development, which is internal and highly individualized, student progression in institutional contexts is shaped by formal enrollment structures, course completion requirements, and program-level checkpoints [9]. Universities rely on structural mechanisms such as placement exams, formal course enrollment, and summative assessment to standardize student evaluation, allocate instructional resources, and maintain academic standards [10,11]. Consequently, a student's observed academic trajectory is not only a product of their intellectual capacity but is also heavily guided by institutional policies that dictate entry requirements, progression rates, and remediation pathways [12]. Thus, navigating this institutional structure requires students to successfully transition through specific pathways, which the model proposed in this study conceptualizes as distinct compartments. Initial placement, for instance, acts as a sorting stage that often determines the trajectory of a student's academic pathway. Highly restrictive placement policies can divert otherwise capable students into remedial sequences, thus creating systemic barriers that significantly reduce their likelihood of graduation [13]. Following successful placement and formal enrollment, a student's active engagement within the system could be tracked through institutional metrics such as attendance, coursework submission, and term-to-term retention. This progression culminates in a formal assessment stage where student competency is evaluated, resulting either in successful completion, attrition, or return to active engagement for remediation. Therefore, mathematically modeling this institutional progression provides a vital macroscopic view of how structural pathways, rather than individual cognitive abilities alone, collectively shape student throughput, attrition, and overall success [14].

Students often progress through prerequisite course sequences in highly variable ways, delaying, repeating, or exiting before encountering advanced material [15]. For example, undergraduate STEM students commonly encounter instructional practices misaligned with their developmental readiness in high-stakes settings, especially in courses that rely on placement examinations to determine enrollment [16]. Research across mathematics and engineering education shows that conceptual and procedural knowledge often progress along distinct developmental pathways and at different rates [17–19]. Students entering placement-determined courses differ in their prior preparation, pace of engagement, and pathways to successful course completion. Although performance on a placement examination determines initial course enrollment, progression toward ultimate success involves several subsequent stages, including exposure to course content, sustained engagement through study and practice, and reaching a formal assessment stage that determines whether a student continues, receives remediation, or completes the course.

Research has shown that student learning and progression can be complex, context-dependent, and non-linear [20]. As such, the progression model should account for the

variability and complexity of students' actual developmental trajectories, which may move from initial placement through active engagement to demonstrated competency and are shaped by factors such as instructional timing, the coherence of instructional support, and individual learning traits [21]. Our conceptualization of student advancement as a series of discrete compartments draws on research on patterns of progressions [4,9,20].

At the institutional level, these stages can be represented as observable transitions among discrete compartments, such as placement, enrollment, and assessment. For example, placement policies and remedial pathways serve as important structural influences on student progression [22]. Meiselman and Schudde [23] and Logue et al. [24] show that institutional rules governing movement from prerequisite to college-level coursework can substantially affect advancement rates. In contrast, Ngo [25] highlights inefficiencies associated with redundant placement pathways. Building on the literature, this study employs mathematical modeling (a deterministic compartmental model framework) to examine how changes in placement, progression, attrition, assessment, and remediation rates may influence student flow through a higher-education course pathway. Our framework applies to course structures that begin with an initial placement examination and culminate in a final summative assessment. Rather than directly modeling the internal development of cognitive understanding, we modeled student progression through observable instructional and administrative states using time-dependent transition rates.

1.1. Theoretical Foundations and Compartmental Modeling Framework

Mathematical modeling offers a quantitative framework for linking micro-level cognitive processes to macro-level educational patterns. This approach has become prominent in the learning sciences and educational data analytics [26]. Although such models utilize a set of differential equations and have a longstanding application in public health and epidemiology for studying transmission processes, population dynamics, and intervention effects [27,28], their potential applications extend to cognitive, behavioral, organizational, and educational systems [7,29,30]. A compartmental model divides a population into a small number of mutually exclusive states, or compartments, and describes the movement of individuals between them using time-dependent transition rates. Rather than tracking individuals, the model tracks the aggregate size of each compartment over time, typically through a system of ordinary differential equations in which each equation balances the inflows and outflows of a given state variable.

Differential-equation models have been applied in educational settings to examine processes such as the development of professional readiness among aerospace engineering students [31] and patterns of student dropout and progression in undergraduate mathematics [32]. The reliability of these models lies in their ability to represent time-dependent transitions and feedback mechanisms, including remediation and re-entry, that may be difficult to capture with static models. Building on this, this study uses continuous-time dynamical modeling to investigate institutional student progression. By formalizing assumptions about student flow, we quantify how structural conditions, such as placement thresholds, attrition rates, and assessment barriers, shape aggregate academic pathways. Also, this model enables the simulation of alternative placement decisions, remedial interventions, and institutional policies.

The proposed framework consists of three compartments: placement (P), active engagement (E), and assessment (A). This compartmental model draws on student-retention theories, particularly Tinto's student integration model [10] and Astin's theory of student involvement [33], both of which posit that a combination of institutional entry requirements, active academic engagement, and formalized assessment outcomes drives academic progression. In this context, we mathematically operationalize several educational constructs:

Placement represents students who have been assigned to a course pathway based on their placement outcome but have not yet begun active course participation. Engagement represents active participation in learning activities, including attendance, assignment completion, study, and practice, that support progression toward formal assessment.

We draw a strict distinction between demonstrated competency, defined as successfully passing an institutional assessment to exit the pathway, and true cognitive mastery, defined as durable and transferable understanding. The proposed model evaluates competency using institutional assessments, while recognizing that passing an examination does not necessarily imply deep conceptual mastery and that failure requires renewed engagement through remediation. Under this framework, completion and student success are defined by successful exit from the course pathway. This allows us to test assumptions about student flow, examine counterfactual scenarios, and identify parameters that separate successful from unsuccessful learning pathways, thereby contributing to the growing body of research linking student retention theory with placement policy and targeted administrative interventions using a mathematical modeling approach.

1.2. Research Questions and Analytical Strategy

This study is theoretical and simulation-based. Rather than analyzing an empirical student dataset, it develops a hypothesis-generating mathematical framework for representing institutional student flow. Using a deterministic compartmental model, we examine how students transition through placement, active engagement, and assessment under alternative hypothetical conditions. The study is guided by the following research questions:

1. How can student flow through placement, active course engagement, and assessment be represented as a continuous-time compartmental dynamical system?
2. What fundamental mathematical properties, including positivity, boundedness, equilibrium states, and stability conditions, characterize the proposed student-flow system?
3. Based on scenario-driven numerical simulations, what structural insights can be gained regarding how variations in institutional parameters (e.g., attrition rates, assessment pass rates, and progression rates) create bottlenecks and alter the overall efficiency and throughput of the academic pipeline?

To address these questions, this study is structured into two main parts:

1. Mathematical analysis: We establish the positivity and boundedness of solutions, derive the equilibrium states, and determine the stability conditions governing the long-term behavior of the system.
2. Numerical simulation: We solve the model under five literature-informed, hypothetical scenarios designed to illustrate institutional conditions that may affect student progression in placement-based course pathways:
 - High Inflow (HI): A large number of students enter the pathway, increasing pressure on instructional capacity and affecting system throughput.
 - Low Placement (LP): This represents restrictive institutional entry or low placement-to-enrollment success. This is driven by a high placement failure rate (ϕ), which restricts the transition from placement stage to active engagement ($P \rightarrow E$).
 - High Attrition (HA): Students leave the pathway at elevated rates through withdrawal, dropout, or disengagement.
 - Slow Progression (SL): Students transition between stages at reduced rates because of instructional, academic, or institutional constraints.
 - Low Pass Rate (LR): A scenario defined by strict assessment outcomes, where a small proportion of assessed students demonstrate competency on their first attempt, increasing remediation and repeated course participation.

Note that these scenarios do not represent empirically validated student cohorts. Instead, they are theoretical archetypes designed to illustrate and observe specific systemic stressors. By comparing these hypothetical cases, we can observe how specific bottlenecks (e.g., an overly strict assessment vs. a high dropout rate) uniquely disrupt the institutional pipeline.

2. Methods

2.1. Educational Rationale and Definition of Constructs

The categorization of the student population into three distinct compartments, Placed (P), Engaged (E), and Assessed (A), is designed to reflect the macro-level structural pathways of higher education institutions, rather than the micro-level cognitive processes of individual learners. We acknowledge that this is a mathematical simplification of a highly complex reality. The model's structural choice compresses intermediate variables, such as variation in assessment formats, transient misconceptions, and daily fluctuations in student motivation, into aggregated transition rates. The rationale for selecting P , E , and A is that they represent the most critical, observable administrative and academic checkpoints in a student's journey in higher education systems: initial placement, active course engagement, and the formal evaluation of competency through assessment that determines whether a student successfully exits the pathway or requires remediation.

Accordingly, we define the three compartments representing distinct stages of student progression as follows:

- P : Students placed but not yet actively engaged. These students have completed the initial placement evaluation but have not transitioned into active coursework.
- E : Students formally enrolled and engaged in the course.
- A : Students undergoing formal assessment to demonstrate competency. From this stage, students either successfully exit the pathway or, in the case of failed assessment attempts, return to active engagement (E) for remediation.

As illustrated in Figure 1, the transitions among these compartments capture the dynamic flow of students through the academic system. Students enter the placement stage (P) before progressing into active engagement (E), and eventually reaching the assessment stage (A). The diagram also highlights attrition rates (d_1, d_2, d_3), external inflows, and feedback mechanisms (remediation), providing a comprehensive representation of the institutional trajectory.

The compartmental differential equation framework relies on the following key assumptions:

A1: Sequential progression through academic states.

Students progress through three distinct academic states: placement (P), active enrollment and engagement (E), and assessment (A). Transitions within this model characterize the macroscopic flow of each student's institutional pathway.

A2: Measurable engagement as a proxy for progression intensity.

Student engagement in course activities (e.g., lecture attendance, problem-solving practice, and use of instructional resources) is a valid indicator of active participation. Higher engagement levels correlate with increased progression rates and subsequent academic performance.

A3: Transitions.

Transition rates between states depend only on the current state and the aggregate characteristics of students, not on their previous academic history. Students with similar profiles in a given compartment share comparable progression probabilities, independent of how they arrived at that state.

A4: Time-invariant parameters.

Transition rates, attrition rates, and progression parameters remain constant during the observation period (one academic term). The system exhibits homogeneous dynamics, with no significant temporal trends observed within this time frame.

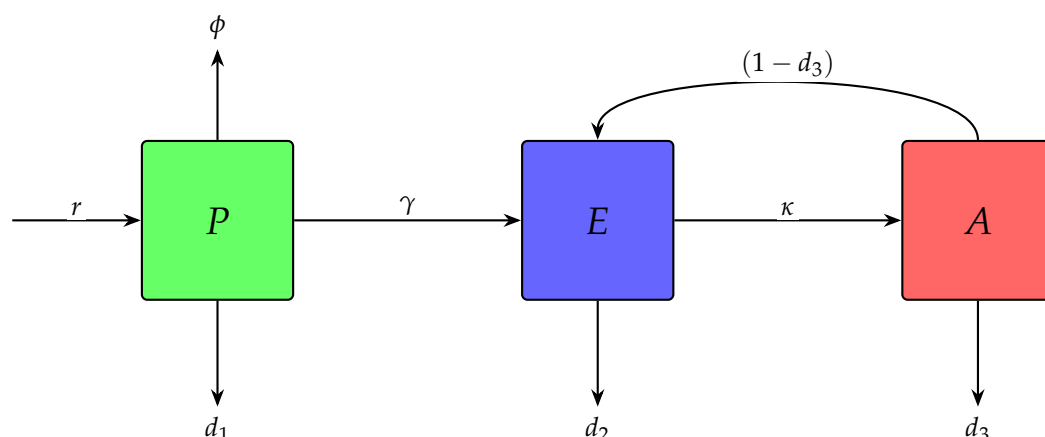


Figure 1. The schematic representation of institutional student flow.

2.2. Compartmental Model Equations

The system of differential equations governing the student dynamics flow is given by

$$\frac{dP}{dt} = r - (\gamma + \phi + d_1)P, \quad (1)$$

$$\frac{dE}{dt} = \gamma P - (\kappa + d_2)E + (1 - d_3)A, \quad (2)$$

$$\frac{dA}{dt} = \kappa E - A, \quad (3)$$

subject to the initial conditions: $P(0) = P_0$, $E(0) = 0$, and $A(0) = 0$.

2.3. Parameter Definitions and Notation

In Table 1, we provide the definitions of the parameters used in the governing equations. We also note that all rates are non-negative and interpreted per unit time (e.g., per week or per term, depending on the chosen time scale).

Table 1. Definitions and descriptions of the rate parameters governing student flow, attrition, and remediation between the placement (P), engagement (E), and assessment (A) compartments.

Parameter	Definition
r	Recruitment inflow rate into the placement compartment (P), estimated as average intake per time unit.
γ	Placement success rate governing the transition $P \rightarrow E$.
ϕ	Placement failure rate driving exit from P without enrollment.
d_1	Placement-stage attrition rate (students who are assessed but do not proceed to enrollment for other reasons).
κ	Assessment progression rate at which engaged students (E) advance to the formal assessment stage (A) through course instruction and learning activities.
d_2	Within-course attrition rate (withdrawal, disengagement, or failure during the instructional process).
d_3	Assessment success rate (proportion of students in A who successfully pass formal evaluation and exit the system).
$1 - d_3$	Remediation/Recycling rate (proportion of students in A who fail formal assessments and are returned to E for further instruction).

3. Mathematical Analysis of the Model

This section provides a mathematical analysis of the dynamical system described by Equations (1)–(3) to characterize its fundamental properties and long-term behavior. First, we establish well-posedness by proving that all solutions remain bounded and positive, thereby ensuring that the model variables remain physically meaningful. Next, we identify the equilibrium points at which the system reaches a steady state. Finally, we conduct a stability analysis to determine the conditions under which the system converges to or diverges from these equilibria, thereby characterizing its macroscopic behavior.

3.1. Boundedness of Solution

Theorem 1 (Boundedness). *Let $N(t) = P(t) + E(t) + A(t)$ represent the total student population within the institutional pathway at time t . Given the non-negative initial conditions, $(P(0), E(0), A(0)) \in \mathbb{R}_+^3$, the solutions of the system are confined to the compact region Ω defined by*

$$\Omega = \left\{ (P, E, A) \in \mathbb{R}_+^3 : P + E + A \leq \frac{r}{\delta} \right\}. \quad (4)$$

Proof. The rate of change of the total population is the sum of the individual compartment rates:

$$\frac{dN(t)}{dt} = \frac{dP(t)}{dt} + \frac{dE(t)}{dt} + \frac{dA(t)}{dt}.$$

Substituting the given differential Equations (1)–(3) and simplifying the terms yields

$$\frac{dN}{dt} = r - (\phi + d_1)P - d_2E - d_3A. \quad (5)$$

Let $\delta = \min(\phi + d_1, d_2, d_3)$. Since all rate parameters are non-negative, $\delta \geq 0$, we establish the following inequality:

$$\frac{dN}{dt} \leq r - \delta(P + E + A) = r - \delta N. \quad (6)$$

Equation (6) is a linear differential inequality. By considering the corresponding differential equation $\frac{dN}{dt} = r - \delta N$, its solution approaches the equilibrium value $\frac{r}{\delta}$ as $t \rightarrow \infty$. By the comparison theorem for differential inequalities, the solution for $N(t)$ is strictly bounded by the solution of the corresponding ordinary differential equation, implying that:

$$N(t) \leq \frac{r}{\delta} + \left(N(0) - \frac{r}{\delta} \right) e^{-\delta t}.$$

As $t \rightarrow \infty$, the exponential term decays to zero, which ensures that

$$\limsup_{t \rightarrow \infty} N(t) \leq \frac{r}{\delta}.$$

□

3.2. Positivity of Solution

Theorem 2 (Positivity and Invariance). *Given the initial data $(P(0), E(0), A(0)) \in \mathbb{R}_+^3$, the solution trajectories $(P(t), E(t), A(t))$ of the dynamical system (1)–(3) remain non-negative for all $t \geq 0$. Consequently, the region Ω is a positively invariant set for the system.*

Proof. To demonstrate that the solutions of the system remain non-negative for all $t \geq 0$, we analyze each differential equation separately.

Positivity of $P(t)$:

Consider the placement compartment equation:

$$\frac{dP}{dt} = r - (\gamma + \phi + d_1)P \triangleq r - \mu P, \quad \text{where } \mu = \gamma + \phi + d_1. \quad (7)$$

Equation (7) is a linear first-order ODE, whose general solution is as follows:

$$P(t) = \left(P_0 - \frac{r}{\mu}\right)e^{-\mu t} + \frac{r}{\mu}.$$

For $P_0 \geq 0$, and $r, \mu > 0$, we consider two cases:

- Case 1: If $P_0 \geq \frac{r}{\mu}$, then $P(t) \geq \frac{r}{\mu}$ for all $t \geq 0$.
- Case 2: If $P_0 < \frac{r}{\mu}$, then $P(t) \geq 0$ for all $t \geq 0$.

Hence, $P(t) \geq 0$ for all $t \geq 0$.

Positivity of $E(t)$:

Consider the enrollment compartment equation:

$$\frac{dE}{dt} = \gamma P + (1 - d_3)A - (\kappa + d_2)E =: f(t) - \alpha E, \quad (8)$$

where $\alpha := \kappa + d_2 > 0$ and $f(t) := \gamma P(t) + (1 - d_3)A(t) \geq 0$ since $P(t), A(t) \geq 0$.

We proceed by contradiction. Suppose $E(t)$ becomes negative for some $t > 0$. Then there must exist a first time $t^* > 0$ such that $E(t^*) = 0$ and $\frac{dE(t^*)}{dt} \leq 0$. However, at $t = t^*$,

$$\left.\frac{dE}{dt}\right|_{t=t^*} = f(t^*) - \alpha \cdot 0 = f(t^*) \geq 0, \quad (9)$$

which is a contradiction. Therefore, $E(t) \geq 0$ for all $t \geq 0$.

Positivity of $A(t)$:

Consider the assessment compartment equation:

$$\frac{dA}{dt} = \kappa E - A. \quad (10)$$

Multiplying by the integrator factor e^t yields:

$$\frac{d}{dt} \left(e^t A(t) \right) = \kappa e^t E(t) \geq 0 \implies A(t) = e^{-t} A_0 + \int_0^t \kappa e^{-(t-s)} E(s) ds. \quad (11)$$

Since $E(s) \geq 0$ and $A_0 \geq 0$, it strictly follows that $A(t) \geq 0$ for all $t \geq 0$. $\square$

3.3. Active Academic Equilibrium (AAE)

In an established institutional pathway, the system will eventually stabilize. We denote the educational equilibrium state as $\mathcal{X}^+ = (P^+, E^+, A^+)$, representing the steady state where the flow of students into and out of the compartment is balanced. At this equilibrium, all time derivatives in Equations (1)–(3) equal zero:

$$\frac{dP(t)}{dt} = \frac{dE(t)}{dt} = \frac{dA(t)}{dt} = 0.$$

Solving the resulting algebraic system yields the steady-state population distribution:

$$(P^+, E^+, A^+) = \left(\frac{r}{\gamma + \phi + d_1}, \frac{r\gamma}{(\gamma + \phi + d_1)(\kappa d_3 + d_2)}, \frac{r\gamma\kappa}{(\gamma + \phi + d_1)(\kappa d_3 + d_2)} \right). \quad (12)$$

3.4. Vacant Pathway Equilibrium (VPE)

The vacant pathway equilibrium (VPE) is a hypothetical state in which no students are actively enrolled or assessed within the instructional compartments. In the proposed model, this corresponds to a state where $E = 0$ and $A = 0$. Let the VPE be denoted as $\mathcal{X}_0 = (P_0, E_0, A_0)$. Setting $E_0 = 0$ and $A_0 = 0$ and solving for the steady state of the P compartment yields the VPE for the system:

$$\mathcal{X}_0 = \left(\frac{r}{\gamma + \phi + d_1}, 0, 0 \right). \quad (13)$$

3.5. Basic Progression Number and Stability Analysis

Adapting the next-generation matrix method from epidemiology to the proposed educational model, we define the basic progression number, denoted $\mathcal{R}_+$. This metric represents the expected number of successful assessment completions a single enrolled student will achieve throughout their academic journey. The relevant transition compartments are E and A , representing students who are actively engaged and those who have reached the assessment stage, respectively. We define the vector of these active compartments as

$$\mathcal{X} = \begin{bmatrix} E \\ A \end{bmatrix}. \quad (14)$$

We decompose the system as

$$\frac{d\mathcal{X}}{dt} = \mathcal{F}(\mathcal{X}) - \mathcal{V}(\mathcal{X}), \quad (15)$$

where $\mathcal{F}(\mathcal{X})$ denotes the rate of new entries into the progression compartments (in this case, students recycled through remediation), and $\mathcal{V}(\mathcal{X})$ represents the transfers between and out of these compartments. That is,

$$\mathcal{F} = \begin{bmatrix} (1 - d_3)A \\ 0 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} (\kappa + d_2)E - \gamma P \\ -\kappa E + A \end{bmatrix}. \quad (16)$$

Linearizing the system by computing the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ with respect to the variables E and A , and evaluating them at the vacant pathway equilibrium $\mathcal{X}_0$, we obtain

$$F = \frac{\partial \mathcal{F}}{\partial \mathcal{X}} \Big|_{\mathcal{X}_0} = \begin{pmatrix} 0 & 1 - d_3 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad V = \frac{\partial \mathcal{V}}{\partial \mathcal{X}} \Big|_{\mathcal{X}_0} = \begin{pmatrix} \kappa + d_2 & 0 \\ -\kappa & 1 \end{pmatrix}. \quad (17)$$

The next-generation matrix is given by $K = FV^{-1}$. First, we compute the inverse of V :

$$V^{-1} = \frac{1}{(\kappa + d_2)(1) - (0)(-\kappa)} \begin{pmatrix} 1 & 0 \\ \kappa & \kappa + d_2 \end{pmatrix} = \frac{1}{\kappa + d_2} \begin{pmatrix} 1 & 0 \\ \kappa & \kappa + d_2 \end{pmatrix}. \quad (18)$$

Multiplying F and V^{-1} yields the next-generation matrix, K :

$$K = FV^{-1} = \begin{pmatrix} 0 & 1-d_3 \\ 0 & 0 \end{pmatrix} \frac{1}{\kappa+d_2} \begin{pmatrix} 1 & 0 \\ \kappa & \kappa+d_2 \end{pmatrix} = \frac{1}{\kappa+d_2} \begin{pmatrix} \kappa(1-d_3) & (\kappa+d_2)(1-d_3) \\ 0 & 0 \end{pmatrix}. \quad (19)$$

The basic progression number, $\mathcal{R}_+$, is the spectral radius of the matrix K . The eigenvalues are $\lambda_1 = \frac{\kappa(1-d_3)}{\kappa+d_2}$ and $\lambda_2 = 0$. Therefore,

$$\mathcal{R}_+ = \rho(K) = \frac{\kappa(1-d_3)}{\kappa+d_2}. \quad (20)$$

Remark 1. The vacant pathway equilibrium (VPE) is locally asymptotically stable if $\mathcal{R}_+ < 1$ and unstable if $\mathcal{R}_+ > 1$. When $\mathcal{R}_+ < 1$, the structural dynamics are insufficient to maintain a steady flow of enrolled students; any initial cohort will eventually exit the system without establishing a sustained, self-replenishing population of engaged learners. Conversely, when $\mathcal{R}_+ > 1$, introducing students into the active course pathway initiates sustained engagement, moving the system away from the VPE toward the active academic equilibrium. Thus, $\mathcal{R}_+ = 1$ represents the critical threshold separating dying-out student flow from a sustainable, active educational pipeline.

Stability Analysis of the AAE

The overall stability of the system's equilibria determines the long-term macroscopic behavior of the student population. Since the system of differential equations is linear, we can determine its stability by analyzing the eigenvalues of the Jacobian matrix of the entire system. The Jacobian matrix, J , is as follows:

$$J = \begin{pmatrix} \frac{\partial}{\partial P} \left(\frac{dP}{dt} \right) & \frac{\partial}{\partial E} \left(\frac{dP}{dt} \right) & \frac{\partial}{\partial A} \left(\frac{dP}{dt} \right) \\ \frac{\partial}{\partial P} \left(\frac{dE}{dt} \right) & \frac{\partial}{\partial E} \left(\frac{dE}{dt} \right) & \frac{\partial}{\partial A} \left(\frac{dE}{dt} \right) \\ \frac{\partial}{\partial P} \left(\frac{dA}{dt} \right) & \frac{\partial}{\partial E} \left(\frac{dA}{dt} \right) & \frac{\partial}{\partial A} \left(\frac{dA}{dt} \right) \end{pmatrix} = \begin{pmatrix} -(\gamma + \phi + d_1) & 0 & 0 \\ \gamma & -(\kappa + d_2) & 1 - d_3 \\ 0 & \kappa & -1 \end{pmatrix}. \quad (21)$$

The eigenvalues (λ) are the roots of the characteristic equation

$$\begin{vmatrix} -(\gamma + \phi + d_1) - \lambda & 0 & 0 \\ \gamma & -(\kappa + d_2) - \lambda & 1 - d_3 \\ 0 & \kappa & -1 - \lambda \end{vmatrix} = 0. \quad (22)$$

Expanding the determinant simplifies Equation (22) to

$$(-(\gamma + \phi + d_1) - \lambda)[(-(\kappa + d_2) - \lambda)(-1 - \lambda) - \kappa(1 - d_3)] = 0. \quad (23)$$

Equation (23) directly indicates that the first eigenvalue is

$$\lambda_1 = -(\gamma + \phi + d_1). \quad (24)$$

Since all parameters are non-negative, it strictly follows that $\lambda_1 < 0$. The remaining two eigenvalues are the roots of the quadratic equation:

$$\lambda^2 + (\kappa + d_2 + 1)\lambda + (d_2 + \kappa d_3) = 0. \quad (25)$$

For a general quadratic equation $\lambda^2 + b\lambda + c = 0$, the roots have negative real parts if and only if $b > 0$ and $c > 0$, according to the Routh–Hurwitz stability criterion. In this case,

- $b = \kappa + d_2 + 1$. Since all parameters are non-negative rates, $b > 0$.
- $c = d_2 + \kappa d_3$. Since all parameters are non-negative, $c \geq 0$.

Assuming standard operating conditions where not all progression and attrition parameters are zero (i.e., $d_2 > 0$ or both $\kappa > 0$ and $d_3 > 0$), we guarantee that $c > 0$. Consequently, both eigenvalues λ_2 and λ_3 will have negative real parts. Because all three eigenvalues of the Jacobian matrix have negative real parts, the system is globally asymptotically stable, indicating that student flow will reliably converge to the calculated steady-state equilibria regardless of initial perturbations.

4. Numerical Method and Simulation

To explore the behavior of the proposed model across a range of pedagogical scenarios, we perform numerical simulations of the governing system of ordinary differential equations. Because the model evolves over multiple interacting time-dependent compartments, reliable simulation requires an accurate and stable time-integration method that can resolve both transient and steady-state dynamics. In this section, we describe the numerical approach used to approximate the system, outline the adaptive hybrid block method for time integration, and present simulation results and performance indicators illustrating the system's qualitative and quantitative behavior across different parameter regimes.

4.1. Numerical Solution of the Model

The system of ordinary differential equations describing the temporal dynamics of the state variables is given by

$$P'(t) = r - (\gamma + \phi + d_1)P, \quad E'(t) = \gamma P - (\kappa + d_2)E + (1 - d_3)A, \quad A'(t) = \kappa E - A, \quad (26)$$

supplemented with initial conditions $P(0) = P_0$, $E(0) = 0$, and $A(0) = 0$. This system is autonomous, linear, and first-order in time, and may be compactly written as

$$x'(t) = g(t, x(t)), \quad x(t) = [P(t), E(t), A(t)]^T. \quad (27)$$

where $\mathbf{g} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represents the right-hand side vector field.

To approximate the solution of this system of initial-value problems on a finite time interval $[0, T]$, we use the adaptive hybrid block method. The method belongs to the class of collocation-based block methods, which compute multiple solution points simultaneously in each integration block and dynamically adapt the step size based on continuous error monitoring.

4.1.1. Hybrid Block Discretization of the Model

The hybrid block method begins by partitioning the interval $[0, T]$ into a sequence of non-overlapping subintervals

$$0 = t_0 < t_1 < \dots < t_N = T, \quad (28)$$

where each block has a length of $h_n = t_{n+1} - t_n$. Within each block, $[t_n, t_{n+1}]$, the method evaluates and updates the solution not only at the endpoints, but at a set of intrastep collocation points

$$t_{n+p_i} = t_n + h_n p_i, \quad i = 0, 1, \dots, M. \quad (29)$$

For this implementation, we define p_i to be the set of equispaced points between 0 and 1, such that

$$p_i = \frac{i}{M}, \quad 0 = p_0 < p_1 < \dots < p_M = 1. \quad (30)$$

This block-wise formulation enables the approximation of the solution across the entire subinterval in a single unified computational step. Within a block, the right-hand side function $g(t, x)$ is approximated using Lagrange polynomial interpolation in the normalized coordinate $\tau \in [0, 1]$. Specifically,

$$x'(t_n + h_n \tau) = \sum_{j=0}^M g(t_{n+p_j}, x_{n+p_j}) \ell_j(\tau), \quad (31)$$

where $\ell_j(\tau)$ denotes the Lagrange basis polynomial associated with the node p_j . Integrating the approximation (31) from $\tau = 0$ to $\tau = p_i$ yields

$$x_{n+p_i} = x_n + h_n \sum_{j=0}^M \mathcal{A}_{i,j} g(t_{n+p_j}, x_{n+p_j}), \quad i = 1, 2, \dots, M, \quad (32)$$

where

$$\mathcal{A}_{i,j} = \int_0^{p_i} \ell_j(\tau) d\tau. \quad (33)$$

Equation (32) defines a system of algebraic equations for the unknown intrablock approximations

$$x_{n+p_1}, x_{n+p_2}, \dots, x_{n+p_M}. \quad (34)$$

For general non-linear initial value problems, the block algebraic system (32) is non-linear and would typically be solved using iterative linearization techniques (e.g., Newton–Raphson). However, for the proposed model, the right-hand side function

$$g(t, x) = \begin{bmatrix} r - (\gamma + \phi + d_1)P \\ \gamma P - (k + d_2)E + (1 - d_3)A \\ kE - A \end{bmatrix}, \quad (35)$$

is strictly linear in the state variables. Therefore, the block Equation (32) forms a linear algebraic system, which can be solved without iteration.

4.1.2. Adaptive Step Size Control

The fixed-step-size hybrid block method described above is embedded within an adaptive scheme. The main idea is to estimate the local truncation error in each block and use it to adjust the next step size, thereby preserving accuracy while minimizing overall computational cost. Under the smoothness assumptions of the true solution $x(t)$, the hybrid block method described above has a local truncation error in a block of order $\mathcal{L}_i(x_n; h_n) = \mathcal{O}(h_n^{M+2})$. Consequently, the method possesses a global order of $M + 1$ in the usual sense (see [34,35] and other related studies).

The embedded lower-order scheme obtained by removing the final collocation node has a local order $\mathcal{O}(h_n^{M+1})$, which provides an $\mathcal{O}(h_n^{M+2})$ -accurate difference to estimate the error of the full scheme. The adaptive procedure within each block proceeds as follows:

1. Full block solution: The method first computes the intrablock solution x_{n+p_i} using the full $(M + 1)$ -point scheme (32).
2. Lower-order embedded solution: A second solution $\tilde{x}_{n+p_i}$ is computed using a reduced M -point formula

$$\tilde{x}_{n+p_i} = x_n + h_n \sum_{j=0}^{M-1} \tilde{A}_{i,j} g(t_{n+p_j}, x_{n+p_j}), \quad (36)$$

where $\tilde{A}_{i,j}$ are the corresponding Lagrange-integral coefficients for the reduced stencil. This embedded solution has a truncation error of order $\mathcal{O}(h_n^{M+1})$, which is one order lower than that of the full block method.

3. Error estimation: The difference between the two approximations yields the vector-valued error estimator

$$\varepsilon = \|x_{n+p_i} - \tilde{x}_{n+p_i}\|. \quad (37)$$

This error is evaluated at the collocation nodes within each block.

4. Step acceptance/rejection: If $\varepsilon < tol$, the computed block solution is accepted and the solver proceeds. If the error exceeds the tolerance, the block is rejected and recomputed using a smaller step size. For rejected block solutions, the new step size is selected according to the controller

$$h_{n,\text{new}} = e \cdot h_{n,\text{old}} \left(\frac{tol}{\varepsilon} \right)^{1/(M+1)}, \quad (38)$$

where $0 < e < 1$ is a safety factor that ensures damping of the step size.

5. Advancement: Once a block solution is affected, the endpoint value $x_{n+1} = x_{n+p_M}$ is stored, and the solver advances to the next block.

The method was implemented using MATLAB, with the error tolerance set to $tol = 10^{-15}$, initial step size $h_0 = 0.01$ and $M = 4$. This numerical approach provides accurate approximations to the solution trajectories $P(t)$, $E(t)$, and $A(t)$ while efficiently handling the system's linearity and maintaining computational efficiency across the scenarios presented in Table 2.

Table 2. Parameter values for all simulated scenarios.

Scenario	r	γ	ϕ	d_1	κ	d_2	d_3
High inflow (HI)	20	0.65	0.06	0.05	0.10	0.012	0.92
Low pass rate (LR)	15	0.65	0.06	0.05	0.10	0.012	0.40
High attrition (HA)	12	0.65	0.06	0.10	0.10	0.080	0.75
Slow progression (SL)	8	0.65	0.06	0.05	0.03	0.012	0.92
Low placement (LP)	5	0.40	0.20	0.10	0.10	0.012	0.80

4.2. Simulated Institutional Scenarios and Performance Metrics

To examine the model's qualitative and quantitative behavior under different structural conditions, we evaluate the state variables and associated performance metrics across a set of representative hypothetical scenarios. These scenarios capture distinct administrative and institutional settings, including variation in recruitment intensity, placement effectiveness, institutional progression, systemic attrition, and examination outcomes. Each scenario is defined by a specific parameter set that determines inflow, throughput, attrition, and the dynamics of demonstrated competency. The parameter configurations in Table 2 are informed by existing studies [11,13,14,36] and are used in the illustrative simulations. The parameters (r , γ , ϕ , d_1 , κ , d_2 , d_3) are as defined in Table 1.

In addition to the primary state variables $P(t)$, $E(t)$, and $A(t)$, we examine derived quantities (presented below) that provide insight into administrative efficiency, systemic throughput, and overall student flow.

- **Assessment Flow Rate:** The rate of change of students in the assessment stage is given by

$$\frac{dA}{dt} = \kappa E(t) - A(t). \quad (39)$$

The term $\kappa E(t)$ represents the inflow of students reaching the assessment stage, where a higher value of κ accelerates the transition from active engagement to formal evaluation. The term $-A(t)$ denotes the outflow of students from the assessment stage, including those who successfully demonstrate competency and those who require remediation after failed attempts.

- **Placement Attrition Rate:** The instantaneous institutional loss rate during the placement phase, combining both placement failure (ϕ) and other non-enrollment losses (d_1), is defined as

$$\text{Placement attrition}(t) = (\phi + d_1)P(t). \quad (40)$$

This metric highlights inefficiencies and barriers at the entry point of the educational pipeline.

- **Course Attrition Rate:** The instantaneous loss rate during the active enrollment phase is defined as

$$\text{Course attrition}(t) = d_2 E(t). \quad (41)$$

This metric measures student disengagement, withdrawal, or failure to progress while enrolled. The cumulative attrition up to time t , which captures the total structural loss of students across both placement and instruction phases, is given by

$$\text{Cumulative attrition}(t) = \int_0^t [(\phi + d_1)P(s) + d_2 E(s)] ds. \quad (42)$$

- **Assessment Outcomes:** Upon completing the active enrollment phase, not all students who complete instruction pass the final institutional assessment. The dynamics yield instantaneous failure and pass rates, $(1 - d_3)A(t)$ and $d_3 A(t)$, respectively. The corresponding cumulative metrics are as follows:

$$\text{Cumulative failures}(t) = \int_0^t (1 - d_3)A(s) ds, \quad (43)$$

$$\text{Cumulative passes}(t) = \int_0^t d_3 A(s) ds. \quad (44)$$

These metrics quantify the overall administrative throughput at the conclusion of the course pathway.

- **Performance Indicators:** We define two macroscopic performance indicators:
 - The completion ratio, $\frac{A(t)}{E(t)}$, measures pathway productivity. A high value indicates that a large proportion of enrolled students successfully advance to the formal assessment stage.
 - The efficiency index, $\frac{A(t)}{P(t) + E(t)}$, evaluates the fraction of total active students in the system who have reached the point of evaluation. Higher values signify

a more effective administrative conversion of student enrollment into the final assessment phase.

5. Results

This section presents the results for five simulated institutional scenarios: high inflow (HI), low pass rate (LR), high attrition (HA), slow progression (SL), and low placement (LP). These scenarios represent cases designed to understand how specific factors influence aggregate student flow through the modeled pathway. Discussion of the results in relation to educational theory and prior research is provided in Section 6.

Figure 2 shows the temporal evolution of the three compartments toward their respective steady-state equilibria. As shown in Figure 2a, the placement compartment P reaches equilibrium rapidly across all scenarios. As expected, the high inflow (HI) scenario yields the highest P (population), driven by the elevated recruitment rate $r = 20$. Conversely, the low placement (LP) scenario yields the lowest P levels; despite a standard inflow, this is due to the high placement failure rate ($\phi = 0.20$), which results in substantial pre-enrollment losses by removing students from the system before they can transition to active enrollment. Figure 2b shows the dynamics of the enrollment compartment in the low pass rate (LR) scenario. Despite standard recruitment, E experiences substantial growth, second only to the HI and SL scenarios. This is a direct numerical consequence of the remediation term $(1 - d_3)A$, which re-routes students from the assessment stage back into active enrollment. Figure 2c shows that the SL scenario is the most significant barrier to demonstrating competency through successful completion of the assessment stage. Even with adequate placement and low attrition, a low learning rate ($\kappa = 0.03$) severely limits the transition from enrollment (E) to assessment (A). This shows the model's sensitivity to the rate of successful completion: when the transition rate κ is low, the system produces few students who reach the assessment stage, regardless of the enrollment level.

Figure 3 decomposes the system's throughput and losses across the simulated scenarios. In Figure 3a, the high inflow (HI) scenario yields the highest absolute rate of competency production. This is mathematically driven by the high number of active enrollments (E) rather than an increased per-capita transition rate, demonstrating how scaling the recruitment input (r) inflates absolute throughput independently of pathway efficiency. In contrast, the slow progression (SL) scenario results in flat and minimal competency production over time. Under this parameter configuration, expanding the size of the active student pool is insufficient to drive throughput because the low competency acquisition rate ($\kappa = 0.03$) acts as a severe systemic bottleneck, limiting the transition from active enrollment (E) to assessment (A).

Figure 3c displays the instantaneous course attrition rate, defined by the outflow $d_2E(t)$, which peaks rapidly under the high attrition (HA) scenario. This structural drain contrasts sharply with the behavior observed in the low-pass-rate (LR) scenario: while the LR configuration retains students within the active pipeline by continuously recycling them through the closed-loop remediation pathway $(1 - d_3)A$, the HA scenario permanently removes students from the system via direct attrition.

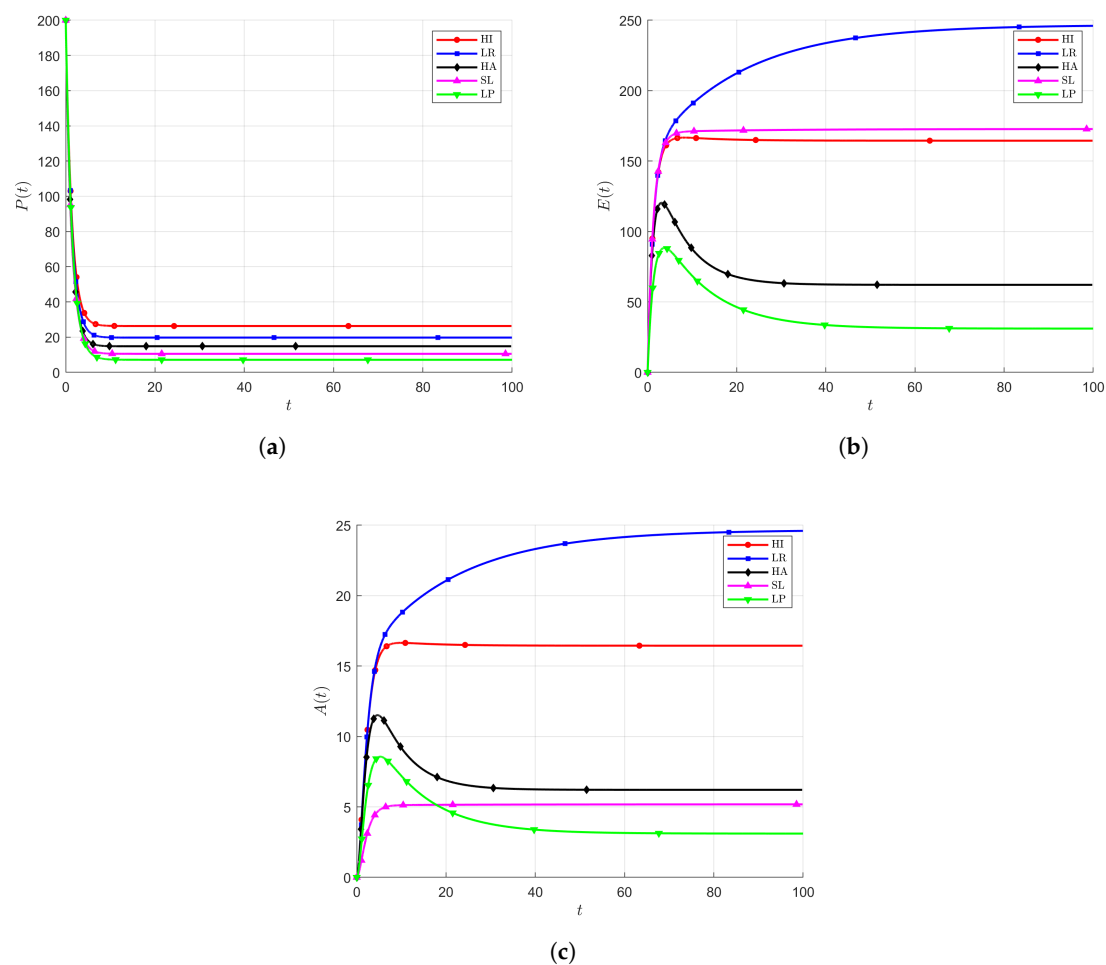


Figure 2. Temporal dynamics of the placement $P(t)$, enrollment $E(t)$, and assessment $A(t)$ compartments under the different parameter scenarios. Each subplot shows how the corresponding state variable evolves as the system approaches its scenario-dependent equilibrium. (a) Placement $P(t)$; (b) Enrollment $E(t)$; (c) Assessment $A(t)$.

Figure 3d presents the cumulative attrition across the entire academic pathway over time. The HA scenario exhibits the steepest initial trajectory among all simulated cases, indicating that under high-attrition conditions, the bulk of systemic losses is concentrated during the early phase of the enrollment pathway.

Figure 4 contrasts the cumulative passes with the cumulative failures generated across the simulated scenarios. As shown in Figure 4a, the cumulative number of passes exhibits almost linear growth for most scenarios. This linear trajectory indicates a relatively stable, continuous rate at which successful students transition out of compartment A and exit the system. On the other hand, Figure 4b reveals a highly divergent pattern in the low-pass-rate (LR) scenario, which displays an accelerated, near-exponential accumulation of cumulative failures. This rapid growth is mathematically consistent with the model's closed-loop remediation cycle. Because failing students are continuously re-routed from A back to E via the $(1 - d_3)A$ feedback term, the same circulating cohort of students is repeatedly reprocessed through the enrollment-to-assessment pathway, causing multiple failure events to accumulate rapidly.

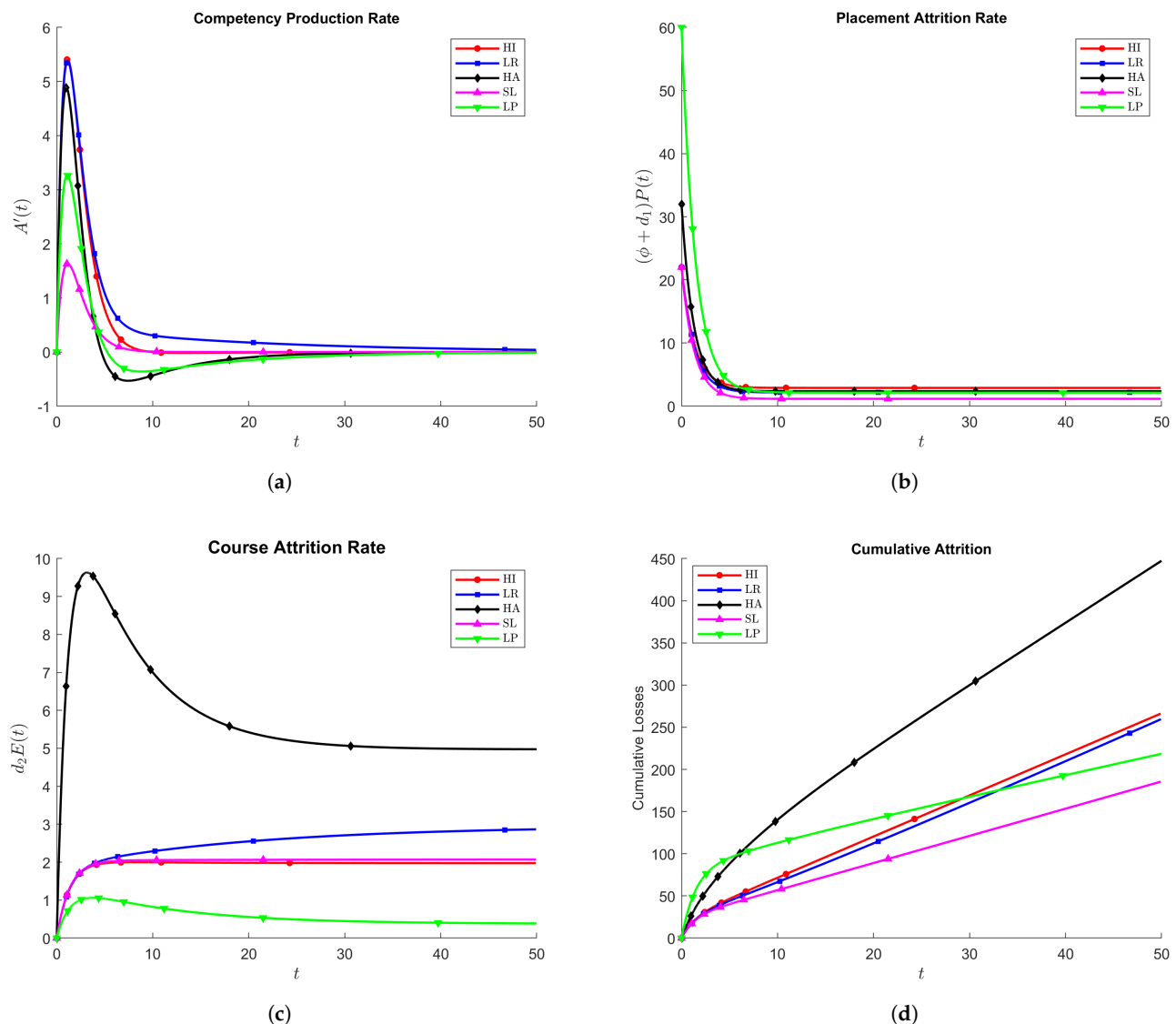


Figure 3. Performance and attrition indicators across scenarios. (a) Competency production rate $A'(t)$; (b) Placement attrition $(\phi + d_1)P(t)$; (c) Course attrition $d_2E(t)$; (d) Cumulative attrition over time.

Figure 5 normalizes the simulated output to allow comparison across different scenarios independent of initial cohort sizes. In Figure 5a, the slow progression (SL) scenario emerges as a distinct outlier, with the completion ratio $A(t)/E(t)$ stabilizing at a low level compared to all other scenarios. A low value of $A(t)/E(t)$ indicates that, at any given time, only a small fraction of actively enrolled students have transitioned into the assessment stage. This completion ratio serves as a descriptive measure of systemic throughput; while a low value indicates a slow conversion of enrolled students to assessed status, it does not, by itself, isolate the underlying pedagogical causes or directly measure instructional quality. A theoretical contextualization of this metric is detailed in Section 6.

Figure 5b reports the efficiency index, $A(t)/(P(t) + E(t))$, which measures how effectively the system converts the active student population (both awaiting placement and currently enrolled) into successfully assessed. The SL scenario yields the lowest efficiency values, followed by the HA scenario. In the SL scenario, students remain in the E stage for much longer periods, generating a large denominator $(P + E)$ while contributing very little to $A(t)$; creating a congested pipeline where students accumulate without progress-

ing. In the HA scenario, the low efficiency is driven by a different mechanism: the large structural outflows through d_1 and d_2 permanently remove students before they can reach the assessment compartment A , thereby reducing the throughput relative to the active population. These two scenarios reduce overall efficiency through different mechanisms, low progression (small κ) versus high systemic attrition (large d_1, d_2); so, the comparative severity depends on the specific administrative criteria adopted and is further analyzed in Section 6.

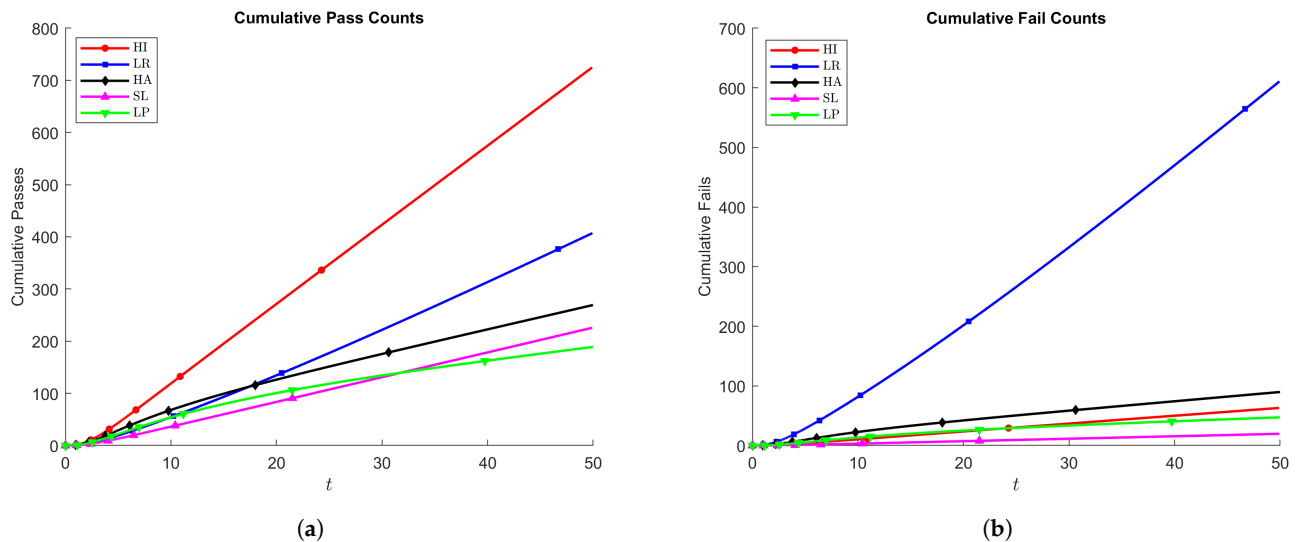


Figure 4. Cumulative completion and failure outcomes. (a) Cumulative exam passes; (b) cumulative exam failures.

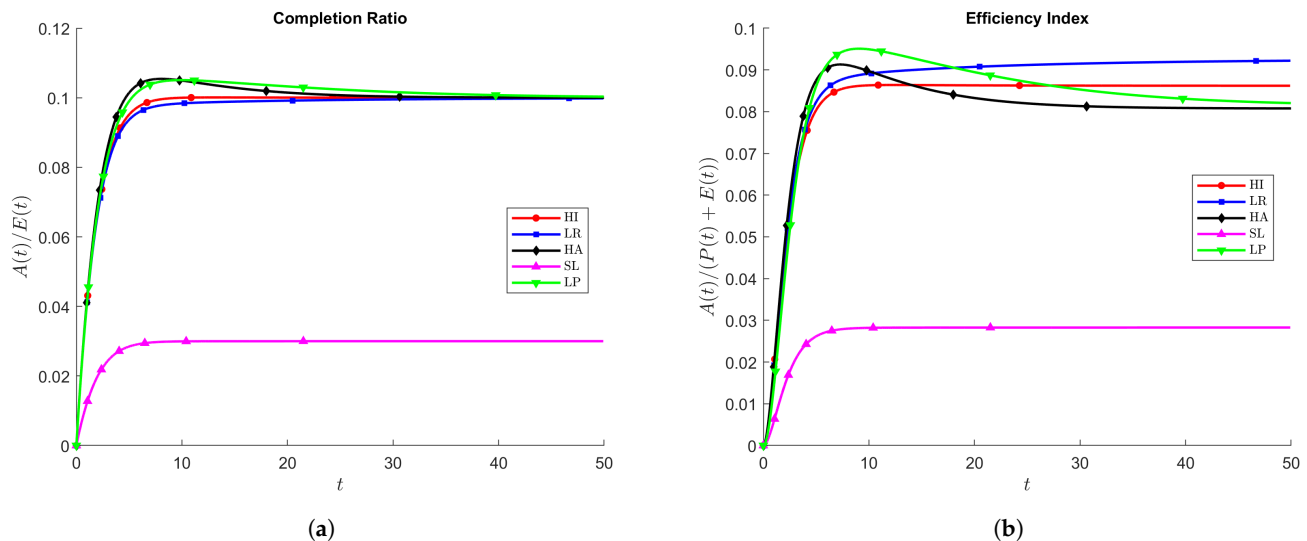


Figure 5. Completion ratio and efficiency index for each scenario. (a) Completion ratio; (b) efficiency index.

The LP scenario is comparatively competitive on the efficiency index, even exceeding some scenarios with higher recruitment (SL and HA). This occurs because the small cohort of students transitioning past the placement stage progresses through the remaining stages with relatively high success; that is, the system is efficient conditional on entry. However, this metric-based efficiency here is distinct from total systemic output. Despite its high efficiency index, the LP scenario yields a very low total competency output because $P(t)$ is heavily suppressed by the strict placement failure constraint ($\phi = 0.20$), which severely limits the active enrollment pool (E) and keeps the numerator $A(t)$ small. Meanwhile, the

HI and LR scenarios settle at moderately high efficiency for different reasons: a large inflow r maintains a large pool of actively enrolled students, and provided that within-course attrition is low, this sustains a steady $E \rightarrow A$ flow even when the competency completion rate d_3 is low, so inefficiencies at the final assessment stage do not strongly impede overall movement through $P \rightarrow E \rightarrow A$.

Therefore, Figure 5b reveals three key structural patterns in the simulated student flow. First, systemic throughput is governed primarily by how quickly students progress through the enrollment stage: the competency acquisition rate κ and the structural attrition rates d_1 and d_2 are the dominant drivers of the efficiency index. Second, restrictive placement policies (LP) can artificially inflate the efficiency index while lowering total competency output. Third, high recruitment volume (HI) combined with stable, unobstructed progression yields the highest absolute output among all illustrated scenarios.

Finally, these results show that the three-compartment model successfully distinguishes between structurally distinct low-output regimes. The slow-progression regime (SL), driven by a small κ , limits the $E \rightarrow A$ transition, whereas the low-pass regime (LR), driven by a small d_3 , restricts exit from the assessment stage and instead accumulates students in E through the remediation loop. The remediation term $(1 - d_3)A$ is essential to this structural distinction: without it, students who fail assessment would be misclassified as permanent attrition, and the significant cyclical accumulation visible in the active enrollment E compartment (Figure 2b) would remain completely hidden.

6. Discussion

In this section, we interpret the observable patterns generated by the model via the following lenses: what the simulations show, how prior research may help explain those patterns, what alternative interpretations remain plausible, and what empirical evidence would be needed to evaluate those interpretations. Because the model was not calibrated or validated against observed student data, the discussion is intended to generate hypotheses rather than to make confirmatory claims.

6.1. Positioning of the Compartmental Model

The three compartments represent institutional positions along a placement-based pathway (placement, engagement, assessment), not stages of conceptual mastery. Where the model touches learning theory, for example, through the remediation term or the progression rate κ , it does so by analogy, and the cognitive constructs (consolidation, transfer, cognitive load) provide interpretations of simulated student flow from placement to assessment. This positioning clarifies what the model can and cannot contribute: it can characterize how structural parameters reshape aggregate flow, but it cannot adjudicate the cognitive or social processes that underlie individual student behavior.

6.2. Remediation Cycles: Instrumental vs. Relational Understanding

The simulation shows that the low-pass rate (LR) scenario produces near-exponential growth in cumulative failures (Figure 4b) alongside accumulation in the enrollment compartment (Figure 2b), driven by the remediation term $(1 - d_3)A$ that routes students who do not pass back into instruction.

Educational research offers several interpretations of this pattern. One, drawn from the learning sciences, is that repeated assessment failure can reflect knowledge that remains at the level of instrumental rather than relational understanding, so that students succeed on isolated tasks but fail on integrative or transfer-oriented assessment; unstable or fragmentary knowledge structures are then susceptible to recurrent breakdown under high-stakes evaluation [7,37]. Consequently, when evaluated at the assessment

stage A , these students experience cognitive overload and fail, triggering the remediation cycle. The repeated cycling of these students back to E reflects a pedagogical reliance on teaching to the test or simple repetition, which fails to resolve the underlying conceptual gaps [38]. This is consistent with mastery-learning accounts in which advancement should follow attained competency, and with dynamic-systems views of learning as recursive re-engagement [39,40].

An equally plausible alternative explanation is that the high rate of failure accumulation is not indicative of a cognitive issue on the part of the students. Instead, it could be driven by poor alignment between course instruction and the final assessment, high test anxiety under stringent evaluation conditions, a sudden, unexplained shift in grading standards at the end of term, or external constraints (employment, caring responsibilities, health) that students face and that limit preparation between attempts. The model cannot distinguish between these because d_3 is a single lumped parameter. Testing the cognitive interpretation specifically would require student-level data linking assessment attempts to measures of conceptual versus procedural understanding and to instructional alignment—evidence that the present study lacks. Evaluating these alternative hypotheses requires detailed learning analytics and assessment data; specifically, researchers would need to analyze item-response theory (IRT) data from both formative and summative assessments to assess diagnostic alignment, alongside metrics that track prior assessment attempts, test anxiety, and student confidence.

6.3. Slow Progression and the Completion Ratio

The simulation shows that the slow learning (SL) scenario yields a persistently low completion ratio $A(t)/E(t)$ (Figure 5a) and the lowest efficiency index (Figure 5b), with students accumulating in E under a small κ .

From an educational research perspective, the slow institutional progression in the SL scenario can be a factor in the failure of academic momentum. According to retention and involvement theories, such as those by Tinto [10] and Astin [33], when students spend extended periods in low-competency states without visible progress, their academic momentum stalls. This lack of momentum increases cognitive fatigue and the likelihood of eventual disengagement, effectively turning slow progress into a form of delayed attrition. Some learners may require more repetition or processing time before their understanding stabilizes, so that time on task does not translate into demonstrated competency at the expected rate [38]. In this case, a low completion ratio would be consistent with insufficiently scaffolded instruction or with pacing that exceeds consolidation.

An alternative explanation for these simulated patterns is that slow progression does not necessarily represent instructional weakness or cognitive stalling. Instead, it may reflect a highly rigorous curriculum with strict grading standards, in which students ought to be given more time to consolidate complex material. Indeed, a low completion ratio is merely a descriptive measure of throughput, not a direct indicator of instructional quality. The same ratio could reflect heterogeneous student preparation, measurement conventions around what counts as reaching the assessment stage, or demanding assessments. Ultimately, interpreting κ as a property of instruction, rather than of the student population or the instruction-population interaction, is an assumption of the interpretation, not a result of the model; distinguishing these factors would require individual-level data on instructional design, student preparation, and time-to-competency.

6.4. Early-Stage Attrition and the Limits of Causal Attribution

Although high attrition (HA) reduces the overall number of students who successfully exit the pathway as academically successful, the simulation results show that it does so

through a structurally different route than in the low-pass-rate scenario. Whereas low pass rates retain students within the system by returning them to engagement for further instruction, high attrition removes them permanently. These losses cluster early in the pathway (Figure 3d), leading to permanent student removal rather than recycling. It is tempting to interpret early attrition of this kind as a failure of foundational instructional support, such as inadequate scaffolding from intuitive to formal reasoning. This is a plausible interpretation, but it is only one of several, and we caution against treating it as evidence of instructional deficiency given the nature of our theoretical study. In real higher-education settings, student withdrawal is shaped by a wide range of factors that the present model does not account for, such as prior academic preparation, financial circumstances, employment, health and well-being, language barrier, special education status, lack of sense of institutional belonging, attendance, motivation, etc. [10,41]. The parameters d_1 and d_2 aggregate all such influences into single rates. Consequently, the model can show that early attrition is costly to overall flow, but it cannot identify why students leave, and no causal attribution to instruction should be drawn from it. Establishing cause would require longitudinal, student-level data with demographic, financial, and engagement covariates over the course of the academic term.

6.5. Accounting for Socioeconomic Heterogeneity and Equity

A major structural limitation of the model bears directly on how its results should be interpreted. Within each compartment, students are treated as homogeneous, whereas students entering placement-based courses differ substantially in preparation, socioeconomic background, language, disability, prior educational opportunity, motivation, and access to institutional resources. Because the model averages over this heterogeneity, its aggregate ratios can obscure unequal effects on different groups.

In real higher education systems, student populations are highly heterogeneous. Attrition and progression rates are influenced by socioeconomic status, financial aid eligibility, first-generation student status, employment hours, grades, etc. [14,41]. A simulated drop in the competency acquisition rate (κ) might be interpreted as an instructional weakness, but in an empirical setting, it could reflect a cohort of students who work part-time jobs and thus have less time on task for study.

Alternatively, differences in completion rates might be attributed to varying student motivation or academic preparation, when they are actually driven by unequal access to institutional resources, such as peer tutoring, advising, or stable internet and study spaces.

Empirically validating these dynamics requires integrating academic registry data with student demographic and socioeconomic datasets. By disaggregating the transition rates (γ, κ) and attrition rates (d_i) using survival analysis across different demographic cohorts, we could move beyond the homogeneity assumption and identify how systemic inequities uniquely shape the flow of diverse student populations through the educational pipeline.

6.6. The Selective Entry Paradox: Efficiency vs. Access

The simulation of the low placement (LP) scenario reveals a structural paradox: the system achieves a highly competitive efficiency index (Figure 5b) despite producing a very low total number of competent graduates. This occurs because the severe placement barrier ($\phi = 0.20$) filters out the vast majority of students, allowing only a highly prepared, resilient cohort to enter the enrollment phase.

If interpreted solely through efficiency, this looks like a well-functioning system; and if interpreted through access and institutional mission, the same result describes a system that appears efficient, precisely because it excludes most potential students at entry. In the context of higher education policy, this pattern is often interpreted as the

“creaming effect” of selective admissions. Research in developmental education shows that highly selective programs often boast excellent completion rates not because their instruction is superior, but because they admit only students who were already highly likely to succeed [11,13]. This is a critical distinction between administrative efficiency and equity, as highlighted in the placement literature [12,13]: restrictive placement policies artificially mask systemic weaknesses by excluding students who require instructional scaffolding. Efficiency conditional on entry is therefore not the same as educational productivity or equity, and the LP result should not be interpreted as endorsing restrictive placement. Whether restrictive placement is desirable depends on whether access, fairness, and support are weighed alongside throughput.

An alternative explanation is that the high efficiency of the LP cohort is indeed a result of highly effective, targeted instructional design. It is possible that the low number of admitted students allows for smaller class sizes and more personalized, high-density tutoring, which drastically accelerates the rate of competency acquisition (κ). To resolve this paradox and test these explanations, one would need to implement a regression discontinuity design (RDD). By comparing the academic trajectories of students who scored just above the placement test cutoff with those who scored just below (and were routed to remedial or alternative pathways), we can determine whether the placement test is a valid diagnostic tool or merely an equity-reducing barrier to access.

6.7. On “Mastery” and What the Assessment Compartment Represents

We also caution against interpreting the assessment compartment A as representing mastery in the strong sense of durable, transferable understanding. In the model, entry to A , and passing at assessment represent demonstrated performance on an evaluation, which is evidence of competence but not equivalent to it; a student may pass without retaining or transferring knowledge, while another may understand the material yet perform poorly under assessment conditions [39]. Accordingly, we describe A as an assessment stage and its successful exits as demonstrated competency rather than mastery, and we treat the pass rate d_3 as a property of the assessment process as configured, not as a direct index of learning.

6.8. Comparison with Previous Research

We observe that some of the model’s behavior is consistent with patterns reported in the empirical higher-education literature, though that literature rests on observed data that the present study does not use. Research on developmental and corequisite pathways documents substantial losses at transition points [11]. This is consistent with the model’s suggestion that early-stage parameters have a disproportionate influence on throughput, and that attrition cannot be attributed to a single cause [41]. Similarly, retention theory posits that attrition is a function of both academic and social integration rather than instruction alone [10]. The contribution of our study is therefore hypothetical rather than empirical: it shows how progression κ , attrition d_1, d_2 , pass rate d_3 , and placement ϕ can combine to produce distinct bottlenecks—slow progression, assessment barriers, and early withdrawal—that produce similar completion figures despite requiring different institutional responses.

6.9. Distinguishing Failure Modes: The Value and the Caveats

A useful feature of the proposed model is that it distinguishes structurally distinct, low-output regimes that yield similar end states. The slow-progression regime (SL, small κ) limits the $E \rightarrow A$ transition; the low-pass regime (LR, small d_3) limits exit from assessment and accumulates students in E through remediation; and the high-attrition regime (HA) instead removes them entirely. The remediation term $(1 - d_3)A$ is essential to this

separation: without it, students who fail would be miscounted as permanent attrition, and the resource burden of repeated re-instruction would be invisible. We stress, however, that this separation is a property of the model's structure, not a validated diagnostic. That the model can distinguish instructional-rate limitation from assessment-barrier limitation does not establish that either mechanism operates in any real course; it shows only that, if such mechanisms were present, they would leave distinguishable signatures in aggregate flow.

6.10. *Slow Progression Versus Attrition: A Criterion-Dependent Comparison*

Slow progression and attrition act through different mechanisms and matter against different criteria. High attrition is most damaging to access, equity, student well-being, and opportunity cost, as it removes students from the system entirely. Slow progression is most detrimental to time-to-completion and institutional capacity, and it delays achievement by congesting the pipeline without necessarily excluding anyone. Repeated remediation also burdens the workload and erodes student confidence, while restrictive placement affects opportunity.

7. Conclusions

This theoretical and simulation-based study demonstrates how compartmental mathematical models, adapted from epidemiological modeling frameworks, can capture the dynamics of institutional student progression in placement-determined courses. Through theoretical analysis and numerical simulations, we examined students' transitions from initial placement through active course engagement to demonstrated competency. By modeling student flow as a continuous, dynamic process rather than as discrete, categorical transitions, we address a significant limitation in existing student progression frameworks.

The numerical analysis of the five scenarios commonly arising in placement-based courses—High Inflow (HI), Low Placement (LP), High Attrition (HA), Slow Progression (SL), and Low Pass Rate (LR)—reveals the following structural insights:

- The competency acquisition rate drives throughput. The Slow Progression (SL) scenario demonstrates that low values of the competency development parameter κ severely restrict the transition from active enrollment (E) to assessment (A), preventing the effective conversion of enrolled students into successful completers regardless of the enrollment volume.
- Assessment structure drives systemic recycling. The Low Pass Rate (LR) scenario demonstrates that restrictive assessment policies (low d_3) generate persistent remediation cycles. Here, students repeatedly return to active instruction without achieving competency, resulting in near-exponential growth in cumulative failures and compounding the institution's resource demands.
- Scale does not equate to pathway efficiency. The High Inflow (HI) scenario produces the largest number of successful completers. However, this is driven strictly by the sheer volume of the enrolled population rather than superior instructional efficacy, highlighting the critical distinction between raw quantitative output and overall system efficiency.
- Early-stage attrition is highly costly. The High Attrition (HA) scenario exhibits concentrated student attrition during the initial placement and early enrollment phases, indicating that inadequate foundational support leads to permanent departure before students can successfully transition to the assessment stage.
- Selective entry artificially inflates efficiency metrics. The Low Placement (LP) scenario demonstrates that restrictive placement policies can artificially improve completion

ratios by limiting access. This gatekeeping masks underlying structural vulnerabilities in the instructional pathway while drastically reducing overall educational reach.

- The remediation term $(1 - d_3)A$ is essential for the model fidelity. Without this feedback mechanism, the model would misclassify non-passing students as permanent attrition. This would obscure the substantial resource allocation required for students who need additional instruction and fail to capture the cyclical nature of institutional student progression.

8. Limitation and Future Work

This study is based on simulated data and is considered exploratory. Its results are therefore best read as hypotheses for institutions to examine empirically, that interventions targeting the competency acquisition rate and early-stage attrition may influence throughput more than interventions targeting the final assessment pass rate, and that efficiency metrics can be misleading when evaluated in isolation from access and total output. These are theoretical hypotheses, not policy recommendations, and each must be validated against institutional data. We acknowledge micro-level pedagogical processes that are not captured in this current formulation of this compartmental model:

- Intermediate cognitive states and misconceptions. The transition from E to A is modeled as a direct flow, abstracting away the reality that students often develop partial understanding or hold persistent misconceptions requiring unlearning and remediation.
- Varying levels of engagement. The active engagement compartment (E) assumes a homogeneous baseline of participation, omitting granular variation in student motivation, attendance, and psychosocial factors.
- Assessment nuances. Exit from the assessment stage (A) is treated as binary (pass or fail), bypassing the variation introduced by different assessment formats, test anxiety, and grading rubrics.

Although this study analyzes the compartmental model using illustrative, literature-based parameters, future work should empirically calibrate and validate the model using longitudinal student-level data. Such data will contain placement scores, enrollment and withdrawal dates, attendance or engagement indicators, assessment attempts and outcomes, course repetition counts, final course grades, and relevant demographic and contextual variables, ideally organized as longitudinal student trajectories. Such data would allow transition and attrition rates to be estimated directly, relax the within-stage homogeneity assumption by disaggregating rates across demographic cohorts, and enable simulated flow patterns to be compared with observed movement through placement, engagement, and assessment. In addition, it would be possible to confirm, refine, or reject the hypotheses presented in this theoretical and simulation-based study. Furthermore, this compartmental model framework is intended to serve as a baseline for other extensions, which may include incorporating varying levels of engagement, demographic covariates, multiple assessment attempts, stop-out and re-entry pathways, etc.

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