logics

MDPI

Article

Carnap's Problem for Intuitionistic Propositional Logic

Haotian Tong [1,†] and Dag Westerståhl [2,3,*]

[1] Department of Philosophy, Tsinghua University, Beijing 100084, China
[2] Department of Philosophy, Stockholm University, 106 91 Stockholm, Sweden
[3] Tsinghua-UvA Joint Research Center for Logic, Department of Philosophy, Tsinghua University, Beijing 100084, China
* Correspondence: dag.westerstahl@philosophy.su.se
† Current address: Duke-NUS Medical School, Singapore 169857, Singapore.

Abstract: We show that intuitionistic propositional logic is *Carnap categorical*: the only interpretation of the connectives consistent with the intuitionistic consequence relation is the standard interpretation. This holds with respect to the most well-known semantics relative to which intuitionistic logic is sound and complete; among them Kripke semantics, Beth semantics, Dragalin semantics, topological semantics, and algebraic semantics. These facts turn out to be consequences of an observation about interpretations in Heyting algebras.

Keywords: intuitionistic logic; Carnap's problem; Heyting algebras; set-based semantics; nuclear semantics; algebraic semantics; logical constants; consequence relations; categoricity

Citation: Tong, H.; Westerståhl, D. Carnap's Problem for Intuitionistic Propositional Logic. *Logics* **2023**, *1*, 163–181. https://doi.org/10.3390/logics1040009

Academic Editor: Wesley H. Holliday

Received: 26 December 2022
Revised: 18 May 2023
Accepted: 14 August 2023
Published: 22 September 2023

1. Motivation and Background

Carnap [1] was concerned with the existence of 'non-normal' interpretations of the connectives in classical propositional logic (CPC), i.e., interpretations that were different from the usual truth tables but still consistent with $\vdash_{\mathsf{CPC}}$. Though neglected for many decades, the issue has been reopened in recent years. One reason was that non-normal interpretations were seen to clash with inferentialist meaning theories ([2]). It has been countered that inferentialism is not really threatened by non-normal interpretations (e.g., [3]), or that such interpretations are avoided by a proper understanding of inference rules (e.g., [4]).

In this note, we do not discuss the shape or the epistemological status of rules, but we follow the model-theoretic approach to Carnap's problem initiated in [5]. That is, a single-conclusion consequence relation $\vdash$ in a logical language is *given*—no matter in what way—and, relative to a semantics which compositionally assigns *semantic values* (set-theoretic objects of various kinds) to formulas, one investigates to what extent the meaning of the logical constants in that language is *fixed* by consistency with $\vdash$. This is a precise model-theoretic task.

A first observation, which, as pointed out in [5], is hidden already in Carnap's 1943 book, is that *compositionality* already rules out non-normal interpretations of the connectives in CPC, relative to the usual two-valued semantics.[1] More interestingly, the classical first-order consequence $\vdash_{\mathsf{FOL}}$ does *not* by itself fix the meaning of $\forall$, relative to a semantics that interprets quantifier symbols as sets of subsets of the domain (unary generalized quantifiers) but constrains it to be a *principal filter*. However, if *permutation invariance* is also required, then the only consistent interpretation is the standard one: $\vdash_{\mathsf{FOL}}$ forces $\forall x\varphi(x)$ to mean 'for all x in the domain, $\varphi(x)$ holds'.

Carnap's question can be asked for any consequence relation in any logical language, relative to any formal semantics for that language. Bonnay and Westerståhl [6] deals with Carnap's problem in modal logic and [7] discusses it for some logics with generalized quantifiers.

Here, we consider Carnap's problem for intuitionistic propositional logic (IPC). Relative to which formal semantics should we ask the question? There is by now a plethora of semantics for which IPC is sound and complete: Kripke semantics, Beth semantics, topological semantics, algebraic semantics, ...; ideally, one would like an answer for each of them. In this note, we show that IPC is indeed *categorical* with respect to most of these semantics.[2]

The next section presents the various semantics to which our results apply. We contrast the familiar *algebraic* semantics with a notion of *set-based* semantics, of which Kripke, Beth, and topological semantics are instances. Our main reference for IPC semantics will be the article [8]—BH19 in what follows—which surveys and compares a great variety of intuitionistic semantics. We largely follow the notation and terminology in BH19 and refer to that article for technical notions not explained here.

Section 3 presents our main result (Theorem 3.6), a fact about algebraic interpretations of the connectives, which turns out to entail categoricity with respect to all set-based semantics (in our sense): only the standard interpretations are consistent with $\vdash_{\mathsf{IPC}}$. In Section 4, we consider Carnap's question for some fragments of IPC. Section 5 concludes with discussion and open issues.

The propositional language will (with a few exceptions at the end) be generated by

$$\varphi \ := \ p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \qquad (p \in Prop)$$

where *Prop* is a denumerable set of propositional variables. So, $\neg\varphi := \varphi \to \bot$. The proof-theoretically defined (for example, by means of a natural deduction system) single-conclusion consequence relation

$$\vdash_{\mathsf{IPC}}$$

in this language is assumed to be familiar, as well as standard intuitionistic Kripke semantics, for which $\vdash_{\mathsf{IPC}}$ is sound and complete. Similarly for classical consequence, $\vdash_{\mathsf{CPC}}$.

2. Algebraic vs. Set-Based Semantics

To set the stage, we first consider Carnap's question in the context of the most well-known semantics for intuitionistic logic: Kripke semantics.

2.1. Carnap's Problem for Kripke Semantics

Start with a partially ordered set (a *poset*) $\mathcal{F} = (X, \leq)$. The semantic values of the formulas will be *upsets*: sets $U \subseteq X$ such that $x \in U$ and $x \leq y$ implies $y \in U$; let $Up(\mathcal{F})$ be the set of upsets. For $U \subseteq X$, let $U{\uparrow} = \{y \in X : \exists x \in U\, x \leq y\} \in Up(\mathcal{F})$. We write $x{\uparrow} = \{x\}{\uparrow}$.

A *valuation* on $\mathcal{F}$ is a function v from *Prop* to $Up(\mathcal{F})$. For each v, semantic values are assigned according to the standard truth definition in Kripke semantics:

(1) a. $[\![p]\!]_v = v(p)$
b. $[\![\bot]\!]_v = \varnothing$
c. $[\![\top]\!]_v = X$
d. $[\![\varphi \wedge \psi]\!]_v = [\![\varphi]\!]_v \cap [\![\psi]\!]_v$
e. $[\![\varphi \vee \psi]\!]_v = [\![\varphi]\!]_v \cup [\![\psi]\!]_v$
f. $[\![\varphi \to \psi]\!]_v = \{x \in X : \forall y \geq x (y \in [\![\varphi]\!]_v \Rightarrow y \in [\![\psi]\!]_v)\}$
$= \{x \in X : x{\uparrow} \cap [\![\varphi]\!]_v \subseteq [\![\psi]\!]_v\}$

One easily verifies that $[\![\varphi]\!]_v \in Up(\mathcal{F})$ for all φ. Next, we have the corresponding semantic consequence relation:

$$\psi_1, \ldots, \psi_n \models \varphi \ \text{ iff for all } \mathcal{F} \text{ and all valuations } v \text{ on } \mathcal{F}, \bigcap_{i=1}^n [\![\psi_i]\!]_v \subseteq [\![\varphi]\!]_v$$

The (soundness and) completeness theorem for intuitionistic propositional logic relative to Kripke semantics then says that

$$\models \ = \ \vdash_{\mathsf{IPC}}$$

It makes sense to say that (1) *interprets* the connectives (over $\mathcal{F}$), according to what we shall call the *standard* (Kripke) *interpretation* $I_{\mathrm{st}}^{\mathcal{F}}$, which is the following function from the *signature* $\{\bot, \top, \wedge, \vee, \to\}$ to corresponding functions on $Up(\mathcal{F})$: for $U, V \in Up(\mathcal{F})$,

(2)
a. $I_{\mathrm{st}}^{\mathcal{F}}(\bot) = \emptyset$
b. $I_{\mathrm{st}}^{\mathcal{F}}(\top) = X$
c. $I_{\mathrm{st}}^{\mathcal{F}}(\wedge)(U, V) = U \cap V$
d. $I_{\mathrm{st}}^{\mathcal{F}}(\vee)(U, V) = U \cup V$
e. $I_{\mathrm{st}}^{\mathcal{F}}(\to)(U, V) = \{x \in X \colon x{\uparrow} \cap U \subseteq V\}$

Just as the standard truth functions can be said to give the *meaning* of the connectives *according to classical two-valued semantics*, $I_{\mathrm{st}}^{\mathcal{F}}$ gives the *meaning* of the connectives (over $\mathcal{F}$) *according to Kripke semantics*. Carnap asked if the classical meaning is determined by the classical consequence relation $\vdash_{\mathsf{CPC}}$. When compositionality is assumed, this is the question of whether there are *other* truth functions than the standard ones, which interpret the connectives in a way that is still *consistent* with $\vdash_{\mathsf{CPC}}$. The answer is no: $\vdash_{\mathsf{CPC}}$ is *categorical* with respect to two-valued semantics. Now, we ask the same about IPC and Kripke semantics. Are there *other* interpretations, in addition to $I_{\mathrm{st}}^{\mathcal{F}}$, which are also consistent with $\vdash_{\mathsf{IPC}}$?

We can make this precise as follows: given a poset $\mathcal{F}$, an *intuitionistic interpretation* of the connectives is a function $I^{\mathcal{F}}$ from $\{\bot, \top, \wedge, \vee, \to\}$ to corresponding functions on $Up(\mathcal{F})$, from which we calculate the semantic values of the formulas as before: $[\![p]\!]_v^{I^{\mathcal{F}}} = v(p)$, $[\![\bot]\!]_v^{I^{\mathcal{F}}}, [\![\top]\!]_v^{I^{\mathcal{F}}} \in Up(\mathcal{F})$, $[\![\varphi \wedge \psi]\!]_v^{I^{\mathcal{F}}} = I^{\mathcal{F}}(\wedge)([\![\varphi]\!]_v^{I^{\mathcal{F}}}, [\![\psi]\!]_v^{I^{\mathcal{F}}})$, etc. $I^{\mathcal{F}}$ is *consistent* with a consequence relation $\vdash$ if

$$\psi_1, \ldots, \psi_n \vdash \varphi \ \text{ implies that for all } v, \textstyle\bigcap_{i=1}^{n} [\![\psi_i]\!]_v^{I^{\mathcal{F}}} \subseteq [\![\varphi]\!]_v^{I^{\mathcal{F}}}$$

So, $I_{\mathrm{st}}^{\mathcal{F}}$ is consistent with $\vdash_{\mathsf{IPC}}$. Do any *other* interpretations of the connectives (over $\mathcal{F}$) also have this property? This is Carnap's question for intuitionistic propositional logic, relative to Kripke semantics.

Note that the *classical* interpretation $I_c^{\mathcal{F}}$, which only differs from $I_{\mathrm{st}}^{\mathcal{F}}$ in that $I_c^{\mathcal{F}}(\to)(U, V) = (X - U) \cup V$, is of course also consistent with $\vdash_{\mathsf{IPC}}$ (indeed with $\vdash_{\mathsf{CPC}}$). However, it is not an intuitionistic interpretation: $I_c^{\mathcal{F}}(\to)$ does not map upsets to upsets. That semantic values are upsets is a crucial ingredient in Kripke semantics.

We shall see that the answer to Carnap's question about Kripke semantics is a resounding no. Indeed, it turns out that categoricity with respect to Kripke semantics is an instance of a much more general result.

2.2. *A Hierarchy of Semantics for* IPC

BH19 organizes a number of well-known semantics for IPC in the following hierarchy:

(3) Kripke $<$ Beth $<$ Topological $<$ Dragalin $<$ Algebraic

We have already looked at Kripke semantics. At the other end, algebraic semantics uses *Heyting algebras*, i.e., algebras of the form

$$\mathcal{A} = (A, \bot^{\mathcal{A}}, \top^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \to^{\mathcal{A}})$$

where the reduct $(A, \bot^{\mathcal{A}}, \top^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}})$ is a bounded lattice, and, in addition, the following equations hold:[3]

(4)
$$a \to a = 1$$
$$a \wedge (a \to b) = a \wedge b$$
$$(a \to b) \wedge b = b$$
$$a \to (b \wedge c) = (a \to b) \wedge (a \to c)$$

The partial order $\leq^{\mathcal{A}}$ is defined as usual:

(5) $\quad a \leq^{\mathcal{A}} b$ iff $a \wedge^{\mathcal{A}} b = a$

An algebra $\mathcal{A}$ of this signature can be taken to *interpret* propositional formulas: for each *valuation* $v\colon Prop \to A$, a *semantic value*

$$[\varphi]^{\mathcal{A}}_v$$

in A is assigned to each formula φ: $[p]^{\mathcal{A}}_v = v(p)$, $[\bot]^{\mathcal{A}}_v = \bot^{\mathcal{A}}$, $[\top]^{\mathcal{A}}_v = \top^{\mathcal{A}}$, $[\varphi \wedge \psi]^{\mathcal{A}}_v = [\varphi]^{\mathcal{A}}_v \wedge^{\mathcal{A}} [\psi]^{\mathcal{A}}_v$, etc.

The consequence relation $\vdash_{\mathsf{IPC}}$ is *consistent* with each Heyting algebra $\mathcal{A}$: if $\psi_1, \ldots, \psi_n \vdash_{\mathsf{IPC}} \varphi$, then for all v, $[\psi_1]^{\mathcal{A}}_v \wedge^{\mathcal{A}} \ldots \wedge^{\mathcal{A}} [\psi_n]^{\mathcal{A}}_v \leq^{\mathcal{A}} [\varphi]^{\mathcal{A}}_v$. That is, consistency is *soundness*, but for consequence, not just validity (this proves to be crucial; see Section 3.2). BH19 is mainly concerned with classes of algebras (or other interpretations) with respect to which $\vdash_{\mathsf{IPC}}$ is sound and *complete*. However, when we are interested in categoricity, completeness plays no role, as we shall see.

A characteristic of algebraic semantics is that no assumptions are made about the domain of the algebras; it can be any non-empty set. The semantics to the left of 'Algebraic' in (3), on the other hand, are all what we shall call *set-based semantics*, which means, among other things, that over a domain X, semantic values of formulas are *subsets* of X. Before giving the details, let us explain what the strict order in (3) means.

Each semantics in (3) 'produces' Heyting algebras. For example, on a poset $\mathcal{F} = (X, \leq)$, Kripke semantics produces the algebra

(6) $$\begin{aligned}\mathbf{Up}(\mathcal{F}) &= (Up(\mathcal{F}), I^{\mathcal{F}}_{\mathrm{st}}(\bot), I^{\mathcal{F}}_{\mathrm{st}}(\top), I^{\mathcal{F}}_{\mathrm{st}}(\wedge), I^{\mathcal{F}}_{\mathrm{st}}(\vee), I^{\mathcal{F}}_{\mathrm{st}}(\to))\\ &= (Up(\mathcal{F}), \emptyset, X, \cap, \cup, I^{\mathcal{F}}_{\mathrm{st}}(\to))\end{aligned}$$

That is, the *standard Kripke interpretation* $I^{\mathcal{F}}_{\mathrm{st}}$ of the connectives on $Up(\mathcal{F})$ yields a Heyting algebra. Similarly, the standard interpretation of the connectives in Beth semantics yields, on each poset $\mathcal{F}$, a Heyting algebra (this time over a subset of $Up(\mathcal{F})$; see Section 2.4). Now, *Kripke* $<$ *Beth* means that every Heyting algebra produced in Kripke semantics can also be produced in Beth semantics but not vice versa: there are Heyting algebras produced by Beth semantics that are not isomorphic to $\mathbf{Up}(\mathcal{F})$ for any $\mathcal{F}$. Similarly for the other semantics in (3); all these facts are proved in BH19.

2.3. Set-Based Semantics

The following definition is not completely precise but suffices for our purposes.[4]

Definition 2.1 (set-based semantics)**.**

1. A *set-based semantics* for IPC is a class $\mathcal{S}$ of structures of some given type. For $X \neq \emptyset$, let $\mathcal{S}(X)$ be the class of structures in $\mathcal{S}$ with *domain* X. Furthermore, $\mathcal{S}$ associates with each structure in $\mathcal{S}(X)$ a set $SV_{\mathcal{S}}(X)$ of subsets of X (but we drop the subscript $_{\mathcal{S}}$ whenever feasible): the set of possible *semantic values* (over X) of formulas, according to $\mathcal{S}$.
2. An $\mathcal{S}$*-valuation* on X is a map $v\colon Prop \to SV(X)$.
3. An $\mathcal{S}$*-interpretation* on X is a function I from $\{\bot, \top, \wedge, \vee, \to\}$ to corresponding operations on $SV(X)$. I and v recursively associate with each formula φ a semantic value $[\![\varphi]\!]^I_v$, as usual:[5] $[\![p]\!]^I_v = v(p)$, $[\![\bot]\!]^I_v = I(\bot)$, $[\![\top]\!]^I_v = I(\top)$, $[\![\varphi \wedge \psi]\!]^I_v = I(\wedge)([\![\varphi]\!]^I_v, [\![\psi]\!]^I_v)$, *etc.*
4. I is *consistent* with a consequence relation $\vdash$, if $\psi_1, \ldots, \psi_n \vdash \varphi$ implies that for all $\mathcal{S}$-valuations v, $\bigcap_{i=1}^n [\![\psi_i]\!]^I_v \subseteq [\![\varphi]\!]^I_v$.
5. Each $\mathcal{S}$-interpretation I over X produces (and can be identified with) an algebra

$$\mathbf{S}(X)_I = (SV(X), I(\bot), I(\top), I(\wedge), I(\vee), I(\to))$$

6. There is an $\mathcal{S}$-interpretation I_{st} such that $\mathbf{S}(X)_{I_{st}}$ is a Heyting algebra and $I_{st}(\wedge)$ is set intersection. I_{st} is called a *standard* $\mathcal{S}$-interpretation.

Fact 2.2. *Standard $\mathcal{S}$-interpretations are unique.*

Proof. Let I_1 and I_2 be standard $\mathcal{S}$-interpretations. Since $I_1(\wedge) = I_2(\wedge) = \cap$, and $\mathbf{S}(X)_{I_1}$ and $\mathbf{S}(X)_{I_2}$ have the same domain, it follows by (5) that $\leq^{\mathbf{S}(X)_{I_1}} = \leq^{\mathbf{S}(X)_{I_2}} = \subseteq$. Moreover, since bounded lattices can also be defined in terms of the properties of their partial orders, this shows that the reducts of $\mathbf{S}(X)_{I_1}$ and $\mathbf{S}(X)_{I_2}$ to $\{\bot, \top, \wedge, \vee\}$, both assumed to be bounded lattices, are identical. Furthermore, in all Heyting algebras:

(7) $\quad a \wedge x \leq b \text{ iff } x \leq a \to b$

Since $\mathbf{S}(X)_{I_1}$ and $\mathbf{S}(X)_{I_2}$ are Heyting algebras with the same domain, this entails that for all $Z, U, V \in SV(X)$:

$$U \cap Z \subseteq V \Leftrightarrow Z \subseteq I_1(\to)(U, V) \Leftrightarrow Z \subseteq I_2(\to)(U, V)$$

This, in turn, entails that $I_1(\to) = I_2(\to)$, and, hence, that $\mathbf{S}(X)_{I_1} = \mathbf{S}(X)_{I_2}$, i.e., $I_1 = I_2$. □

One often thinks of the elements of X as worlds or states. Kripke semantics, as presented in Section 2.1, with the pertaining notions of (standard) interpretation and of consistency with a consequence relation, is a set-based semantics in the sense of Definition 2.1: $\mathcal{S}$ is the class of posets, and, for each $\mathcal{F} = (X, \leq) \in \mathcal{S}(X)$, $SV(X) = Up(\mathcal{F})$, and $\mathbf{S}(X)_{I_{st}} = \mathbf{Up}(\mathcal{F})$.

Carnap's question for set-based semantics is just as for Kripke semantics: Are there $\mathcal{S}$-interpretations other than the standard one that are also consistent with $\vdash_{\mathsf{IPC}}$? Let us look at some more examples.

2.4. Nuclear Interpretations

The concept of nuclear semantics, introduced in [9], covers several well-known semantics for IPC, including Kripke and Beth semantics. If $\mathcal{A}$ is a Heyting algebra, a *nucleus* on $\mathcal{A}$ is a map $j \colon A \to A$ with the following properties (omitting the superscript $^{\mathcal{A}}$):

(i) $\quad a \leq ja$ $\quad$ (inflationarity)
(ii) $\quad jja \leq ja$ $\quad$ (idempotence)
(iii) $\quad j(a \wedge b) = ja \wedge jb$ $\quad$ (multiplicativity)

It easily follows that $\leq$ in (ii) can be replaced by $=$, and that we have:

(iv) $\quad a \leq b$ implies $ja \leq jb$ $\quad$ (monotonicity)

We say that $a \in A$ is *fixed*, if it is a fixpoint of j, i.e., if $ja = a$. Note that $1^{\mathcal{A}}$ is always fixed, as is every element of the form ja, but it is possible to have $0^{\mathcal{A}} < j0^{\mathcal{A}}$. Let A_j be the set of fixed elements of A and consider the following algebra:

(8) $\quad \mathcal{A}_j = (A_j, j0^{\mathcal{A}}, 1^{\mathcal{A}}, \wedge_j^{\mathcal{A}}, \vee_j^{\mathcal{A}}, \to_j^{\mathcal{A}})$

where $\wedge_j^{\mathcal{A}}$ is $\wedge^{\mathcal{A}}$ restricted to A_j, and similarly for $\to_j^{\mathcal{A}}$, but, for $a, b \in A_j$, $a \vee_j^{\mathcal{A}} b =_{\text{def}} j(a \vee^{\mathcal{A}} b)$. The next result is well known (BH19, section 3.2).

Theorem 2.3. *If $\mathcal{A}$ is a Heyting algebra and j is a nucleus on A, then $\mathcal{A}_j$ is a Heyting algebra.*

Now recall from (6) the Heyting algebras of the form $\mathbf{Up}(\mathcal{F})$, where $\mathcal{F} = (X, \leq)$. When j is a nucleus on this algebra, Bezhanishivili and Holliday call $(\mathcal{F}, j)$ a *nuclear frame*. They show how familiar poset-based semantics for IPC can be recast as semantics over nuclear frames. The truth definition for the *standard* interpretation is then as in (9) below, where $v \colon Prop \to Up(\mathcal{F})_j$, and, for readability, we have written $[\![\varphi]\!]_v^{\mathcal{F},j}$ rather than $[\![\varphi]\!]_v^{I_{st}^{\mathcal{F},j}}$:

(9) a. $[\![p]\!]_v^{\mathcal{F},j} = v(p)$
b. $[\![\bot]\!]_v^{\mathcal{F},j} = j\varnothing$
c. $[\![\top]\!]_v^{\mathcal{F},j} = X$
d. $[\![\varphi \wedge \psi]\!]_v^{\mathcal{F},j} = [\![\varphi]\!]_v^{\mathcal{F},j} \cap [\![\psi]\!]_v^{\mathcal{F},j}$
e. $[\![\varphi \vee \psi]\!]_v^{\mathcal{F},j} = j([\![\varphi]\!]_v^{\mathcal{F},j} \cup [\![\psi]\!]_v^{\mathcal{F},j})$
f. $[\![\varphi \to \psi]\!]_v^{\mathcal{F},j} = \{x \in X \colon x\uparrow \cap [\![\varphi]\!]_v^{\mathcal{F},j} \subseteq [\![\psi]\!]_v^{\mathcal{F},j}\}$

More generally, an arbitrary *nuclear interpretation* (over $\mathcal{F}$) is a map $I^{\mathcal{F},j}$ as in Definition 2.1:3, with semantic values $[\![\varphi]\!]_v^{I^{\mathcal{F},j}}$ calculated as specified there.[6]

Example 2.4 (Kripke semantics)**.** Given $\mathcal{F}$, let j_K be the identity function on $Up(\mathcal{F})$; it is obviously a nucleus on $\mathbf{Up}(\mathcal{F})$. So, (intuitionistic) *Kripke frames* are nuclear frames of the form $(\mathcal{F}, j_K)$ (or simply $\mathcal{F}$), and the *standard Kripke interpretation* $I_{\mathrm{st}}^{\mathcal{F},j_K}$ is now as in (9), with $j = j_K$. These are exactly the usual truth conditions in Kripke semantics for intuitionistic logic.

In general, then, a *Kripke interpretation on* $\mathcal{F}$ is a nuclear interpretation of the form $I^{\mathcal{F},j_K}$.

Example 2.5 (Beth semantics)**.** There are several versions of Beth semantics, but here we follow BH19: a *Beth frame* is a poset $\mathcal{F}$ (rather than a tree), and a *path* is chain C in $\mathcal{F}$ closed under upper bounds: if x is an upper bound of C, then $x \in C$. So, the nuclear setting applies without change. Let, for $U \in Up(\mathcal{F})$,

(10) $j_B U = \{x \in X \colon \text{every path through } x \text{ intersects } U\}$

One verifies that j_B is a nucleus on $\mathbf{Up}(\mathcal{F})$—the *Beth nucleus*—and the *standard Beth interpretation* $I_{\mathrm{st}}^{\mathcal{F},j_B}$ is as in (9) with $j = j_B$. Thus, it differs from the standard Kripke interpretation only in the semantic values of disjunctions and in that propositional atoms (and, hence, all formulas) are evaluated as *fixed* upsets.

In general, we define a *Beth interpretation on* $\mathcal{F}$ to be any nuclear interpretation of the form $I^{\mathcal{F},j_B}$.

Example 2.6 (Dragalin semantics)**.** Dragalin generalized Beth semantics by considering a more general notion of a path, called a *development* by Bezhanishvili and Holliday. Each $x \in X$ is a associated with a set $D(x)$ of subsets of X satisfying certain conditions. Dragalin proved that

$$j_D U = \{x \in X \colon \text{every development in } D(x) \text{ intersects } U\}$$

is a nucleus on $\mathbf{Up}(\mathcal{F})$, so Dragalin semantics is an instance of nuclear semantics. The importance of this kind of nuclear semantics is seen from the result in [9] that every complete Heyting algebra can be realized as the set of fixed upsets of a nuclear frame of the form $(\mathcal{F}, j_D)$ (this does not hold for Beth semantics).[7]

Just as before, one has the *standard Dragalin interpretation* and a notion of an arbitrary *Dragalin interpretation*.

We have now seen three main kinds of *nuclear semantics*, that is, set-based semantics as in Definition 2.1, where $\mathcal{S}$ is some class of nuclear frames, and for $((X, \leq), j) \in \mathcal{S}$, $SV(X) = Up((X, \leq))_j$. For example, *Beth semantics* can be identified with the class $\mathcal{S}_B$ of nuclear frames $(\mathcal{F}, j_B)$, and *Beth interpretations on* $(X, \leq)$ are $\mathcal{S}_B$-interpretations on X.

Since $\mathbf{S}(X)_{I_{\mathrm{st}}^{\mathcal{F},j}} = \mathbf{Up}(\mathcal{F})_j$ is always a Heyting algebra (Theorem 2.3), $\vdash_{\mathsf{IPC}}$ is sound with respect to *any* nuclear semantics. The next semantic framework for IPC is not strictly nuclear but almost.

2.5. Topological Interpretations

Topological semantics is the oldest formal semantics for IPC, going back to [10,11]. Let $\Omega(X)$ be a topology on X, i.e., $\Omega(X)$ is its set of opens.[8] Then $\mathbf{\Omega}(X) = (\Omega(X), \emptyset, X, \cap, \cup, \to)$ is a (complete) Heyting algebra, with

$$U \to V = int((X - U) \cup V)$$

where *int* is the *interior operation*: for $Y \subseteq X$,

$$int(Y) = \bigcup\{U \in \Omega(X) \colon U \subseteq Y\}$$

So, *topological semantics* is a set-based semantics as in Definition 2.1, with $\mathcal{S}$ as the class of topological spaces, or some subclass thereof, and $SV(X) = \Omega(X)$. By the above, the *standard topological interpretation* of the connectives on $\Omega(X)$, $I_{\mathrm{st}}^{\Omega(X)}$, is given by

(11)
a. $I_{\mathrm{st}}^{\Omega(X)}(\bot) = \emptyset$
b. $I_{\mathrm{st}}^{\Omega(X)}(\top) = X$
c. $I_{\mathrm{st}}^{\Omega(X)}(\wedge)(U, V) = U \cap V$
d. $I_{\mathrm{st}}^{\Omega(X)}(\vee)(U, V) = U \cup V$
e. $I_{\mathrm{st}}^{\Omega(X)}(\to)(U, V) = int((X - U) \cup V)$

We can now say that a *topological interpretation* on X is a map $I^{\Omega(X)}$ from $\{\bot, \top, \wedge, \vee, \to\}$ to corresponding functions on $\Omega(X)$, and the semantic values of formulas under $I^{\Omega(X)}$ and $v \colon Prop \to \Omega(X)$ are calculated as usual.

If $\mathcal{F} = (X, \leq)$ is a poset, the *Alexandroff topology* on X has $\Omega(X) = Up(\mathcal{F})$, and it is easy to see that $int((X - U) \cup V) = \{x \in X \colon x{\uparrow} \cap U \subseteq V\}$, so Kripke semantics in effect *is* topological semantics with the Alexandroff topology.

In general, the open sets themselves need not be upsets of a partial order. However, a theorem of Dragalin states that $\mathbf{\Omega}(X)$ is always *isomorphic* to the fixed upset algebra of some nuclear frame; indeed relative to a Dragalin nucleus j_D (see BH19, Theorem 4.23 for a proof). This shows that *Topological* $\leq$ *Dragalin* in the hierarchy (3).

In fact, [9] proved that *every* nuclear frame $(\mathcal{F}, j)$ is equal to a frame $(\mathcal{F}, j_D)$ with the same poset and the Dragalin nucleus (Theorem 4.25 in BH19). This shows that Dragalin semantics (with $\mathcal{S}$ as the class of Dragalin frames) and nuclear semantics (with $\mathcal{S}$ as the class of nuclear frames) are *equivalent* (in terms of producing Heyting algebras). So, in (3), one can replace *Dragalin* with *Nuclear*.

3. Carnap Categoricity

$\vdash_{\mathsf{IPC}}$ is sound and complete for all the semantics (classes of nuclear frames or topological spaces) discussed in the preceding section—relative, of course, to what we are here calling the standard interpretations of the connectives. However, our interest is not completeness but categoricity. For each particular semantics $\mathcal{S}$ mentioned above, we defined the notion of a (local) $\mathcal{S}$-interpretation of the connectives and described how semantic values relative to such an interpretation are computed for each formula.

We also specified, in each case, the *standard* $\mathcal{S}$-interpretations. These are all consistent with $\vdash_{\mathsf{IPC}}$. However, could there be *other* $\mathcal{S}$-interpretations that are *also* consistent with $\vdash_{\mathsf{IPC}}$?

Definition 3.1 (categoricity in set-based semantics)**.** A consequence relation $\vdash$ in the propositional language is (Carnap) *categorical* with respect to a set-based semantics $\mathcal{S}$ if, for every structure in $\mathcal{S}$, the only $\mathcal{S}$-interpretation on that structure's domain consistent with $\vdash$ is the standard interpretation. If $\vdash$ is associated with a particular logic L, we also say that L is *categorical* (with respect to $\mathcal{S}$) when $\vdash$ is.

What about algebraic semantics? In Section 2.2, we noted that any algebra of the signature $\{\bot, \top, \wedge, \vee, \to\}$ could be taken as an interpretation of the connectives. It is also reasonable to say that Heyting algebras are the *standard* interpretations. This leads to *two* distinct Carnap style questions about algebraic semantics.

First, we could ask: why Heyting algebras? The answer may seem obvious, but, to even formulate the question in a non-trivial way, we need to specify a larger class of algebras from which the Heyting algebras are singled out by consistency with $\vdash_{\mathsf{IPC}}$.

Second, we can follow the pattern from set-based semantics: Given a standard 'semantic algebra' $\mathcal{A}$ (normally a Heyting algebra), we can define what it would mean for other algebras with domain A to be consistent with a consequence relation $\vdash$. Then, we ask whether, among such algebras, only $\mathcal{A}$ fits the bill relative to our favored consequence relation. In the case of set-based semantics, however, consistency with $\vdash$ was defined in terms of *intersection* (of the semantic values of the premises) and the *subset* relation. These are not available for algebras whose domains are not sets of sets. Instead, we take $\mathcal{A}$ itself to provide the required meet operation and partial order.

3.1. Algebraic Interpretations

The syntax algebra of propositional logic is a *term algebra* with countably many generators, which means that for *any* algebra $\mathcal{A}$ of the same signature, every map v from propositional atoms to A extends to a unique homomorphism to $\mathcal{A}$. However, it would make no sense to let every algebra of that signature be a putative interpretation of the connectives. Interpretations are relative to a *semantics*, and a semantics needs a notion of *truth*, or of *entailment*. We could of course stipulate that $[\varphi]^{\mathcal{A}}_v = 1^{\mathcal{A}}$ means that φ is true in $\mathcal{A}$ under v. However, nothing can be done with that stipulation unless we know more about the role of $1^{\mathcal{A}}$. For example, why not $0^{\mathcal{A}}$ instead?

On the other hand, we want to make as few assumptions as possible. Without further ado, we propose the following.

Definition 3.2. $\mathcal{A} = (A, 0^{\mathcal{A}}, 1^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, \to^{\mathcal{A}})$ is an *algebraic interpretation* if $\leq^{\mathcal{A}}$ defined by $a \leq^{\mathcal{A}} b \Leftrightarrow a \wedge^{\mathcal{A}} b = a$ is a partial order.

Then, we define consistency with a consequence relation $\vdash$ as follows.[9]

Definition 3.3. An algebraic interpretation $\mathcal{A}$ is *consistent with* $\vdash$, if, whenever $\psi_1, \ldots, \psi_n \vdash \varphi$ holds, we have for all A-valuations v and all $c \in A$, that if $c \leq^{\mathcal{A}} [\psi_i]^{\mathcal{A}}_v$ for $i = 1, \ldots, n$, then $c \leq^{\mathcal{A}} [\varphi]^{\mathcal{A}}_v$. In particular, if $\vdash \varphi$, then for all v and all $c \in A$, $c \leq^{\mathcal{A}} [\varphi]^{\mathcal{A}}_v$.

Thus, if $\vdash$ has theorems, $\mathcal{A}$ has a unique largest element (which under minimal further assumptions must be $1^{\mathcal{A}}$), but not all consequence relations have theorems; see Section 4. We have:

Fact 3.4. *Let $\mathcal{A}$ be an algebraic interpretation.*

(i) *$\mathcal{A}$ is consistent with $\vdash_{\mathsf{IPC}}$ iff $\mathcal{A}$ is a Heyting algebra.*

(ii) *$\mathcal{A}$ is consistent with $\vdash_{\mathsf{CPC}}$ iff $(A, 0^{\mathcal{A}}, 1^{\mathcal{A}}, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}}, '^{\mathcal{A}})$ is a Boolean algebra, where $a'^{\mathcal{A}} := a \to^{\mathcal{A}} 0^{\mathcal{A}}$.*

Proof. Apart from the new terminology, this is essentially well-known; for the record we indicate the proof. (i): We already know that Heyting algebras are consistent with $\vdash_{\mathsf{IPC}}$. In the other direction, suppose $\mathcal{A}$ is consistent with $\vdash_{\mathsf{IPC}}$. Since $\vdash_{\mathsf{IPC}} \top$, we have for all $c \in A$ (and all v) , $c \leq^{\mathcal{A}} [\top]^{\mathcal{A}}_v = 1^{\mathcal{A}}$, so $1^{\mathcal{A}}$ is the unique largest element. Moreover, for all $a, c \in A$, since $\bot \vdash_{\mathsf{IPC}} p$, we obtain (choosing an A-valuation v such that $v(p) = a$) that if $c \leq^{\mathcal{A}} [\bot]^{\mathcal{A}}_v = 0^{\mathcal{A}}$, then $c \leq^{\mathcal{A}} [p]^{\mathcal{A}}_v = a$. Thus, with $c = 0^{\mathcal{A}}$, we have for all $a \in A$, $0^{\mathcal{A}} \leq^{\mathcal{A}} a$.

Next, the commutativity, associativity, and idempotence of $\wedge^{\mathcal{A}}$ and $\vee^{\mathcal{A}}$ follow easily from the corresponding IPC-consequences. For example, to show that for all $a, b, c \in A$,

(a) $\quad a \vee^{\mathcal{A}} (b \vee^{\mathcal{A}} c) = (a \vee^{\mathcal{A}} b) \vee^{\mathcal{A}} c$

we take v such that $v(p) = a$, $v(q) = b$, and $v(r) = c$, and use that $p \vee (q \vee r) \vdash_{\mathsf{IPC}} (p \vee q) \vee r$. By consistency with $\vdash_{\mathsf{IPC}}$, $a \vee^{\mathcal{A}} (b \vee^{\mathcal{A}} c) = [p \vee (q \vee r)]_v^{\mathcal{A}} \leq^{\mathcal{A}} [(p \vee q) \vee r]_v^{\mathcal{A}} = (a \vee^{\mathcal{A}} b) \vee^{\mathcal{A}} c$. Similarly, $(a \vee^{\mathcal{A}} b) \vee^{\mathcal{A}} c \leq^{\mathcal{A}} a \vee^{\mathcal{A}} (b \vee^{\mathcal{A}} c)$, so, since $\leq^{\mathcal{A}}$ is assumed to be a partial order, (a) follows.

Similarly for the absorption equations (dropping superscripts) $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$, as well as the equations (4) involving $\to$. For example, that $(a \to b) \wedge b = b$ follows from the fact that $(p \to q) \wedge q \vdash_{\mathsf{IPC}} q$ and $q \vdash_{\mathsf{IPC}} (p \to q) \wedge q$. This shows that $\mathcal{A}$ is a Heyting algebra.

(ii): The right-to-left direction is well known. Moreover, if $\mathcal{A}$ is consistent with $\vdash_{\mathsf{CPC}}$, the defining equations of Heyting algebras are still valid. It only remains to show that $a \wedge a' = 0$ and $a \vee a' = 1$. The first is a consequence of $a \wedge (a \to b) = a \wedge b$ and the fact that $a \wedge 0 = 0$. The second identity is the Law of Excluded Middle. □

3.2. Categoricity

To deal with the second kind of algebraic categoricity issue, we proceed in analogy with Definition 2.1.

Definition 3.5 ($\mathcal{A}$-interpretations)**.**

1. Let $\mathcal{A}$ be an algebraic interpretation. An *$\mathcal{A}$-interpretation* is a map I from the signature to appropriate functions on A. We let
$$\mathcal{A}_I = (A, I(\bot), I(\top), I(\wedge), I(\vee), I(\to)) = (A, \bot^{\mathcal{A}_I}, \top^{\mathcal{A}_I}, \wedge^{\mathcal{A}_I}, \vee^{\mathcal{A}_I}, \to^{\mathcal{A}_I})$$
and may identify I and $\mathcal{A}_I$.
2. An *A-valuation* is a map $v\colon Prop \to A$. I and v recursively assign a value $[\varphi]_v^{\mathcal{A}_I}$ to each formula φ as usual.
3. $\mathcal{A}_I$ is *$\mathcal{A}$-consistent* with $\vdash$ if whenever $\psi_1, \ldots, \psi_n \vdash \varphi$, we have for all A-valuations v and all $c \in A$ that $c \leq^{\mathcal{A}} [\psi_1]_v^{\mathcal{A}_I} \wedge^{\mathcal{A}} \ldots \wedge^{\mathcal{A}} [\psi_n]_v^{\mathcal{A}_I}$ implies $c \leq^{\mathcal{A}} [\varphi]_v^{\mathcal{A}_I}$. Thus, $\mathcal{A}$ is consistent with $\vdash$ as in Definition 3.3 if $\mathcal{A}$ is $\mathcal{A}$-consistent with $\vdash$.

In general, $\mathcal{A}_I$ need not be an algebraic interpretation, even if it is consistent with $\vdash_{\mathsf{IPC}}$. However, matters of consistency with consequence relations are decided by $\mathcal{A}$, not $\mathcal{A}_I$. When $\mathcal{A}$ is a Heyting algebra, a strong conclusion about $\mathcal{A}_I$ follows, as we now show.[10]

Theorem 3.6. *If $\mathcal{A}$ is a Heyting algebra, and $\mathcal{A}_I$ is $\mathcal{A}$-consistent with $\vdash_{\mathsf{IPC}}$, then $\mathcal{A}_I = \mathcal{A}$.*

Proof. Let $\mathcal{A}$ be a Heyting algebra, and suppose that $\mathcal{A}_I$ is and $\mathcal{A}$-interpretation which is $\mathcal{A}$-consistent with $\vdash_{\mathsf{IPC}}$. First,

(b) $\quad \wedge^{\mathcal{A}_I} = \wedge^{\mathcal{A}}$

Since $p \wedge q \vdash_{\mathsf{IPC}} p$ and $p \wedge q \vdash_{\mathsf{IPC}} q$, we have, for $a, b \in A$ and v such that $v(p) = a$ and $v(q) = b$, $a \wedge^{\mathcal{A}_I} b = [p \wedge q]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [p]_v^{\mathcal{A}_I} = a$, and, similarly, $a \wedge^{\mathcal{A}_I} b \leq^{\mathcal{A}} b$, so $a \wedge^{\mathcal{A}_I} b \leq^{\mathcal{A}} a \wedge^{\mathcal{A}} b$. Moreover, since $p, q \vdash_{\mathsf{IPC}} p \wedge q$, we also have, by the definition of $\mathcal{A}$-consistency, that $a \wedge^{\mathcal{A}} b \leq^{\mathcal{A}} a \wedge^{\mathcal{A}_I} b$. This proves (b). Note that it also follows that $\leq^{\mathcal{A}_I} = \leq^{\mathcal{A}}$. However, since we do not (yet!) know that $\mathcal{A}_I$ is a Heyting algebra, in fact not even that the reduct of $\mathcal{A}_I$ to $\{\bot, \top, \wedge, \vee\}$ is a bounded lattice,[11] we cannot reason as in the proof of Fact 2.2 to directly conclude that $\mathcal{A}$ and $\mathcal{A}_I$ are identical. Instead, we reach this conclusion using $\mathcal{A}$-consistency with $\vdash_{\mathsf{IPC}}$, in the following steps.

First, reasoning as in the beginning of the proof of Fact 3.4 (i), we see that $\bot^{\mathcal{A}_I}$ is the $\leq^{\mathcal{A}_I}$-smallest element of A, and $\top^{\mathcal{A}_I}$ the $\leq^{\mathcal{A}_I}$-greatest. Thus,

(c) $\quad \bot^{\mathcal{A}_I} = 0^{\mathcal{A}}$ and $\top^{\mathcal{A}_I} = 1^{\mathcal{A}}$.

We now claim:

(d) $\quad a \to^{\mathcal{A}_I} b \ \leq^{\mathcal{A}} \ a \to^{\mathcal{A}} b$

To see this, take v with $v(p) = a$ and $v(q) = b$. Since $p, p \to q \vdash_{\mathsf{IPC}} q$, we have $[p]_v^{\mathcal{A}_I} \wedge^{\mathcal{A}} [p \to q]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [q]_v^{\mathcal{A}_I}$, i.e., $a \wedge^{\mathcal{A}} (a \to^{\mathcal{A}_I} b) \leq^{\mathcal{A}} b$. Now recall (7), i.e., that $a \wedge^{\mathcal{A}} x \leq^{\mathcal{A}} b$ iff $x \leq^{\mathcal{A}} a \to^{\mathcal{A}} b$. So, (d) follows. The next claim follows by a special case of (7): $a \leq b$ iff $a \to b = 1$ (from (7) with $x = 1$, since $a \wedge 1 = a$).

(e) $\quad$ For every A-valuation v, $[\varphi]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [\psi]_v^{\mathcal{A}_I}$ iff $[\varphi \to \psi]_v^{\mathcal{A}_I} = 1^{\mathcal{A}}$.

We are now able to show:

(f) $\quad \vee^{\mathcal{A}_I} = \vee^{\mathcal{A}}$

Take $a, b \in A$. From $p \vdash_{\mathsf{IPC}} p \vee q$ and $q \vdash_{\mathsf{IPC}} p \vee q$, we obtain, with a suitable valuation, that $a \leq^{\mathcal{A}} a \vee^{\mathcal{A}_I} b$ and $b \leq^{\mathcal{A}} a \vee^{\mathcal{A}_I} b$; hence, $a \vee^{\mathcal{A}} b \leq^{\mathcal{A}} a \vee^{\mathcal{A}_I} b$. It remains to show $a \vee^{\mathcal{A}_I} b \leq^{\mathcal{A}} a \vee^{\mathcal{A}} b$. Take a valuation v such that $v(p) = a$, $v(q) = b$, and $v(r) = a \vee^{\mathcal{A}} b$. We claim

$$[p \to r]_v^{\mathcal{A}_I} = 1^{\mathcal{A}}$$

Since $a \leq^{\mathcal{A}} a \vee^{\mathcal{A}} b$, that is, $[p]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [r]_v^{\mathcal{A}_I}$, this follows by (e). Similarly, $[q \to r]_v^{\mathcal{A}_I} = 1^{\mathcal{A}}$. Now, since

$$p \to r, q \to r, p \vee q \vdash_{\mathsf{IPC}} r$$

we have $[p \to r]_v^{\mathcal{A}_I} \wedge^{\mathcal{A}} [q \to r]_v^{\mathcal{A}_I} \wedge^{\mathcal{A}} [p \vee q]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [r]_v^{\mathcal{A}_I}$, that is, $[p \vee q]_v^{\mathcal{A}_I} = a \vee^{\mathcal{A}_I} b \leq^{\mathcal{A}} [r]_v^{\mathcal{A}_I} = a \vee^{\mathcal{A}} b$. This proves (f). Finally:

(g) $\quad \to^{\mathcal{A}_I} = \to^{\mathcal{A}}$

By (d), we only need to check that for $a, b \in A$, $a \to^{\mathcal{A}} b \leq^{\mathcal{A}} a \to^{\mathcal{A}_I} b$. Suppose $x \leq^{\mathcal{A}} a \to^{\mathcal{A}} b$. By (7), $x \wedge^{\mathcal{A}} a \leq^{\mathcal{A}} b$. Let v be such that $v(p) = x$, $v(q) = a$, and $v(r) = b$. Then, we have $x \wedge^{\mathcal{A}} a = [p \wedge q]_v^{\mathcal{A}_I} \leq^{\mathcal{A}} [r]_v^{\mathcal{A}_I} = b$, so by (e), $[p \wedge q \to r]_v^{\mathcal{A}_I} = 1^{\mathcal{A}}$. Now use the fact that

$$p, p \wedge q \to r \vdash_{\mathsf{IPC}} q \to r$$

By consistency, $x = x \wedge^{\mathcal{A}} 1^{\mathcal{A}} \leq^{\mathcal{A}} [q \to r]_v^{\mathcal{A}_I} = a \to^{\mathcal{A}_I} b$. We have shown that for all $x \in A$, $x \leq^{\mathcal{A}} a \to^{\mathcal{A}} b$ implies $x \leq^{\mathcal{A}} a \to^{\mathcal{A}_I} b$. Letting $x = a \to^{\mathcal{A}} b$, we have a proof of (g). This concludes the proof of the theorem. □

Theorem 3.6 is our main categoricity result, as will be seen from the corollaries in the next subsection. Fact 3.4 is unsurprising, but it does show why Heyting algebras (Boolean algebras) are *exactly* the right choice for intuitionistic (classical) logic. An interesting class of algebras in this connection are the *Dummett algebras* defined in BH19. If H is a Heyting algebra and j a nucleus on H, the Dummett algebra $D(H, j)$ of the same signature is the same as the Heyting algebra of the fixpoints H_j, except that

(12) $\quad a \to^j b =_{\mathrm{def}} a \to jb$

Dummett algebras need not be Heyting algebras—Bezhanishvili and Holliday give a simple example—but they, of course, qualify as algebraic interpretations.[12] BH19 proves the following.

Theorem 3.7 ([8], Theorem 3.25)**.** $\vdash_{\mathsf{IPC}} \varphi$ *iff* φ *is valid in all Dummett algebras.*

This result shows how *consequence*, rather than validity, is crucial for our story about the meaning of the connectives. We need to be able to conclude from $\psi \vdash_{\mathsf{IPC}} \varphi$ that, for all v, $[\psi]_v^{\mathcal{A}} \leq^{\mathcal{A}} [\varphi]_v^{\mathcal{A}}$; it is not enough to know that, for all v, $[\psi \to \varphi]_v^{\mathcal{A}} = 1^{\mathcal{A}}$. By the above and Theorem 3.6, non-Heyting Dummett algebras are not consistent with $\vdash_{\mathsf{IPC}}$, even though they validate all IPC theorems.

3.3. Application to Set-Based Semantics

Let $\mathcal{S}$ be a set-based semantics as in Definition 2.1, and let I be an $\mathcal{S}$-interpretation over X, which is consistent with $\vdash_{\mathsf{IPC}}$. The algebra $\mathbf{S}(X)_{I_{\mathrm{st}}}$ of the standard interpretation is a Heyting algebra, and the algebra $\mathbf{S}(X)_I$ corresponding to I is an $\mathbf{S}(X)_{I_{\mathrm{st}}}$-interpretation in the sense of Definition 3.5. Both algebras have the domain $SV(X)$, and valuations in the set-based sense are identical to valuations is the algebraic sense. So, clearly, we have, for every formula φ,

$$[\![\varphi]\!]_v^I = [\varphi]_v^{\mathbf{S}(X)_I}$$

Since $\wedge^{\mathbf{S}(X)_{I_{\mathrm{st}}}}$ is intersection, and $\leq^{\mathbf{S}(X)_{I_{\mathrm{st}}}}$ is the subset relation, it follows that $\mathbf{S}(X)_I$ is $\mathbf{S}(X)_{I_{\mathrm{st}}}$-consistent with $\vdash_{\mathsf{IPC}}$. Thus, by Theorem 3.6, $\mathbf{S}(X)_I = \mathbf{S}(X)_{I_{\mathrm{st}}}$, that is, $I = I_{\mathrm{st}}$. We have proved the following (see Definition 3.1).

Corollary 3.8. IPC *is (Carnap) categorical with respect to every set-based semantics. In particular, the only Kripke interpretation of the connectives consistent with* $\vdash_{\mathsf{IPC}}$ *is the standard interpretation, and similarly for Beth interpretations (indeed all nuclear interpretations), topological interpretations, etc.*

Remark 3.9. The Carnap categoricity of CPC with respect to classical possible worlds semantics, proved in [5], is a special case of the result for IPC, since on posets that are sets of isolated points, IPC and CPC coincide. In more detail: suppose W is any non-empty set (of 'worlds'), and I^W an interpretation of the connectives over W—i.e., I^W assigns to each connective a function on $\mathcal{P}(W)$ of appropriate arity—which is consistent with $\vdash_{\mathsf{CPC}}$. The upsets of the poset $\mathcal{F} = (W, \{(x,x) : x \in W\})$ are exactly the subsets of W. Thus, $I^{\mathcal{F}} = I^W$ is in fact a Kripke interpretation, in the sense of Example 2.4, which is consistent with $\vdash_{\mathsf{CPC}}$, hence with $\vdash_{\mathsf{IPC}}$. By the Corollary, $I^{\mathcal{F}}$ is standard, and so I^W is standard. That is, $I^W(\bot) = \varnothing$, $I^W(\wedge)$ is intersection, $I^W(\vee)$ is union, and $I^W(\to)(U,V) = (W - U) \cup V$ (so $I^W(\neg)$ is complement).

Similarly, letting $\mathcal{F}$ be a single reflexive point $(\{x\}, \{(x,x)\})$, the categoricity of CPC with respect to classical two-valued semantics, mentioned in Section 1, also follows from Corollary 3.8: in this case, the interpretation functions are truth functions, and the truth values 0 and 1 correspond to the upsets $\varnothing$ and $\{x\}$.

The remark underscores the *local* character of our main result: Corollary 3.8 holds for each single structure in $\mathcal{S}$.

Again, the absence of non-standard interpretations of the connectives consistent with $\vdash_{\mathsf{IPC}}$ depends crucially on the fact that we require consistency for consequence, not just validity. In fact, Wesley Holliday found a counter-example (*p.c*): a non-standard Kripke interpretation of the connectives which validates all IPC theorems.[13] With his kind permission, we present the example here.[14]

Example 3.10 (Holliday)**.** Let $\Omega(X)$ be any topological space. We will define a topological interpretation $I^{\Omega(X)}$ as in Section 2.5, such that (a) if $\vdash_{\mathsf{IPC}} \varphi$, then for all valuations v on $\Omega(X)$, $[\![\varphi]\!]_v^{I^{\Omega(X)}} = X$, but (b) $I^{\Omega(X)} \neq I_{\mathrm{st}}^{\Omega(X)}$. This gives a generic topological counter-example. For more concreteness, we can start with a Kripke frame $\mathcal{F} = (X, \leq)$ and let $\Omega(X)$ be the Alexandroff topology described in that section. Kripke semantics is essentially topological semantics on Alexandroff spaces, and it is easy to see that the counter-example then becomes a non-standard Kripke interpretation $I^{\mathcal{F},j_k}$ (Example 2.4) which validates all theorems of IPC.

For this example, it is easier to use $\neg, \wedge, \vee, \rightarrow$ as primitive connectives. (11-a) is then replaced by

(11-a)′ $I_{st}^{\Omega(X)}(\neg)(U) = int(X - U)$

Recall the *closure* operation *cl*, the dual of *int*: $cl(Y)$ is the smallest closed set (set whose complement is open) containing Y. An easy calculation shows

(13) $$[\![\neg\neg\varphi]\!]_v^{I_{st}^{\Omega(X)}} = int(cl([\![\varphi]\!]_v^{I_{st}^{\Omega(X)}}))$$

and thus

(14) $$[\![\neg\neg\neg\varphi]\!]_v^{I_{st}^{\Omega(X)}} = int(X - int(cl([\![\varphi]\!]_v^{I_{st}^{\Omega(X)}})))$$

Now define $I^{\Omega(X)}$ as follows.

(15)
a. $I^{\Omega(X)}(\wedge)(U,V) = int(cl(U)) \cap int(cl(V))$
b. $I^{\Omega(X)}(\neg)(U) = int(X - int(cl(U)))$
c. $I^{\Omega(X)}(\rightarrow)(U,V) = I^{\Omega(X)}(\neg)(I^{\Omega(X)}(\wedge)(I^{\Omega(X)}(\neg)(U),V))$
d. $I^{\Omega(X)}(\vee)(U,V) = I^{\Omega(X)}(\neg)(I^{\Omega(X)}(\wedge)(I^{\Omega(X)}(\neg)(U), I^{\Omega(X)}(\neg)(V)))$

Thus, we are putting double negations in front of negated formulas and the conjuncts of conjunctions, whereas $\rightarrow$ and $\vee$ are defined classically from $\wedge$ and $\neg$. Clearly, $I^{\Omega(X)}$ is non-standard: for example, just find a space with open sets U, V such that $int(cl(U)) \cap int(cl(V)) \neq U \cap V$.

Now consider the following *negative translation* of classical into intuitionistic propositional logic:

(16)
a. $g(p) = p$
b. $g(\varphi \wedge \psi) = \neg\neg g(\varphi) \wedge \neg\neg g(\psi)$
c. $g(\neg\varphi) = \neg\neg\neg g(\psi)$
d. $g(\varphi \rightarrow \psi) = \neg(g(\varphi) \wedge \neg g(\psi))$
e. $g(\varphi \vee \psi) = \neg(\neg g(\varphi) \wedge \neg g(\psi))$

Using well-known facts about negative translations, it is not hard to show that

(17) $\models_{\mathsf{CPC}} \varphi$ iff $\vdash_{\mathsf{IPC}} g(\varphi)$.[15]

Finally, observe that $I^{\Omega(X)}$ interprets a formula φ just as $g(\varphi)$ is standardly interpreted in topological semantics. In other words, for each topological valuation v we have:

(18) $$[\![\varphi]\!]_v^{I^{\Omega(X)}} = [\![g(\varphi)]\!]_v^{I_{st}^{\Omega(X)}}$$

This is proved by a straightforward inductive argument, using the standard topological truth definition and (13)–(16). Thus, if $\vdash_{\mathsf{IPC}} \varphi$, then $\vdash_{\mathsf{CPC}} \varphi$, hence $\vdash_{\mathsf{IPC}} g(\varphi)$ by (17), and so, for any topological valuation v, $[\![g(\varphi)]\!]_v^{I_{st}^{\Omega(X)}} = X$, which by (18) entails that $[\![\varphi]\!]_v^{I^{\Omega(X)}} = X$. That is, $I^{\Omega(X)}$ validates all IPC theorems.[16]

On the other hand, it is easy to see that $I^{\Omega(X)}$ is not consistent with $\vdash_{\mathsf{IPC}}$. For example, one can have $[\![p]\!]_v^{I^{\Omega(X)}} = [\![p \rightarrow q]\!]_v^{I^{\Omega(X)}} = X$, while $[\![q]\!]_v^{I^{\Omega(X)}} \neq X$.

Remark 3.11. It may be worth noting that, by contrast, categoricity facts about classical *modal* logic (see [6]) only require validating all theorems. This is precisely because it is classical. If L is a modal logic, $\psi_1, \ldots, \psi_n \vdash_{\mathsf{L}} \varphi$ means by definition that $\vdash_{\mathsf{L}} \psi_1 \wedge \ldots \wedge \psi_n \rightarrow \varphi$.

Thus, if it holds for all v that $[\![\psi_1 \wedge \ldots \wedge \psi_n \to \varphi]\!]_v^{(X,F)} = X$, we can conclude, by the fact that $\wedge$ and $\to$ are standard (which, as we saw in Remark 3.9, follows from Corollary 3.8), that $\bigcap_{i=1}^n [\![\psi_i]\!]_v^{(X,F)} \subseteq [\![\varphi]\!]_v^{(X,F)}$.

4. Some Fragments of $\vdash_{\mathsf{IPC}}$

Every *intermediate logic* (logic between IPC and CPC) is categorical with respect to set-based semantics as per Corollary 3.8, but it is natural to ask about the categoricity of logics *weaker* than IPC. Here, a logic/consequence relation $\vdash$ is weaker if it is a proper subset of $\vdash_{\mathsf{IPC}}$. In this section, we look at one class of such logics. Consider any standard natural deduction axiomatization of $\vdash_{\mathsf{IPC}}$. For $\Phi \subseteq \{\bot, \top, \wedge, \vee, \to\}$, let L_Φ be the propositional language with connectives in Φ, and let $\vdash_\Phi$ be the consequence relation obtained by deleting the rules for the connectives left out.[17] Now one can ask if the results corresponding to Theorem 3.6 hold for $\vdash_\Phi$. We look at a few illustrative cases.

4.1. $\vdash_\wedge$

It should be clear that the correspondent of Theorem 3.6 holds for $\vdash_\wedge$, but a quick look at the details is instructive. Thus, an *algebraic interpretation* is an algebra $\mathcal{A} = (A, \wedge^{\mathcal{A}})$ such that $a \leq^{\mathcal{A}} b \Leftrightarrow a \wedge^{\mathcal{A}} b = a$ is a partial order, and $\mathcal{A}$ is *consistent* with $\vdash$ just as in Definition 3.3. Similarly, the notion of an $\mathcal{A}$-interpretation, and of such an interpretation being $\mathcal{A}$-consistent with $\vdash$, is as in Definition 3.5 (restricted to $L_\wedge$).

Clearly, the standard algebraic interpretations are meet semi-lattices ($\vdash_\wedge$ has no theorems, so a largest, or smallest, element need not exist). An algebraic interpretation need not be a meet semi-lattice,[18] but consistency with $\vdash_\wedge$ singles out the meet semi-lattices; see below.

A *set-based semantics* $\mathcal{S}$ for $\vdash_\wedge$ is as in Definition 2.1: the (trivially unique) *standard* interpretation $\mathbf{S}(X)_{I_{\mathrm{st}}}$ is a meet semi-lattice in which $I_{\mathrm{st}}(\wedge)$ is intersection.

Fact 4.1. (i) *$\mathcal{A}$ is consistent with $\vdash_\wedge$ iff $\mathcal{A}$ is a meet semi-lattice.*
(ii) *If $\mathcal{A}$ is a meet semi-lattice, and $\mathcal{A}_I$ is $\mathcal{A}$-consistent with $\vdash_\wedge$, then $\mathcal{A}_I = \mathcal{A}$.*
(iii) *$\vdash_\wedge$ is categorical in every set-based semantics.*

Proof. (i): For the left-to-right direction, use the fact that $p \vdash_\wedge p \wedge p$, $p \wedge p \vdash_\wedge p$, $p \wedge q \vdash q \wedge p$, $p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r$, $(p \wedge q) \wedge r \vdash p \wedge (q \wedge r)$, and that $\leq^{\mathcal{A}}$ is anti-symmetric.

(ii): As in the proof of (b) in the proof of Theorem 3.6, using that $p \wedge q \vdash_\wedge p$, $p \wedge q \vdash_\wedge q$, and $p, q \vdash_\wedge p \wedge q$.

(iii): As in the proof of Corollary 3.8. □

4.2. $\vdash_{\top\wedge\to}$

In contrast with the preceding example, this is an interestingly large fragment of $\vdash_{\mathsf{IPC}} = \vdash_{\bot\top\wedge\vee\to}$.[19] Algebraic interpretations, consistency, $\mathcal{A}$-interpretations, and $\mathcal{A}$-consistency are just as before. The *standard* algebraic interpretations are called *Brouwerian semi-lattices* ([15]), or *implicative semi-lattices* ([16]): $(A, \top^{\mathcal{A}}, \wedge^{\mathcal{A}})$ is a meet semi-lattice with $\top^{\mathcal{A}}$ as largest element, and $\to^{\mathcal{A}}$ satisfies the four axioms in (4) (which use only $\wedge$, $\to$, and 1). There does not have to be a smallest element.[20]

Set-based semantics $\mathcal{S}$ for this language are defined as before, and standard interpretations are again unique.

Fact 4.2. (i) $\mathcal{A}$ is consistent with $\vdash_{\top\wedge\to}$ iff $\mathcal{A}$ is a Brouwerian semi-lattice.
(ii) If $\mathcal{A}$ is a Brouwerian semi-lattice, and $\mathcal{A}_I$ is $\mathcal{A}$-consistent with $\vdash_{\top\wedge\to}$, then $\mathcal{A}_I = \mathcal{A}$.
(iii) $\vdash_{\top\wedge\to}$ is categorical in every set-based semantics.

Proof. (i): For the left-to-right direction, the relevant inequalities (hence identities) are all provable in $\vdash_{\top\wedge\to}$.

(ii): Again, the corresponding parts of the proof of of Theorem 3.6 work: (b), half of (c), (d) (which depends on (7), which holds in Brouwerian semi-lattices), (e) (likewise, note that $1^{\mathcal{A}}$ is used), and (g). The $\vdash_{\mathsf{IPC}}$-consequences used in the proof all hold for $\vdash_{\top\wedge\rightarrow}$.

(iii): As in the proof of Corollary 3.8. $\square$

4.3. $\vdash_{\vee}$

This logic does not exactly fit the previous format, but we can say that an *algebraic interpretation* $\mathcal{A} = (A, \vee^{\mathcal{A}})$ is one where $a \leq^{\mathcal{A}} b \Leftrightarrow a \vee^{\mathcal{A}} b = b$ is a partial order. Then, we define: $\mathcal{A}$ is *consistent with* $\vdash$, if, whenever $\psi_1, \ldots, \psi_n \vdash \varphi$ holds, we have for all A-valuations v and all $c \in A$, that if $c \leq^{\mathcal{A}} [\psi_i]^{\mathcal{A}}_v$ for $i = 1, \ldots, n$, then $c \leq^{\mathcal{A}} [\varphi]^{\mathcal{A}}_v$. The standard interpretations are the join semi-lattices.

We can define $\mathcal{A}$-interpretations, and $\mathcal{A}$-consistency with $\vdash$ (almost) as before. However, now the proof of (f) in the proof of Theorem 3.6 no longer works, since it depends on facts about $\rightarrow$. Indeed, *categoricity fails*: we can find a non-standard Kripke interpretation of $\vee$ that is consistent with $\vdash_{\vee}$.

Fact 4.3. (i) *$\mathcal{A}$ is consistent with $\vdash_{\vee}$ iff $\mathcal{A}$ is a join semi-lattice.*
(ii) *There is a non-standard Kripke interpretation consistent with $\vdash_{\vee}$.*

Proof. (outline) (i): For the left-to-right direction, we use that $p \vee p \vdash_{\vee} p$, $p \vdash_{\vee} p \vee p$, $p \vee q \vdash_{\vee} q \vee p$, $p \vee (q \vee r) \vdash_{\vee} (p \vee q) \vee r$, and $(p \vee q) \vee r \vdash_{\vee} p \vee (q \vee r)$. (Note that these are single-premise consequences, so lower bounds are not needed.)

(ii): Using the derivability facts of $\vdash_{\vee}$ listed above, it is not hard to verify:

(19) $\quad \psi_1, \ldots, \psi_n \vdash_{\vee} \varphi$ iff there is ψ_i such that each atom in ψ_i occurs in φ.

Now let $\mathcal{F}$ be any poset such that there is a nucleus j on $Up(\mathcal{F})$ which is *not* the identity, and let $I^{\mathcal{F}}(\vee)(U, V) = j(U \cup V)$. Using (19) and the monotonicity of j, it is easy to see that $I^{\mathcal{F}}$ is consistent with $\vdash_{\vee}$, but $I^{\mathcal{F}}$ is not the standard Kripke interpretation.[21] $\square$

Interestingly, we have here a failure of categoricity in a set-based semantics for a weak sublanguage of $L_{\bot\top\wedge\vee\rightarrow}$, a failure which disappears when more connectives are added. Note that this is another illustration of the asymmetry between $\wedge$ and $\vee$. Note also that although $\vdash_{\vee}$ is a fragment of both $\vdash_{\mathsf{IPC}}$ and $\vdash_{\mathsf{CPC}}$, the failure of categoricity occurs in a set-based semantics but not in classical two-valued semantics. Indeed, the introduction and elimination rules for $\vee$ uniquely fix the standard truth table for $\vee$.[22]

4.4. $\vdash_{\wedge\vee}$ *and* $\vdash_{\neg\wedge\vee}$

The logic $\vdash_{\neg\wedge\vee}$ is another interesting fragment of $\vdash_{\mathsf{IPC}}$. Since implication is missing, we use $\neg$ in place of $\bot$ (see note 21). However, conjunction is present, so the definitions of *algebraic interpretations, consistency, $\mathcal{A}$-interpretations*, and *$\mathcal{A}$-consistency* are just as before, in both cases. The *standard* algebraic interpretations are *distributive lattices* in the case of $\vdash_{\wedge\vee}$, and *pseudo-complemented distributive lattices* in the case of $\vdash_{\neg\wedge\vee}$: pseudo-complementation means that for each $a \in A$, the maximum of $\{b\colon a \wedge b = 0\}$ belongs to A.[23]

The distributive lattices consistent with $\vdash_{\wedge\vee}$ need not be bounded, for example, any linearly ordered set is consistent with $\vdash_{\wedge\vee}$; note that this logic has no theorems. Pseudo-complemented distributive lattices are bounded.

The relevant facts about distributivity and pseudo-complementation are easily seen to be derivable in the respective logics, and we have:

Fact 4.4. (i) *$\mathcal{A}$ is consistent with $\vdash_{\wedge\vee}$ iff $\mathcal{A}$ is a distributive lattice.*
(ii) *$\mathcal{A}$ is consistent with $\vdash_{\neg\wedge\vee}$ iff $\mathcal{A}$ is a pseudo-complemented distributive lattice.*

However, as in the case of $\vdash_{\vee}$, there seems to be no way to prove categoricity for $\vee$ in these logics, at least not along the lines we have tried so far.

(20) **Open problem**
Are the logics $\vdash_{\wedge\vee}$ *and (more interestingly)* $\vdash_{\neg\wedge\vee}$ *categorical for set-based semantics?*

5. Discussion

We find it rather remarkable that IPC is Carnap categorical. Many scholars think that the intuitionistic meaning of the connectives is best captured by the informal Brouwer–Heyting–Kolmogorov explanation, and that the familiar formal semantics for which IPC is sound and complete fail in various degrees to do justice to that explanation. For example, the BHK explanation of the meaning of $\to$ uses notions such as 'proof' and 'construction': a proof of $\varphi \to \psi$ is a construction that takes any proof of φ to a proof of ψ. The usual set-based semantics, such as nuclear semantics, involve no similar notions. Nor do they rely on facts about verification or assertion, although one can argue quite convincingly, as BH19 does building on Dummett and others, that some instances of nuclear semantics *represent* such facts rather accurately.

5.1. Carnap's Question and the Meaning of the Connectives

However, Carnap's question, as we construe it here, is a model-theoretic question. If you will, it is about the *extension* of the meaning of the connectives. Then, relative to certain set-theoretic objects taken to be *semantic values* of sentences (the 'extensional part' of sentence meanings), there is nothing more to say about those extensions than how they determine the semantic value of a complex sentence from the semantic values of its (immediate) constituents. In set-based semantics for intuitionistic logic, the values are certain subsets of some domain, so the connectives must be interpreted as functions on those values. Moreover, what we find remarkable is that, in every set-based semantics as defined here, regardless of how the appropriate semantic values are selected—for example, by choosing a nucleus on the set of upsets of a poset or by choosing a topology—the consequence relation $\vdash_{\mathsf{IPC}}$ *uniquely fixes*, among all the in-principle available options, the functions interpreting the connectives on those values.

Interestingly, these facts are all explained by a single semantic result about Heyting algebras, i.e., Theorem 3.6, which has nothing per se to do with subsets of domains.

Much less surprising is the fact that the class *HA* of Heyting algebras is exactly the right choice: Fact 3.4. Still, facing the extensive literature on algebraic semantics for intuitionistic logic, someone might naively ask: why Heyting algebras? Are there no other possible algebraic interpretations? No: If you want to work with a class C of algebras—algebraic interpretations in our sense—all of whose members are consistent with $\vdash_{\mathsf{IPC}}$, then $C \subseteq HA$. Indeed, *HA* is the *largest* such class, and similarly for $\vdash_{\mathsf{CPC}}$ and Boolean algebras.

These facts have nothing to do with completeness. To see this, define for each algebraic interpretation $\mathcal{A}$ the semantic consequence relation $\models_{\mathcal{A}}$ by

(21) $\psi_1, \ldots, \psi_n \models_{\mathcal{A}} \varphi$ iff for all valuations v on A and all $c \in A$, $c \leq^{\mathcal{A}} [\psi_i]^{\mathcal{A}}_v$ for $i = 1, \ldots, n$ implies $c \leq^{\mathcal{A}} [\varphi]^{\mathcal{A}}_v$.

Thus, $\mathcal{A}$ is *consistent* with $\vdash$ iff $\vdash \subseteq \models_{\mathcal{A}}$, and $\vdash$ is *sound and complete* for a class C of algebras iff $\vdash \subseteq \bigcap_{\mathcal{A}\in C} \models_{\mathcal{A}}$ (soundness) and $\bigcap_{\mathcal{A}\in C} \models_{\mathcal{A}} \subseteq \vdash$ (completeness). So, while it happens to be true that *HA* is the largest class for which $\vdash_{\mathsf{IPC}}$ is sound and complete, it is also the largest class for which $\vdash_{\mathsf{IPC}}$ is sound, by Theorem 3.6(i). Completeness is not needed to single out *HA*.

Further, the abstract completeness of a logic, in the sense of its set of theorems (or its consequence relation) being recursively enumerable, is irrelevant to categoricity. For example, every intermediate logic is categorical with respect to set-based semantics, but there are uncountably many intermediate logics, and, hence, uncountably many whose set of theorems is not recursively enumerable.

Going beyond propositional logic, a concrete example of an incomplete logic that is Carnap categorical (the meaning of the logical constants is fixed by the standard consequence relation) is the logic $\mathcal{L}(\mathcal{Q}_0)$, which is a classical first-order logic with the additional quantifier 'there are infinitely many'.[24]

5.2. Further Questions

Here are some final observations and open questions.[25]

1. One can weaken the logics $\vdash_\Phi$ defined in Section 4 by constraining the inference rules in various ways. A case in point is [20], which studies a weaker logic in $L_{\neg\wedge\vee}$, called $\vdash_\mathsf{F}$, with restrictions on the $\vee$E and $\neg$I rules. Among other things, distributivity no longer holds.[26] Holliday shows how $\vdash_\mathsf{F}$ can be seen as a neutral base logic, from which intuitionistic logic, versions of the *orthologic* studied in [21], and classical logic, can be obtained by suitable additions or changes to the rules or by corresponding constraints in the algebraic or the relational semantics he provides for $\vdash_\mathsf{F}$. Finding a non-standard interpretation consistent with $\vdash_\mathsf{F}$—if there is such an interpretation—might be easier than finding one for $\vdash_{\neg\wedge\vee}$.

In the same spirit, we may ask whether there is a proper fragment of $\vdash_{\mathsf{IPC}}$ in the language $L_{\bot\top\wedge\vee\to}$ that is Carnap categorical. If no such fragment exists, that would be a new kind of functional completeness property of IPC.

2. Theorem 3.6 is an existence and uniqueness result: *there is* an interpretation consistent with $\vdash_{\mathsf{IPC}}$ (namely, the standard interpretation), and it is in fact the *only* one. In the proof-theoretic tradition, the existence and uniqueness of propositional connectives is a well-established topic; see [12], Ch. 4, for a comprehensive overview. Here, the existence and uniqueness is relative to a set of *rules*. For example, $\wedge$ and $\vee$ are *unique* relative to a natural deduction presentation of IPC, in the sense that if we introduce new connectives $\wedge'$ and $\vee'$, with the same rules as for $\wedge$ and $\vee$, respectively, then $\varphi \wedge \psi$ is equivalent to $\varphi \wedge' \psi$, and similarly for $\vee$ and $\vee'$. Indeed, we only need the introduction and elimination rules for these two connectives. With the notation from Section 4, we have: $\varphi \wedge \psi \dashv\vdash_{\wedge\wedge'} \varphi \wedge' \psi$ and $\varphi \vee \psi \dashv\vdash_{\vee\vee'} \varphi \vee' \psi$.
As Humberstone noted, and [22] spells out in detail, there is a difference between $\wedge$ and $\vee$ here: while the former is *implicitly definable* in a precise sense, the latter, although unique, is not. Došen and Schroeder-Heister explore connections to Beth's Definability Theorem.[27]
It is an intriguing question whether, and in that case how, their proof-theoretic notion of implicit definability relates to our semantic notion of categoricity (which exhibits a similar contrast between $\wedge$ and $\vee$).

3. We have so far followed the lead of [5] and other papers in considering consequence relations $\vdash$ where the first argument is a (possibly empty) *set* of formulas and the second is a *formula*; the SET-FMLA setting in the terminology of [12]. One could look at other settings, such as FMLA (no premises), SET-SET, FMLA-FMLA, or SEQ-FMLA (with a sequence of premises). As we have seen, the categoricity results would differ significantly. We think SET-FMLA most naturally corresponds to intuitive ideas about 'what follows from what', but one would like a better argument for why the others are less suitable. Alternatively, and less contentiously, an overview of what the results would be in those settings.[28]

4. In all categoricity results (in our sense) that we are aware of, it is practically immediate that any interpretation consistent with the relevant consequence relation $\vdash$ must interpret conjunction ($\wedge$) standardly. It is the other logical constants that need some work (except for $\bot$ if *Ex Falso* holds). However, one could argue that this is built into our definition of consistency with $\vdash$ in Definition 2.1: the *intersection* of the values of the premises must be included in the value of the conclusion. This guarantees that $U \cap V \subseteq I(\wedge)(U, V)$; the converse inclusion is from single-premise $\wedge$-elimination.
By contrast, the semantics for IPC presented in [24] might well be called a set-based semantics, since the semantic values are sets of states, but it does not fit our

Definition 2.1.[29] The relevant class of structures is the class of *complete, residuated partial orders* $\mathcal{F} = (X, \leq)$, where completeness, as usual, means that least the upper bounds ($\sqcup$) and greatest lower bounds ($\sqcap$) of all subsets of X exist. [30] Fine's (standard) interpretation of conjunction is as follows:

(22) $\llbracket \varphi \wedge \psi \rrbracket^{\mathcal{F}}_{v} = \{x \sqcup y \colon x \in \llbracket \varphi \rrbracket^{\mathcal{F}}_{v} \text{ and } y \in \llbracket \psi \rrbracket^{\mathcal{F}}_{v}\}$

$x \sqcup y$ is thought of as the 'fusion' of x and y. Accordingly, φ is called an *exact consequence* of $\psi_1, \ldots, \psi_n$ if (for all $\mathcal{F}$ and v) $\sqcup\{x_1, \ldots, x_n\} \in \llbracket \varphi \rrbracket^{\mathcal{F}}_{v}$ whenever $x_i \in \llbracket \psi_i \rrbracket^{\mathcal{F}}_{v}$, for $i = 1, \ldots, n$. Fine shows that $\vdash_{\mathsf{IPC}}$ is sound and complete for this notion of semantic consequence. Although Definition 2.1 does not apply here (since it *intersects* the semantic values of the premises, rather than 'fusing' their members), one could well raise the issue of categoricity with respect to Fine's semantics. However, the results in this note say nothing about that.

5. Another natural line of inquiry would be to investigate categoricity for intuitionistic *predicate* logic. To what extent do the results of classical predicate logic ([5]) carry over?

We feel these issues deserve more thought but leave them for future work.

Author Contributions: Main result, H.T. and D.W.; original draft preparation, review and editing, D.W. all authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: We thank Valentin Goranko and Lloyd Humberstone for their helpful comments at a (much) earlier stage of this work and Johan van Benthem, Denis Bonnay, Valentin Goranko (again), Wesley Holliday, and two anonymous referees for their constructive remarks on the present paper. Special thanks to Wes for several helpful conversations and for allowing us to present his Example 3.10, and to one of the referees for a comment that led to a substantial rewrite of the first version.

Conflicts of Interest: The authors declare no conflict of interest.

Notes

1 Compositionality is by now an accepted requirement on formal semantics, but the idea was not around in 1943. The result as stated presupposes that propositional atoms are treated as *variables*, as we do here. Otherwise, one would have to exclude by stipulation the non-standard compositional interpretations making all sentences true, since these would also be consistent with $\vdash_{\mathsf{CPC}}$.

2 Most but not all: a semantics for IPC, to which our results do not apply, is given in Section 5.

3 There are several equivalent ways of defining Heyting algebras; see BH19, Section 2.1.

4 In earlier work we used 'possible worlds semantics' rather than 'set-based semantics', but this is potentially misleading. Indeed, the elements of the domains in the structures in $\mathcal{S}$ need not have anything to do with worlds, or states, or possibilities. A more accurate label, in view of clause 6 of the definition, would be *intersective set-based semantics*, but we find this too clumsy. An example of a semantics for IPC where semantic values are sets but which is not set-based in our sense is mentioned Section 5.2:4.

5 We use $[\cdot]^{\cdot}_{v}$ for semantic values in algebraic semantics, and $\llbracket\cdot\rrbracket^{\cdot}_{v}$ for values in set-based semantics.

6 BH19 lets valuations be functions v from *Prop* to $Up(\mathcal{F})$ (which they write as $Up(X)$) and then defines $\llbracket p \rrbracket^{\mathcal{F},j}_{v} = jv(p)$. Since the standard interpretation of the connectives takes pairs of fixed upsets to fixed upsets, all semantic values belong to $Up(\mathcal{F})_j$. So, we obtain the same result by taking valuations to map propositional variables directly to fixed upsets, as done here.

7 A Heyting algebra $\mathcal{A}$ is *complete* if $\bigvee^{\mathcal{A}} B$ exists for every $B \subseteq A$ (and hence $\bigwedge^{\mathcal{A}} B$ also exists). The Heyting algebras exemplified in this note are in fact complete, but this will not play a role for our purposes here.

8 So $\Omega(X) \subseteq \mathcal{P}(X), \varnothing, X \in \Omega(X)$, and $\Omega(X)$ is closed under finite intersections and arbitrary unions.

9 We are now in the framework that Humberstone [12] calls $\leq$*-based algebraic semantics*. The definition is suggested by his Remark 2.14.7(i).

10 Theorem 3.6 generalizes our original formulation of Carnap categoricity for IPC, and was suggested to us by an anonymous referee, to whom we are most grateful.

11 It is easy to find examples of algebras $\mathcal{A}_1, \mathcal{A}_2$ of signature $\{\bot, \top, \wedge, \vee\}$ such that $\mathcal{A}_1$ is a bounded lattice, $A_1 = A_2$, $\wedge^{\mathcal{A}_1} = \wedge^{\mathcal{A}_2}$, and $\leq^{\mathcal{A}_1} = \leq^{\mathcal{A}_2}$, but $\vee^{\mathcal{A}_2}$ is not the least upper bound operator.

12 Dummett algebras have an interesting connection to a proposal in [13] to explain the significance of Beth semantics in terms of a distinction between a formula being *verified* and *assertible* at a stage x (BH19, end of section 3.2)).

13 BH19 gives a simple example of a Dummett algebra that is not a Heyting algebra (just before Theorem 3.25), using the Beth nucleus j_B on $Up(\mathcal{F})$, where $\mathcal{F}$ is the poset $\{0,1\}$ with the standard order, so $Up(\mathcal{F})$ is $\{\emptyset, \{1\}, \{0,1\}\}$ with the inclusion order. However, this does not yield a non-standard *Beth interpretation*, since the non-standard interpretation of $\to$ as in (12) does not carry over to $Up(\mathcal{F})_{j_B} = \{\emptyset, \{0,1\}\}$.

14 The example was originally intended to show that the stronger notion of consistency is needed for the result in [5] about CPC; see Remark 3.9. Here, we have adapted it to IPC.

15 g is similar to the *Kolmogorov translation*, say, G, which puts a double negation in front of all subformulas. One can show that if φ is not an atom, $\vdash_{\mathsf{IPC}} g(\varphi) \leftrightarrow G(\varphi)$. It is well known that $\vdash_{\mathsf{CPC}} \varphi \Leftrightarrow \vdash_{\mathsf{IPC}} G(\varphi)$, and since no atom is a theorem, (17) follows. [14] is a survey of various negative translations.

16 Indeed, it validates all CPC theorems, by (17), and since the above argument in fact works for all valuations $v\colon Prop \to \mathcal{P}(X)$. As Wesley Holliday pointed out, $I^{\Omega(X)}$ evaluates complex formulas using the *double negation nucleus*, i.e., in the Heyting algebra of *regular open* sets (sets U such that $U = int(cl(U))$), which is a Boolean algebra.

17 In most (all?) cases, $\vdash_\Phi$ turns out to be the restriction of $\vdash_{\mathsf{IPC}}$ to L_Φ.

18 For example, let $\mathcal{A} = (\{a,b\}, \wedge^{\mathcal{A}})$, where (dropping superscripts) $a \wedge a = a$, $b \wedge b = b$, $a \wedge b = b$, $b \wedge a = a$. Then $x \leq y$, i.e., $x \wedge y = x$, holds iff $x = y$, so $\leq$ is a partial order, but $\wedge$ is not commutative in $\mathcal{A}$.

19 We skip the brackets and commas when writing $\vdash_\Phi$.

20 If we add $\bot$ to the signature as the smallest element, we have negation, and contradictions, in the language, so e.g., $(a \to a) \to \bot = \bot$. The results corresponding to Fact 4.2 below still hold, by an easy addition to the proof.

21 This simple non-standard interpretation was suggested by Wesley Holliday, and replaces a more ad hoc construction than we originally gave. Using another ad hoc construction, one can give a non-standard Kripke interpretation of $\neg$, which is consistent with $\vdash_\neg$, but we omit the proof of that here. Here, $\vdash_\neg$ is defined with the two natural deduction rules

$$\frac{\psi \quad \neg\psi}{\varphi} \qquad\qquad \frac{\begin{array}{cc}[\varphi] & [\varphi] \\ \vdots & \vdots \\ \psi & \neg\psi\end{array}}{\neg\varphi}$$

22 The rows 111, 101, 011, are fixed by $p \vdash_\vee p \vee q$ and $q \vdash_\vee p \vee q$. The row 000 is fixed by $p \vee p \vdash_\vee p$, which is an instance of $\vee$-elimination.

23 Here 0 is identified with $a \wedge \neg a$. The variety of pseudo-complemented lattices can also be defined by adding the following equations to those for distributive lattices:

a. $0 \wedge a = 0$
b. $a \wedge \neg(a \wedge b) = a \wedge \neg b$
c. $a \wedge \neg 0 = a$
d. $\neg\neg 0 = 0$

See [17] for $\vdash_{\neg\wedge\vee}$, and [18] for $\vdash_{\wedge\vee}$, or consult [19].

24 This was observed by the second author and is stated and generalized in [7]. The set of valid sentences in $\mathcal{L}(\mathcal{Q}_0)$ is not recursively enumerable.

25 Questions and comments from several people inspired the remarks in this subsection, in particular from Wes Holliday for part 1 and from Johan van Benthem for part 2.

26 Holliday presents $\vdash_{\mathsf{F}}$ with a Fitch-style natural deduction system, where the added constraint becomes a requirement on the Reiteration rule. $\vdash_{\mathsf{F}}$ is also extended to a first-order language with the quantifiers $\forall$ and $\exists$ (but without $\to$ and $=$). Failure of distributivity is a characteristic of quantum logic, but Holliday argues that it also accords with certain facts about natural language semantics, in particular facts about epistemic modals.

27 Humberstone [12], Chapter 4.35, proves a similar result for $\vee$: it is not uniquely characterizable by what he calls *zero-premise rules*. See also the discussion in [23], pp. 110–114.

28 In fact, the solution proposed in Carnap [1] to the existence of non-normal interpretations was in effect to use the SET-SET framework instead. This makes good sense for classical logic. However, for intuitionistic logic there is, as far we know, no obviously correct candidate for the SET-SET version of IPC. For example, $p \vee q \vdash p, q$ is not valid in standard Beth semantics (since we can have $j_b(U \cup V) \not\subseteq U \cup V$), although it holds in Kripke semantics.

29 We thank an anonymous referee for directing our attention to Fine's semantics, which is based on partial orders, but where semantic values are not upsets.

30 Fine defines, for $x, y \in X$,

$$x \to y = \sqcap\{z \in X \colon y \leq x \sqcup z\}$$

and calls $\mathcal{F}$ *residuated* if it always holds that $y \leq x \sqcup (x \to y)$. As pointed out in BH19 (Section 2.3), Fine's frames are not complete Heyting algebras but complete *co-Heyting* algebras. Fine argues that his semantics comes closer to the BHK interpretation of the connectives than other formal semantics for IPC, in view of the clause (22) for conjunction, as well as the clause for implication:

$$[\![\varphi \to \psi]\!]_v^{\mathcal{F}} = \{\sqcup\{x \to f(x) \colon x \in [\![\varphi]\!]_v^{\mathcal{F}}\} \colon f \text{ is a function from } [\![\varphi]\!]_v^{\mathcal{F}} \text{ to } [\![\psi]\!]_v^{\mathcal{F}}\}$$

References

1. Carnap, R. Formalization of Logic. In *Studies in Semantics*; Cambridge University Press: Cambridge, MA, USA, 1943; Volume 2.
2. Raatikainen, P. On the rules of inference and the meaning of logical constants. *Analysis* **2008**, *68*, 282–287. [CrossRef]
3. Murzi, J.; Hjortland, O. Inferentialism and the categoricity problem: Reply to Raatikainen. *Analysis* **2009**, *69*, 480–488. [CrossRef]
4. Murzi, J.; Topey, B. Categoricity by convention. *Philos. Stud.* **2021**, *178*, 3391–3420. [CrossRef] [PubMed]
5. Bonnay, D.; Westerståhl, D. Compositionality solves Carnap's Problem. *Erkenntnis* **2016**, *81*, 721–739. [CrossRef]
6. Bonnay, D.; Westerståhl, D. Carnap's Problem for modal logic. *Rev. Symb. Log.* **2021**, *16*, 578–602. [CrossRef]
7. Speitel, S. Logical Constants between Inference and Reference. Ph.D. Thesis, UCLA at San Diego, San Diego, CA, USA, 2020.
8. Bezhanishvili, G.; Holliday, W.H. A semantic hierarchy for intuitionistic logic. *Indag. Math.* **2019**, *30*, 403–469. [CrossRef]
9. Bezhanishvili, G.; Holliday, W.H. Locales, nuclei, and Dragalin frames. In *Advances in Modal Logic*; Beklemishev, L., Demri, S., Maté, A., Eds.; College Publications: London, UK, 2016; Volume 11, pp. 177–196.
10. Stone, M. Topological representation of distributive lattices and Brouwerian logics. *Časopis Pro Pěstování Mat. A Fys.* **1937**, *67*, 1–25. [CrossRef]
11. Tarski, A. Der Aussagenkalkül und die Topologie. *Fundam. Math.* **1938**, *31*, 103–134. [CrossRef]
12. Humberstone, L. *The Connectives*; MIT Press: Cambridge, MA, USA, 2011.
13. Dummett, M. *Elements of Intuitionism*, 2nd ed.; Clarendon Press: Oxford, UK, 2000.
14. Ferreira, G.; Oliva, P. On various negative translations. In *Logic, Construction, Computation*; Berger, U., Diener, H., Schuster, P., Seisenberger, M., Eds.; De Gruyter: New York, NY, USA, 2012.
15. Köhler, P. Brouwerian semi-lattices. *Trans. Am. Math. Soc.* **1981**, *268*, 103–126. [CrossRef]
16. Nemitz, W.C. Implicative semi-lattices. *Trans. Am. Math. Soc.* **1965**, *117*, 128–142. [CrossRef]
17. Rebagliato, J.; Verdú, V. On the algebraization of some Gentzen systems. *Fundam. Inform.* **1993**, *18*, 319–338. [CrossRef]
18. Font, J.M.; Verdú, V. Algebraic logic for classical conjunction and disjunction. *Stud. Log.* **1991**, *50*, 393–419. [CrossRef]
19. Font, J.M. *Abstract Algebraic Logic*; College Publications: London, UK , 2016.
20. Holliday, W.H. A fundamental non-classical logic. *Logics* **2023**, *1*, 36–79. [CrossRef]
21. Goldblatt, R. Semantic analysis of orthologic. *J. Philos. Log.* **1974**, *3*, 19–35. [CrossRef]
22. Došen, K.; Schroeder-Heister, P. Uniqueness, definability and interpolation. *J. Symb. Log.* **1988**, *53*, 554–570. [CrossRef]
23. Williamson, T. Equivocation and existence. *Proc. Aristot. Soc.* **1988**, *88*, 109–127. [CrossRef]
24. Fine, K. Truth-maker semantics for intuitionistic logic. *J. Philos. Log.* **2014**, *43*, 549–577. [CrossRef]