Nonlinear Extended State Observer Based Prescribed Performance Control for Quadrotor UAV with Attitude and Input Saturation Constraints

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Abstract: In this paper, a prescribed performance control scheme of the quadrotor unmanned aerial vehicle (UAV) under attitude and input saturation constraints is introduced. According to the underactuated feature, the quadrotor UAV system can be decomposed into an underactuated subsystem and a fully actuated subsystem. With the feedback linearization technique, a single nonlinear extended state observer (ESO) is proposed, and multiple observations are utilized to estimate both matched and unmatched disturbances, which not only can obtain a uniform convergence, but also reduces the complexity of the observer’s parameter adjustment. To improve system stability, an input saturation algorithm for each single rotor is introduced to modify the final control output. In addition, the limited attitude for the quadrotor UAV is also considered as a saturation constraint in the control scheme with a compensation auxiliary system. On this basis, dynamic surface control (DSC) with prescribed performance is adopted to guarantee the bounded convergence and steady-state error. All state errors of the closed-loop system are proven to be uniformly bounded using the Lyapunov theory, and the simulation results are given to demonstrate the stability, effectiveness, and superiority of the proposed control strategies at last.

Keywords: quadrotor UAV; extended-state observer; prescribed performance control; input saturation; state constraint

1. Introduction

Unmanned aerial vehicles (UAVs) have attracted significant attention from research institutes, companies and industries over the last decade, owing to their wide applications in environment mapping, safety monitoring, disaster preventive management, service delivery, etc. [1]. The quadrotor UAV is the most widely used vertical take-off and landing aircraft with flexible maneuverability.

For the quadrotor UAV, the highly nonlinear underactuated system has led to essential challenges in designing effective flight controller with robust stability and high performance. Many linear techniques have been designed to reach a full control of the quadrotor UAV, such as the cascade PID control [2,3] and LQR control [4]. However, for such a highly nonlinear system, the linearized control methods have great limitations for the analysis of system robustness and anti-interference ability. Therefore, to overcome the limitations of the linear control methods, nonlinear control methods, such as feedback linearization [5–7], backstepping approach [8,9], sliding mode control (SMC) [10–12], observer-based robust control [13–15] or some adaptive control approaches [16–19] have been investigated to achieve good stability and tracking performance. Additionally, research works, such as refs. [20–22], also introduced the prescribed performance method to manipulate the
convergence rate and the steady-state error inside the prescribed bounds. Among the afore-mentioned methods, observer-based backstepping approaches have been widely adopted by researchers and demonstrate great advantages in the control of the quadrotor UAV system [9,23]. However, there is a significant drawback in that the backstepping control is sensitive to the initial error due to the repeated differentiation of the virtual controls [24]. To solve this limitation, the dynamic surface control (DSC) [25,26] is represented in this work and integrated with the prescribed performance technique to guarantee the bounded convergence and steady-state errors.

Integrated with the microelectronic mechanical system (MEMS), positioning technologies and fusion algorithms, the real-time translational and rotational states of the quadrotor UAV can be obtained for state and attitude estimation [27–29]. Various state or disturbance observers were integrated with controllers to enhance stability and performance robustness [30]. For instance, in ref. [31], Liu et al. designed two novel finite-time disturbance observers for position dynamics and attitude dynamics separately. The selection of control gains is improved to be mildly greater than the observation error instead of disturbances, which is more practical for implementation, but the system states are directly obtained from measurements. To observe both the states and disturbances, in [32], Xi et al. proposed an adaptive sliding mode disturbances observer for a robot manipulator system. The backstepping-based auxiliary system with error feedback is used as a system state observer. Compared to the disturbance observer, the extended state observer (ESO) [33], with the attractive advantages of having a concise structure and the ability to estimate both the states of the dynamics and the lumped disturbances via the system outputs, is widely studied [22,34,35]. In ref. [15], Zhang et al. applied six conventional ESOs to estimate the disturbances exist on quadrotor UAV system. Liu et al. [36] designed a finite-time ESO for attitude tracking of a quadrotor UAV using angular rates as the observer feedback. In the work of [37], Niu et al. applied the finite-time ESO to estimate the disturbance for the terminal sliding mode surface. Although the above-mentioned schemes have achieved the observation purpose, all these ESOs are only effective on integral-chain systems with matched disturbance using linear observer gains and a single measurement. The quadrotor UAV dynamics is usually decomposed into an underactuated subsystem and a fully actuated (UF) subsystem [38], where the former is a fourth-order pure-feedback system with both matched and unmatched disturbances. To compensate for the matched or unmatched disturbance from output, Chen et al. proposed two generalized approaches on unmatched disturbances situation based on ESO [39] and nonlinear disturbance observer [40]: the unmatched disturbance is effectively attenuated from output via a disturbance compensation gain. However, the estimation of both matched and unmatched disturbances with only one observer is still challenging, especially for the quadrotor UAV system. Consequently, in this work, we proposed a nonlinear ESO to approximate both the matched and unmatched disturbance estimation problems of the quadrotor UAV system.

The input saturation caused by limited propeller thrust forces often constrains the performance severely, and even dominates the stability of the quadrotor UAV [41]. Thus, the input constraints are practically important issues for control problems, and extensive research works have considered stabilizing a nonlinear system with input saturation [42–44]. In ref. [41], Wang et al. introduced the hyperbolic function and Nussbaum function to designed priori-bounded control inputs for the trajectory tracking control of quadrotor UAV. Liu et al. in [23] applied two input saturation functions for position dynamics and attitude dynamics separately with two auxiliary control systems for compensation of the saturation effects. However, all these researchers only focused on the synthetic torque and force control input, but for the quadrotor UAV system, the limited force generated from each single rotor presents a more practical constraint rather than the synthetic torque and force control input. Thus, an input saturation algorithm on each single rotor of quadrotor UAV is proposed and an auxiliary system is designed in this work to compensate for the saturation effects.
A quadrotor UAV mounted with precise inspection instrument could fit the requirements for conducting healthy monitoring in limited spaces, such as tunnel safety patrolling, as shown in Figure 1. Highly aggressive maneuvers are not allowed, and the body inclination angle $\phi$ has to be within safety ranges. The large inclination angle of the quadrotor UAV will also cause insufficient lift force and further affect the system stability and performance. Thus, the attitude constraints turned out to be an important issue. Nevertheless, the input saturation for the position dynamics were considered in refs. [23,41], which can be seen as a bound for calculating the desired attitude angles, but did not directly consider the attitude constraints in designing the trajectory tracking controller for the quadrotor UAV. To realize the attitude constraint for quadrotor UAV, a saturation function is added to the inclination angle of the body with an auxiliary system.

Motivated by the above observations, the input saturation and prescribe performance control were taken into account in most existing research works. It is of practical significance to consider the attitude constraint for the control of the quadrotor UAV. Compared to other control schemes, a backstepping-typed DSC approach is more suitable for implementing the attitude constraint with a compensation auxiliary system. In this work, we focus on the prescribed performance control issue for the quadrotor UAV in the presence on both input saturation and attitude constraint, where the uncertainties and disturbances are approximated and attenuated through a novel nonlinear ESO base DSC. The main contributions of this paper are as follows:

1. The control scheme is developed by the DSC technique with two auxiliary systems designed for attitude and input saturation constraints. Additionally, the prescribed performance method provides a more intuitive way to adjust the tracking speed and steady-state error.
2. Considering the limitations of existing ESOs, two nonlinear ESOs are developed for approximating the pure-feedback subsystems of quadrotor UAV. Under such scheme, only one ESO is utilized for each quadrotor UF subsystem to estimate both the matched and unmatched disturbances with multiple state observations. Thus, a uniform convergence speed can be obtained, and the complexity of the observer’s parameter adjustment are reduced compared to conventional ESO designs.
3. To improve the control stability of quadrotor UAV, the input saturation constraint is modified to exert on the thrust force generated by each rotor rather than the synthetic torque and force control inputs. Furthermore, the attitude constraint is firstly taken into account for stabilizing a quadrotor UAV. The constraint is realized by a saturation function with an auxiliary system as compensation to keep the inclination of quadrotor UAV within safety region.

The rest of this paper is organized as follows. In Section 2, problem formulation and preliminaries are illustrated for a typical configuration of quadrotor UAV. Section 3 introduces and proves a nonlinear ESO with multi-measurement feedback and varying-observer gain. The main results, which are the design of proposed controller, with stability analysis of the closed-loop system including the ESO system, are provided in Section 4.
Then, several cases are simulated to validate the effectiveness of the proposed method in Section 5. The conclusion is given in Section 6.

2. Problem Formulation and Preliminaries

The dynamics model of a typically configured quadrotor UAV, shown in Figure 2, is well established in many works in the literature [45]. The position and attitude vectors are represented by \( \xi \equiv [x, y, z]^T \) and \( \eta \equiv [\phi, \theta, \psi]^T \), separately. The three orientation angles are referred to as roll \((-\frac{\pi}{2} < \phi < \frac{\pi}{2})\), pitch \((-\frac{\pi}{2} < \theta < \frac{\pi}{2})\) and yaw \((-\pi \leq \psi < \pi)\). The translational velocity vector \( \dot{v} \equiv [v_x, v_y, v_z]^T \) is given in an earth-fixed coordinate, and the angular velocity vector \( \dot{\omega} \equiv [p, q, r]^T \) is defined according to the body-fixed coordinate. Then, the dynamics of the quadrotor UAV are described as follows:

\[
\begin{align*}
\dot{\xi} &= v \\
\dot{v} &= m^{-1}R_t u_\xi - g e_3 + d_\xi \\
\eta &= R_r \omega \\
\dot{\omega} &= J^{-1} u_\eta + C_\eta + d_\eta
\end{align*}
\]  

(1)

where \( R_t \) and \( R_r \) are the translation velocity matrix and rotation velocity matrix [45], \( m \) and \( J = \text{diag}(J_x, J_y, J_z) \) are the body mass and inertia of the quadrotor UAV, \( g \) is the gravity constant, \( d_\xi = [d_x, d_y, d_z]^T \) and \( d_\eta = [d_\phi, d_\theta, d_\psi]^T \) represent the lumped variables of force disturbances and torque disturbances [46], \( C_\eta = -\omega \times J \omega \) denotes the body gyroscopic effect term, and \( u_\xi = [0, 0, \tau_f]^T \) and \( u_\eta = [\tau_p, \tau_q, \tau_r]^T \) are the force and torque with respect to the body-fixed coordinate, respectively.

![Figure 2. Quadrotor UAV configuration.](image)

2.1. Problem Formulation

In order to implement control strategies for quadrotor UAV, the UF decomposition is adopted with state vectors \( x_1 = [x, y]^T \), \( x_2 = [v_x, v_y]^T \), \( x_3 = [\phi, \theta]^T \), \( x_4 = [p, q]^T \), \( x_5 = [\psi, z]^T \) and \( x_6 = [r, v_z]^T \). Accordingly, the underactuated subsystem is defined as

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g_1 u(\varphi_1) + d_1 \\
\dot{x}_3 &= g_2 x_4 + d_2 \\
\dot{x}_4 &= g_3 q_2 + d_3
\end{align*}
\]  

(2)

and a fully actuated subsystem is defined as

\[
\begin{align*}
\dot{x}_5 &= g_4 x_6 + d_4 \\
\dot{x}_6 &= g_5 q_3 + d_5
\end{align*}
\]  

(3)
where the parameters \( g_i, i = 1 \ldots 5 \) are

\[
\begin{align*}
g_1 &= \frac{\tau_f}{m} \begin{bmatrix} S_{\phi} & C_{\phi} \\ -C_{\phi} & S_{\phi} \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & S_{\phi} T_0 \\ 0 & C_{\phi} \end{bmatrix}, \quad g_3 = \begin{bmatrix} \frac{1}{\tau_r} & 0 \\ 0 & \frac{1}{\tau_f} \end{bmatrix}, \\
g_4 &= \begin{bmatrix} C_{\phi} S_{\theta} & 0 \\ 0 & 1 \end{bmatrix}, \quad g_5 = \begin{bmatrix} \frac{1}{\tau_c} & 0 \\ 0 & \frac{c - \phi}{m} \end{bmatrix},
\end{align*}
\]

and the lumped disturbances \( d_i, i = 1 \ldots 5 \) are

\[
\begin{align*}
d_1 &= \begin{bmatrix} -d_x \\ -d_y \end{bmatrix}, \\
d_2 &= \begin{bmatrix} C_{\phi} T_0 r \\ -S_{\phi} r \end{bmatrix}, \\
d_3 &= \begin{bmatrix} l_x - \frac{l_y}{2} q r - d_{\phi} \\ l_x \frac{l_y}{2} p r - d_{\theta} \end{bmatrix}, \\
d_4 &= \begin{bmatrix} S_{\phi} S_{\theta} q \\ 0 \end{bmatrix}, \\
d_5 &= \begin{bmatrix} l_x - \frac{l_y}{2} p q - d_{\psi} \\ -g - d_z \end{bmatrix},
\end{align*}
\]

and the control inputs \( \varphi_i, i = 1, 2, 3 \) are

\[
\varphi_1 = \begin{bmatrix} S_{\phi} \\ C_{\phi} S_{\theta} \end{bmatrix}, \quad \varphi_2 = \begin{bmatrix} \frac{T_p}{\tau_q} \\ \frac{T_r}{\tau_q} \end{bmatrix}, \quad \varphi_3 = \begin{bmatrix} \frac{T_r}{\tau_q} \end{bmatrix},
\]

the symbols \( S(\cdot), C(\cdot), T(\cdot) \) and \( S(\cdot) \) are the short form of trigonometric function \( \sin(\cdot), \cos(\cdot), \tan(\cdot) \) and \( \sec(\cdot) \).

**Remark 1.** From \( (2) \) and \( (3) \), it can be seen that the subsystem does not satisfy the matching condition because translational and rotational dynamics both involve uncertainties and external disturbances, and the translational motions of quadrotor UAV are achieved through body rotational motions.

The rotor dynamic can be approximated by a first-order low-pass filter. However, for non-aggressive maneuvers, the fast rotor dynamics can be regarded as a conventional assumption, and the rotor dynamic effects can also be diminished by the high bandwidth of the controller \([19, 47]\). Therefore, the rotor dynamics are ignored and the four virtual control variables \( \{ \tau_f, \tau_p, \tau_r, \tau_q \} \) are obtained from thrust force as

\[
\begin{bmatrix} \tau_p, \tau_q, \tau_r, \tau_f \end{bmatrix}^T = \Phi u(\varphi_4)
\]

(4)

where

\[
\Phi = \begin{bmatrix} 1 & -1 & -l & 1 \\ -l & -1 & 1 & 1 \\ c & -c & c & -c \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \varphi_4 = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}
\]

(5)

and \( u(\cdot) \) represents the amount of control after the saturation constraints, \( F_i, i = 1 \ldots 4 \) is the thrust force generated by \( i \)th rotor, \( l \) is the arm length shown in Figure 2, and \( c \) is the torque-thrust coefficient.

**Assumption 1.** All state vectors position, \( \xi \), velocity, \( v \), attitude angle, \( \eta \), and angular rate, \( \omega \), of quadrotor UAV are available from measurement during flight.

**Assumption 2.** The lumped disturbances are bounded to the first-order derivative \( \|d_i\| \leq \bar{d}_i \) and have constant values in steady state \( \lim_{t \to \infty} d_i(t) = 0 \) and \( \lim_{t \to \infty} d_i(t) = D_{i,\infty} \) where \( i = 1 \ldots 5 \).

**Assumption 3.** The orientation angle \( \{ \phi, \theta, \psi \} \) are all within valid ranges as \( (-\frac{\pi}{2} < \phi < \frac{\pi}{2}) \), \( (-\frac{\pi}{4} < \theta < \frac{\pi}{4}) \) and \( (-\pi \leq \psi < \pi) \).
Assumption 4. The desired x-y-z position \( \{x_d, y_d, z_d\} \) and yaw angle \( \psi_d \) and their derivatives are assumed to be piecewise uniformly bounded.

2.2. Input and State Constraints

Due to the limited force generated by each rotor, the input saturation constraint is expressed using saturation function as follows:

\[
u(F_i) = \begin{cases} 
F_{\text{max}}, & \text{if } F_i > F_{\text{max}} \\
F_i, & \text{if } F_{\text{min}} \leq F_i \leq F_{\text{max}} \\
F_{\text{min}}, & \text{if } F_i < F_{\text{min}}
\end{cases}
\]  

(6)

where \( F_i, i = 1 \ldots 4 \) are the thrust force generated by each rotor, and \( F_{\text{max}} > F_{\text{min}} \in (0, \infty) \) are the maximum and minimum force values.

The attitude constraint is designed with respect to the inclination angle of the quadrotor UAV, thus the saturation function for control variables \( \phi_1 \) in (2) is

\[
u(\phi_1) = \begin{cases} 
\frac{M_\eta}{\|\phi_1\|} \phi_1, & \text{if } \|\phi_1\| > M_\eta \\
\phi_1, & \text{if } \|\phi_1\| \leq M_\eta
\end{cases}
\]

(7)

where \( M_\eta \) is the maximum allowed inclination value of the quadrotor UAV with respect to the horizontal level.

2.3. Prescribe Performance

Prescribe performance is achieved by ensuring the tracking error evolves within the following predefined region [20]:

\[-\delta_i \rho_i(t) < e_i(t) < \delta_i \rho_i(t), \quad \forall t \geq 0\]

(8)

where \( e_i, i = x, y, z, \psi \) are the errors, \( \delta_i \) and \( \delta_i \) are positive constants and \( \rho_i(t) \) is a performance function which satisfies smooth, strictly positive and decaying bounded with \( \lim_{t \to \infty} \rho_i(t) = \rho_{i0} > 0 \). Thus, the performance function is defined as

\[\rho_i(t) = (\rho_{i0} - \rho_{i0o})e^{-l_1 t} + \rho_{i0o}, \quad \forall t \geq 0\]

(9)

where \( \rho_{i0}, \rho_{i0o} \) and \( l_1, i = x, y, z, \psi \) are positive constants and the initial condition \(-\delta_i \rho_i(0) < e_i(0) < \delta_i \rho_i(0)\) are satisfied. Then, transforming the original constrained tracking error behavior into an equivalent unconstrained one as

\[\alpha_i = \Phi_i \left( \frac{e_i}{\rho_i(t)} \right)\]

(10)

where \( \Phi_i(\cdot) : (-\delta_i, \delta_i) \to (-\infty, +\infty) \) is a strictly increasing smooth function and chosen [21] as

\[\Phi_i \left( \frac{e_i}{\rho_i(t)} \right) = \frac{1}{2} \ln \left( \frac{\delta_i + \frac{e_i}{\rho_i(t)}}{\delta_i - \frac{e_i}{\rho_i(t)}} \right).\]

(11)

Accordingly, the transformed error dynamics is derived as

\[\dot{\alpha}_i = \gamma_i \left( \dot{e}_i - \frac{\rho_i(t)}{\rho_i(t)} e_i \right)\]

(12)

with,

\[\gamma_i = \frac{1}{2\rho_i(t)} \left( \frac{1}{\delta_i + \frac{e_i}{\rho_i(t)}} + \frac{1}{\delta_i - \frac{e_i}{\rho_i(t)}} \right).\]

(13)
Define $|γ_i|_{\text{min}} = γ_{\text{in}}$, from (8), one can conclude that

$$0 < γ_{\text{in}} = \frac{2}{\rho_i(\dot{\delta}_i + \dot{\hat{\delta}}_i)} ≤ γ_i, \quad -l_i < \frac{\rho_i(\dot{t})}{\rho_i(t)} < 0$$

(14)

which will be used in the following proof of stability.

**Lemma 1** ([48]). Consider the original constrained error $e_i(t)$ and transformed error $α_i(t)$ with error transformation defined in (10). If $α_i(t)$ is bounded, $e_i(t)$ in (8) is satisfied for $∀t ≥ 0$.

For clarity, the following two notations are defined for this paper:

1. $I$ without a superscript represents a $I_{2×2}$ identity matrix;
2. $0$ represents a zero matrix with proper dimensions;
3. $\text{col}(\cdot)$ represents the column vector.

§3. ESO Design

Most existing studies on ESOs are only effective on integral-chain systems with matched disturbance using linear observer gains and single measurement. However, considering the UF subsystems of quadrotor UAV, the difficulties are the approximations of both the matched and unmatched disturbances. By taking the advantages of coordinate transform from the feedback linearization technique, the quadrotor UF subsystems can be reformulated into Brunovsky systems. Then, the matched and unmatched disturbances from the quadrotor system are added as the augmented states, and the whole system is estimated using the ESO technique.

Considering UF subsystems (2) and (3), augment the dynamics with lumped disturbances and rewritten in ESO from as

$$\begin{align*}
\dot{x}_1 &= f_1(x_1, \varphi_2) + D_1x_1^a + L_{11}(y_{1m} - C_1x_1) \\
\dot{x}_2 &= L_{12}(y_{1m} - C_1x_1) \\
\dot{x}_2 &= f_2(x_2, \varphi_3) + D_2x_2^a + L_{21}(y_{2m} - C_2x_2) \\
\dot{x}_2 &= L_{22}(y_{2m} - C_2x_2)
\end{align*}$$

(15)

(16)

with

$$\begin{align*}
x_1 &= \text{col}(x_1 \ldots x_4), \quad x_1^a = \text{col}(d_1, d_2, d_3) \\
x_2 &= \text{col}(x_5, x_6), \quad x_2^a = \text{col}(d_4, d_5) \\
f_1(\cdot) &= \text{col}(x_2, g_1\varphi_1, g_2x_4, g_3\varphi_2) \\
f_2(\cdot) &= \text{col}(g_4x_6, g_5\varphi_3)
\end{align*}$$

where $\{C_1, C_2\}$ are the observation matrices, $\{D_1, D_2\}$ are the distribution matrix of disturbances, $\{y_{1m}, y_{2m}\}$ are the observations, $\{L_{11}, L_{12}\}$ and $\{L_{21}, L_{22}\}$ are the observer feedback matrices.

**Remark 2.** From the state space dynamics, one can find that (2) is an affine-in-control pure-feedback system, and (3) is a strict-feedback system. Additionally, (2) and (3) contain unmatched disturbances.

In order to compute the nonlinear feedback matrices, coordinate transformation is proposed to transform the pure-feedback system (2) and strict-feedback system (3) to
canonical systems. According to the feedback linearization method [49], define alternative state variables \( \bar{z}_1 = \text{col}(z_1 \ldots z_4) \) and \( \bar{z}_2 = \text{col}(z_5, z_6) \) as

\[
\begin{align*}
\bar{z}_1 &= \bar{x}_1 \\
\bar{z}_2 &= \frac{\partial f_1}{\partial x_1} \bar{x}_1 + \frac{\partial f_1}{\partial x_2} \bar{x}_2 \\
\bar{z}_3 &= \frac{\partial f_2}{\partial x_1} \bar{x}_1 + \frac{\partial f_2}{\partial x_2} \bar{x}_2 + \frac{\partial f_2}{\partial x_3} \bar{x}_3 \\
\bar{z}_4 &= \frac{\partial f_3}{\partial x_1} \bar{x}_1 + \frac{\partial f_3}{\partial x_2} \bar{x}_2 + \frac{\partial f_3}{\partial x_3} \bar{x}_3 + \frac{\partial f_3}{\partial x_4} \bar{x}_4 \\
\bar{z}_5 &= \bar{x}_5 \\
\bar{z}_6 &= \frac{\partial f_4}{\partial x_5} \bar{x}_5 + \frac{\partial f_5}{\partial x_6} \bar{x}_6
\end{align*}
\]

(17)

where \( f_1 = x_2, f_2 = g_1 \varphi_1, f_3 = \frac{\partial (g_1 \varphi_1)}{\partial x_3} g_2 x_4 \) and \( f_5 = g_4 x_6 \). The lumped disturbances \( \{d_1 \ldots d_5\} \) are considered independent to state variables \( \{x_1 \ldots x_6\} \). Therefore, according to (17) and (18), the state transform matrices for which transfer \( \{\bar{x}_1, \bar{x}_2\} \) to \( \{\bar{z}_1, \bar{z}_2\} \) can be obtained as

\[
S_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & J_2 & J_4 \end{bmatrix}, \quad S_2 = \begin{bmatrix} I & 0 \\ 0 & J_5 \end{bmatrix}
\]

(19)

where \( J_1 = \frac{\partial f_1}{\partial z_2} = I, J_2 = \frac{\partial f_2}{\partial z_3} = g_1 I_{\varphi \psi}, J_3 = \frac{\partial f_3}{\partial z_3} = J_3(x_3, x_4), J_4 = \frac{\partial f_4}{\partial z_4} = g_4 I_{\varphi \theta} \varphi_2 \) and \( J_5 = \frac{\partial f_5}{\partial z_6} = g_4 \). \( I_{\varphi \theta} \) is given as

\[
I_{\varphi \theta} = \begin{bmatrix} C_{\varphi} & 0 \\ -S_{\varphi} C_{\theta} & C_{\varphi} C_{\theta} \end{bmatrix}
\]

(20)

and the details of variables \( \{f_1, f_2, f_3, f_4, f_5\} \) are shown in Appendix A.

Introducing the observer bandwidth \( w_1, w_2 > 0 \) in the high-gain observer method [50], the bandwidth matrices are defined as

\[
W_1 = \text{diag} \left( 1, 1, w_1, w_1^2, w_1^3, w_1^3 \right), \quad W_2 = \text{diag} (1, 1, w_2, w_2^2).
\]

(21)

Using the coordinate transformation matrices (19) and bandwidth matrices (21), the nonlinear ESO observer feedback matrices in (15) and (16) are obtained as

\[
\begin{bmatrix} L_{11} \\ L_{12} \end{bmatrix} = \begin{bmatrix} w_1 S_1^{-1} W_1 \chi_1 \\ w_1^2 D_1^T S_1^{-1} W_1 D_1 \chi_2 \end{bmatrix} C_1 W_1^{-1} S_1 C_1^T
\]

(22)

\[
\begin{bmatrix} L_{21} \\ L_{22} \end{bmatrix} = \begin{bmatrix} w_2^2 S_2^{-1} W_2 \chi_3 \\ w_2^2 D_2^T S_2^{-1} W_2 D_2 \chi_4 \end{bmatrix} C_2 W_2^{-1} S_2 C_2^T
\]

(23)

where \( \{\chi_1 \ldots \chi_4\} \) are gain matrices with positive entries.

To prove the convergence of the proposed ESO, the canonical form of (15) and (16) can be computed using the following relationship as

\[
\begin{align*}
\dot{z}_1 &= S_1 W_1^{-1} \bar{x}_1 \\
\dot{z}_2 &= S_2 W_2^{-1} \bar{x}_2 \\
\dot{\bar{x}}_1 &= D_1^T S_1 \left( w_1 W_1 \right)^{-1} D_1 \dot{\bar{x}}_1 \\
\dot{\bar{x}}_2 &= D_2^T S_2 \left( w_2 W_2 \right)^{-1} D_2 \dot{\bar{x}}_2
\end{align*}
\]

(24)
Define the observer estimation error as
\[
\varepsilon_1 = \begin{bmatrix} z_1 - \hat{z}_1 \\ \varepsilon_1 \end{bmatrix}, \quad \varepsilon_2 = \begin{bmatrix} \varepsilon_2 - \hat{z}_2 \\ \varepsilon_2 \end{bmatrix}
\] (26)
using the mean value theorem [51], the error dynamics of augmented subsystem (2) and (3) under ESO (15), (16) can be obtained from (15)–(26) as
\[
\dot{\varepsilon}_1 = w_1 \hat{A}_1 \varepsilon_1 + D_1 \frac{d_1}{w_1} + D_2 \frac{d_2}{w_1} + D_3 \frac{d_3}{w_1}
\] (27)
\[
\dot{\varepsilon}_2 = w_2 \hat{A}_2 \varepsilon_2 + D_4 \frac{d_4}{w_2} + D_5 \frac{d_5}{w_2}
\] (28)
with
\[
\hat{A}_1 = \begin{bmatrix} A_1 - \chi_1 C_1 & D_1 \\ -\chi_2 C_1 & 0^{3 \times 3} \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} A_2 - \chi_3 C_2 & D_2 \\ -\chi_4 C_2 & 0^{2 \times 2} \end{bmatrix}
\]
where \(\{D_1, D_2, D_3\}\) are the columns of \(0^{8 \times 6}\) and \(\{D_4, D_5\}\) are the columns of \(0^{4 \times 4}\).
\(\{A_1, A_2\}\) are the transfer matrices in canonical form as
\[
A_1 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}
\]

**Remark 3.** The result of adding the observer bandwidth is a scalar amplification of the location of poles or a scalar amplification for all eigenvalues, which can be seen from (27) and (28).

Since multiple disturbances appeared in the subsystem (2) and (3), observability is essential and can be checked from
\[
A_i' = \begin{bmatrix} A_i & D_i \\ 0 & 0 \end{bmatrix}, \quad C_i' = [C_i, \ 0], \quad i = 1, 2
\]
where \(A_i'\) and \(C_i'\) are the transfer matrix and observation matrix of the augmented systems, separately. From subsystems (2), (3) and Assumption 1, the observability matrices can be calculated and have a rank of 7 for (2) and 4 for (3). Therefore, the corresponding states and lumped disturbances are observable. Note that the observations can be reduced as long as the observability is satisfied.

The gain matrices \(\{\chi_1, \ldots, \chi_4\}\) are selected as follows such that \(\hat{A}_1\) and \(\hat{A}_2\) are Hurwitz matrices.
\[
\chi_1 = \begin{bmatrix} I & I & 0 & 0 \\ 0 & 2I & I & 0 \\ 0 & 0 & 2I & I \\ 0 & 0 & 0 & 2I \end{bmatrix}, \quad \chi_2 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \chi_3 = \begin{bmatrix} 2I & I \\ 0 & 2I \end{bmatrix}, \quad \chi_4 = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}
\] (29)

Hence, the observer bandwidth becomes the only tuning parameter of the observer and is determined by the system dynamics. Now, the following is the proof of stability of proposed ESO method.

**Theorem 1.** Considering the ESO error dynamics in (27) and (28), there exist two positive definite matrices \(P_1\) and \(P_2\) satisfying \(\hat{A}_1^T P_1 + P_1 \hat{A}_1 = -I\) and \(\hat{A}_2^T P_2 + P_2 \hat{A}_2 = -I\). Under Assumptions 1, 2 and 3, the uniformly bounded stability of proposed nonlinear ESOs for quadrotor UAV system can be guaranteed if the observer bandwidth \(\{w_1, w_2\}\) is selected such that \(l_1 = \ldots \ldots\)

\[
A = \begin{bmatrix} a \\ \cdots \\ a \end{bmatrix}
\]
\[
\min \left( \frac{m_1}{\lambda_{\max}(P_1)}, \frac{m_2}{\lambda_{\max}(P_2)} \right) > 0 \text{ where } m_1 = \left( w_1 - \frac{\lambda_1}{w_1} - \frac{\lambda_2}{w_1} - \frac{\lambda_3}{w_1} \right), \quad m_2 = \left( w_2 - \frac{\lambda_4}{w_2} - \frac{\lambda_5}{w_2} \right) \text{ and } \lambda_i = \{ \|P_1 D_i\|, \|P_2 D_i\|, \|P_1 D_3\|, \|P_2 D_3\| \}. 
\]

**Proof of Theorem 1.** Choose the following Lyapunov candidate function as

\[
V = \varepsilon_1^T P_1 \varepsilon_1 + \varepsilon_2^T P_2 \varepsilon_2. \quad (30)
\]

Substituting (27) and (28) into the derivative of \( V \) yields

\[
\dot{V} = -w_1 \| e_1 \|^2 - w_2 \| e_2 \|^2 + 2 \varepsilon_1^T P_1 \dot{D}_1 \frac{d_1}{w_1} + 2 \varepsilon_1^T P_2 \dot{D}_2 \frac{d_2}{w_2} + 2 \varepsilon_2^T P_1 \dot{D}_3 \frac{d_3}{w_1} \\
+ 2 \varepsilon_2^T P_2 \dot{D}_4 \frac{d_4}{w_2} + 2 \varepsilon_2^T P_2 \dot{D}_5 \frac{d_5}{w_2} \\
\leq -w_1 \| e_1 \|^2 - w_2 \| e_2 \|^2 + 2 \lambda_1 \frac{\varepsilon_1^T}{w_1} \| e_1 \| d_1 + 2 \lambda_2 \frac{\varepsilon_1^T}{w_1} \| e_1 \| d_2 + 2 \lambda_3 \frac{\varepsilon_1^T}{w_1} \| e_1 \| d_3 \\
+ 2 \lambda_4 \frac{\varepsilon_2^T}{w_2} \| e_2 \| d_4 + 2 \lambda_5 \frac{\varepsilon_2^T}{w_2} \| e_2 \| d_5 \\
\leq -l_1 V + l_2
\]

where \( l_1 = \min \left( \frac{m_1}{\lambda_{\max}(P_1)}, \frac{m_2}{\lambda_{\max}(P_2)} \right) \) with \( m_1 = \left( w_1 - \frac{\lambda_1}{w_1} - \frac{\lambda_2}{w_1} - \frac{\lambda_3}{w_1} \right), \quad m_2 = \left( w_2 - \frac{\lambda_4}{w_2} - \frac{\lambda_5}{w_2} \right) \)

and \( l_2 = \frac{\lambda_1}{w_1} \| e_1 \|^2 + \frac{\lambda_2}{w_1} \| e_2 \|^2 + \frac{\lambda_3}{w_1} \| e_1 \|^2 + \frac{\lambda_4}{w_2} \| e_2 \|^2 + \frac{\lambda_5}{w_2} \| e_2 \|^2 \). From (31), \( \dot{V} < 0 \) if \( V > \frac{l_2}{l_1} \), and therefore, it can be concluded that the error dynamics (27) and (28) are bounded stably as

\[
\| \dot{e}_1 \| + \| \dot{e}_2 \| \leq \sqrt{\frac{2l_2}{l_1} \min(\lambda_{\max}(P_1), \lambda_{\max}(P_2))}
\]

where \( \lambda_{\max}(P) \) and \( \lambda_{\min}(P) \) denote the maximum and minimum eigenvalues of the matrix \( P \). The proof is finished. \( \square \)

### 4. Controller Design

In this section, the prescribed performance tracking control strategy for the quadrotor UAV is proposed based on the DSC technique with attitude and input constraints. The estimated state vector \( \hat{x}_i, i = 1 \ldots 6 \), and lumped disturbances, \( \delta_i, i = 1 \ldots 5 \), are obtained from the proposed ESO in previous section. For clarity, the following notations are defined in this section:

1. \( \hat{x}_i, i = 1 \ldots 6 \) are the estimated variables,
2. \( \hat{x}_i = x_i - \hat{x}_i, i = 1 \ldots 6 \) are the estimated errors,
3. \( \lambda_m(A) \) and \( \lambda_n(A) \) are the maximum and minimum eigenvalues of matrix \( A \).

#### 4.1. Controller Design for Underactuated System

Define \( e_1 = x_{1d} - x_1 \) as the first surface error. To achieve guaranteed tracking performance of \{ \( x, y \) \}, the transformed error in (10) and its derivative are

\[
\dot{\hat{x}}_1 = \gamma_1 (\hat{x}_{1d} - x_2 - \rho_1^{-1} \hat{p}_1 e_1) \quad (33)
\]

where \( \rho_1 = \text{diag}(\rho_x, \rho_y) \) and \( \gamma_1 = \text{diag}(\gamma_x, \gamma_y) \).

The virtual control variables \( \bar{x}_2 \) is designed as

\[
\bar{x}_2 = K_1 \hat{a}_1 + \hat{x}_{1d} - \rho_1^{-1} \hat{p}_1 \dot{e}_1 \quad (34)
\]

where \( K_1 \in \mathbb{R}^{2 \times 2} \) is a positive definite matrix.
Following the principle DSC technique, introduce a variable $x_{2d} \in \mathbb{R}^2$ and pass the virtual control $\bar{z}_2$ through a first-order low-pass filter with the positive definite time constant matrix $\tau_1 \in \mathbb{R}^{2 \times 2}$ as

$$\tau_1 x_{2d} + x_{2d} = \bar{x}_2, \quad x_{2d}(0) = \bar{x}_2(0). \quad (35)$$

Let $z_1 = \bar{x}_2 - x_{2d}$ denote the filtering error, then the filtering error dynamics can be derived as

$$\dot{z}_1 = -\tau_1^{-1} z_1 + \dot{\bar{x}}_2$$
$$= -\tau_1^{-1} z_1 + O_1(\bar{x}_{1d}, \bar{\alpha}, \bar{e}_1, \bar{\bar{e}}_1, \bar{\bar{\bar{e}}}_1) \quad (36)$$

where $O_1(\cdot)$ is a continuous function.

Consider the Lyapunov function candidate $V_1 = \frac{1}{2} \bar{x}_{1d}^T \gamma_1^{-1} \bar{x}_1 + \frac{1}{2} z_1^T z_1$, and its derivative with respect to time is

$$V_1 = \frac{1}{2} \bar{x}_{1d}^T (K_1 - \frac{1}{2} I) \bar{x}_1 - z_1^T \left( \tau_1^{-1} - \frac{1}{2} I \right) z_1$$
$$+ \frac{1}{2} O_1^T O_1 + \frac{1}{2} e_2^T e_2 + \|\bar{\alpha}_1\| \|z_1\| + \hat{\rho}_1^2 \|\bar{\alpha}_1\| \|e_1\|. \quad (37)$$

Invoking Young's inequality, error transform (26) and bounds (14), the inequality above can be further expressed as

$$V_1 \leq -\frac{1}{2} \bar{x}_{1d}^T (K_1 - \frac{1}{2} I) \bar{x}_1 - z_1^T \left( \tau_1^{-1} - \frac{1}{2} I \right) z_1$$
$$+ \frac{1}{2} O_1^T O_1 + \frac{1}{2} e_2^T e_2 + \|\bar{\alpha}_1\| \|z_1\| + \hat{\rho}_1^2 \|\bar{\alpha}_1\| \|e_1\|. \quad (38)$$

Define $e_2 = x_{2d} - \bar{x}_2$ as the second surface error, and its derivative is

$$\dot{e}_2 = \dot{x}_{2d} - \bar{g}_1 \bar{\varphi}_1 - \bar{d}_1. \quad (39)$$

The virtual control variables $\tilde{\varphi}_1$ are designed as

$$\bar{\varphi}_1 = \bar{g}_1^{-1} (K_2 \bar{e}_2 + x_{2d} - \bar{d}_1) - K_{\zeta} \tilde{\zeta}_1 \quad (40)$$

where $K_2, K_{\zeta} \in \mathbb{R}^{2 \times 2}$ are positive definite matrices, and $\zeta_1 \in \mathbb{R}^2$ is a variable from the following auxiliary system for compensating the effect of state constraints.

\[
\dot{\zeta}_1 = \begin{cases} 
-\kappa_1 \zeta_1 - \frac{|\zeta_1^T e_1| + \frac{1}{2} e_1^T e_1}{\|e_1\|^2} \zeta_1 + e_1, & \text{if } \|\zeta_1\| \geq \delta_1 \\
0, & \text{if } \|\zeta_1\| < \delta_1 
\end{cases} \quad (41)
\]

where $e_1 = \bar{\varphi}_1 - u(\bar{\varphi}_1)$ is the error of the state constraints, $\kappa_1 \in \mathbb{R}^{2 \times 2}$ is a positive definite matrix, and $\delta_1$ is a small positive constant.

**Remark 4.** As for the variable $\zeta_1$ in (40), which is defined in (41), its value is dependent on $e_1$. If the derivative of $\zeta_1$ is not equal to zero in the auxiliary system, the result of $\zeta_1$ might render the virtual control $\bar{\varphi}_1$ smaller and closed to $u(\bar{\varphi}_1)$. Therefore, the saturation error can be compensated by $\zeta_1$. When the derivative of $\zeta_1$ is equal to zero, the result of $\zeta_1$ is a small constant value and it may affect the virtual control $\bar{\varphi}_1$ slightly since $\delta_1$ is a small constant.

**Remark 5.** The combination of Euler angle $\phi$, $\theta$ and $\psi$ have orders to form the rotation matrix $R(t)$ in (1). Thus, the actual inclination angles along the x-axis and y-axis with respect to the earth-frame are $\phi$ and $\sin^{-1}(C_{\psi} S_{\theta})$, separately. Here, the trigonometric variables are used as $S_{\phi}$ and $C_{\phi} S_{\theta}$, which is $\phi_1$ in (2). The state constraint exerts to $\|\varphi_1\|$ to ensure the total inclination angle within $\sin^{-1}(M_{\psi})$. 
Introducing a variable $x_{3d} \in \mathbb{R}^2$, we pass the constrained virtual control $u(\varphi_1)$ through a first-order low-pass filter with the positive definite time constant matrix $\tau_2 \in \mathbb{R}^{2 \times 2}$ as

$$\tau_2 x_{3d} + x_{3d} = u(\varphi_1), \quad x_{3d}(0) = u(\varphi_1(0)). \tag{42}$$

Let $z_2 = u(\varphi_1) - x_{3d}$ denote the filtering error, then the filtering error dynamics can be derived as

$$\dot{z}_2 = -\tau_2^{-1} z_2 + \varphi_1$$

$$= -\tau_2^{-1} z_2 + O_2 (\varepsilon_2, x_{2d}, d_1, d_2, \dot{z}_2, \ddot{z}_1) \tag{43}$$

where $O_2(\cdot)$ is a continuous function.

Consider the Lyapunov function candidate $V_2 = \frac{1}{2} z_2^T e_2 + \frac{1}{2} \dot{z}_2^T z_2 + \frac{1}{2} \xi_1^T \xi_1$, when $\|\xi_1\| \geq \delta_1$. Its derivative with respect to time is

$$\dot{V}_2 = e_2^T \left(-K_z e_2 + g_1 z_2 + g_1 e_3 + g_1 K_{\xi_1} \dot{\xi}_1 + g_1 e_1 - K_z d_2 - d_1\right)$$

$$+ \dot{z}_2^T \left(\frac{1}{2} \tau_2^{-1} z_2 + O_2 \right) + \xi_1^T \left(\frac{1}{2} \tau_2^{-1} z_2 + O_2 \right) + \delta_1^T \frac{1}{2} \tau_2^{-1} \xi_1 + e_1 \tag{44}$$

According to the definition of matrix $g_1$, one can conclude that $\|g_1\|_{\text{max}} = g_{1m} = g$.

Invoking Young’s inequality, the $V_2$ can be further expressed as

$$\dot{V}_2 \leq -e_2^T \left(K_z - \frac{g_{1m}}{2} I\right) e_2 - z_2^T \left(\tau_2^{-1} - \frac{1}{2} I\right) z_2 - \xi_1^T \left(K_z - \frac{g_{1m}}{2} I\right) \xi_1$$

$$+ \frac{1}{2} \xi_1^T O_2 \xi_1 + \frac{1}{2} \dot{z}_2^T \left(\tau_2^{-1} - \frac{1}{2} I\right) z_2$$

$$+ \frac{1}{2} \dot{e}_1^T e_1 + \frac{1}{2} \dot{e}_2^T e_2 + g_{1m} \|e_2\| \|z_2\| + g_{1m} \lambda_m (K_{\xi_1}) \|e_2\| \|\xi_1\|$$

$$+ \frac{w_1 \lambda_m (K_z)}{\lambda_n (j_1)} \|e_2\| \|e_2\| + \frac{w_2}{\lambda_n (j_1)} \|e_2\| \|e_1\|. \tag{45}$$

When $\|\xi_1\| < \delta_1$, the last term is $\xi_1^T \xi_1 = 0$, thus, its derivative with respect to time is

$$\dot{V}_2 = e_2^T \left(-K_z e_2 + g_1 z_2 + g_1 e_3 + g_1 K_{\xi_1} \dot{\xi}_1 + g_1 e_1 - K_z d_2 - d_1\right)$$

$$+ \dot{z}_2^T \left(\frac{1}{2} \tau_2^{-1} z_2 + O_2 \right) \leq -e_2^T \left(K_z - \frac{g_{1m}}{2} I\right) e_2 - z_2^T \left(\tau_2^{-1} - \frac{1}{2} I\right) z_2$$

$$+ \frac{1}{2} \dot{e}_1^T e_1 + \frac{1}{2} \dot{e}_2^T e_2 + g_{1m} \|e_2\| \|z_2\| + g_{1m} \lambda_m (K_{\xi_1}) \|e_2\| \|\xi_1\|$$

$$+ \frac{w_1 \lambda_m (K_z)}{\lambda_n (j_1)} \|e_2\| \|e_2\| + \frac{w_2}{\lambda_n (j_1)} \|e_2\| \|e_1\|. \tag{46}$$

Synthesizing (45) and (46), the inequality becomes

$$\dot{V}_2 \leq -e_2^T \left(K_z - \frac{g_{1m}}{2} I\right) e_2 - z_2^T \left(\tau_2^{-1} - \frac{1}{2} I\right) z_2 - \xi_1^T \left(K_z - \frac{g_{1m}}{2} I\right) \xi_1$$

$$+ \frac{1}{2} \dot{e}_1^T e_1 + \frac{1}{2} \dot{e}_2^T e_2 + g_{1m} \|e_2\| \|z_2\| + g_{1m} \lambda_m (K_{\xi_1}) \|e_2\| \|\xi_1\|$$

$$+ \frac{w_1 \lambda_m (K_z)}{\lambda_n (j_1)} \|e_2\| \|e_2\| + \frac{w_2}{\lambda_n (j_1)} \|e_2\| \|e_1\|. \tag{47}$$

Define $e_3 = x_{3d} - \varphi_1$ as the third surface error, and its derivative is

$$\dot{e}_3 = x_{3d} - I_{\varphi \theta} g_2 x_4 - I_{\varphi \theta} d_2 \tag{48}$$

where $I_{\varphi \theta}$ is the Jacobian matrix of $\varphi_1$ in (20) with determinant equals to $C_{\varphi}^2 C_{\theta}$. Therefore, $I_{\varphi}$ is non-singular under Assumption 3.
The virtual control variable $\xi_4$ is designed as

$$\xi_4 = g_2^{-1}(J^{-1}_3 (\hat{K}_3 \hat{e}_3 + \hat{x}_{3d}) - \hat{d}_2)$$  (49)

where $K_3 \in \mathbb{R}^{2 \times 2}$ is a positive definite matrix.

Introducing a variable $x_{4d} \in \mathbb{R}^2$, we pass the constrained virtual control $g(x_4)$ through a first-order low-pass filter with the positive definite time constant matrix $\tau_3 \in \mathbb{R}^{2 \times 2}$ as

$$\tau_3 \dot{x}_{4d} + x_{4d} = g(x_4), \quad x_{4d}(0) = g(x_4)(0).$$  (50)

Let $z_3 = \hat{x}_4 - x_{4d}$ denote the filtering error, then the filtering error dynamics can be derived as

$$\dot{z}_3 = -\tau_3^{-1}z_3 + \hat{x}_4$$
$$= -\tau_3^{-1}z_3 + \phi_3 + \hat{x}_{3d}, \hat{e}_3, \hat{d}_2$$  (51)

where $\phi_3(\cdot)$ is a continuous function.

Consider the Lyapunov function candidate $V_3 = \frac{1}{2}e_3^T e_3 + \frac{1}{2}z_3^T z_3$. Its derivative is

$$\dot{V}_3 = e_3^T (-K_3 e_3 + J_{\psi g_2} z_3 + J_{\theta g_2} e_4 - K_3 \hat{e}_3 - J_{\theta d_2})$$
$$+ z_3^T (-\tau^{-1} z_3 + \phi_3).$$  (52)

From (2) and (20), after the matrix operation, one can conclude that $(J_{\theta})_{ij} < 1$ and $(J_{\psi g_2})_{ij} < 1$ are satisfied. Invoking Young’s inequality, $V_3$ can be further expressed as

$$\dot{V}_3 \leq -e_3^T (K_3 - \frac{1}{2} I) e_3 - z_3^T (\tau_3^{-1} - \frac{1}{2} I) z_3 + \frac{1}{2} \phi_3^T \phi_3 + \frac{1}{2} e_4^T e_4$$
$$+ ||e_3|| ||z_3|| + \frac{w_3^3 \lambda_{max}(K_3)}{\lambda_n(I_2)} ||e_3|| ||e_3|| + \frac{w_3^2}{\lambda_n(I_2)} ||e_3|| ||e_2||.$$  (53)

Define $e_4 = x_{4d} - x_4$ as the fourth surface error, and its derivative is

$$\dot{e}_4 = \dot{x}_{4d} - \phi_3 e_2 - d_3.$$  (54)

The virtual control variables $\phi_2$ is designed as

$$\phi_2 = g_3^{-1}(K_4 \hat{e}_4 + \dot{x}_{4d} - \hat{d}_3)$$  (55)

where $K_4 \in \mathbb{R}^{2 \times 2}$ are positive definite matrices. The control input $\phi_2$ represents the input torque along x-axis and y-axis with respect to body frame of quadrotor UAV which are compound control variables of four thrust force generated by rotors shown in (5). Thus, the saturation constraints cannot be implemented directly. The design of the input saturation is introduced in the following Section C.

4.2. Controller Design for Fully Actuated System

Define $e_5 = x_{5d} - x_5$ as the fifth surface error. To achieve guaranteed tracking performance of $\{\psi, z\}$, the transformed error in (10) and its derivative are

$$\dot{a}_5 = g_2(\hat{x}_{5d} - g_4 x_6 - d_4 - \rho_2^{-1} \rho_2 e_5)$$  (56)

where $\rho_2 = \text{diag}(\rho_\psi, \rho_\theta)$ and $\gamma_2 = \text{diag}(\gamma_\psi, \gamma_\theta)$.

The virtual control variables $\hat{x}_6$ is designed as

$$\hat{x}_6 = g_4^{-1}(K_5 \hat{a}_5 + \hat{x}_{5d} - \hat{d}_4 - \rho_2^{-1} \rho_2 \hat{e}_5)$$  (57)

where $K_5 \in \mathbb{R}^{2 \times 2}$ is a positive definite matrix.
Introduce a variable $x_{6d} \in \mathbb{R}^2$, and pass the virtual control $\dot{x}_6$ through a first-order low-pass filter with the positive definite time constant matrix $\tau_4 \in \mathbb{R}^{2 \times 2}$ as

$$\tau_4 \dot{x}_{6d} + x_{6d} = x_6, \quad x_{6d}(0) = \dot{x}_6(0).$$

(58)

Let $z_4 = \dot{x}_6 - x_{6d}$ denote the filtering error, and following the same procedure in (36), the filtering error dynamics can be derived with (12) and (57) as

$$\dot{z}_4 = -\tau_4^{-1} z_4 + \dot{x}_6$$

$$= -\tau_4^{-1} z_4 + O_4(\dot{x}_{6d}, \dot{x}_5, e_5, e_6, \dot{x}_5, \dot{d}_4, \dot{d}_4)$$

(59)

where $O_4(\cdot)$ is a continuous function.

Consider the Lyapunov function candidate $V_5 = \frac{1}{2} \dot{x}_6^T \gamma_5^{-1} x_5 + \frac{1}{2} z_4^T z_4$. Its derivative with respect to time is

$$\dot{V}_5 = a_5^T ( -K_5 x_5 + g_4 x_4 + g_4 e_6 + \rho_2^{-1} \rho_2 x_5 - \dot{d}_4 ) + z_4^T ( -\tau_4^{-1} z_4 + O_4 ).$$

(60)

Define $\|g_4\|_{\text{max}} = \|g_{4m}\|$ and $\|g_{4m}\|_{\text{max}} = \max \{ \|C_b S \theta_6\|_{\text{max}}, 1\}$ since the entry $C_b S \theta_6$ in matrix $g_4$ is bounded under the state constraints. Invoking Young’s inequality, the $\dot{V}_5$ can be further expressed as

$$\dot{V}_5 \leq -a_5^T \left( K_5 - \frac{\|g_{4m}\|_{\text{max}}}{2} \right) x_5 - z_4^T \left( \tau_4^{-1} - \frac{1}{2} I \right) z_4 + \frac{1}{2} O_4^T O_4 + \frac{1}{2} e_6^T e_6$$

$$+ g_4 \|x_5\| \|z_4\| + e_6^T g_4 e_6 + \| \dot{d}_4 \| = \| \dot{d}_3 \|.$$}

(61)

Define $e_6 = x_{6d} - x_6$ as the sixth surface error, and its derivative is

$$\dot{e}_6 = x_{6d} - \dot{g}_5 \phi_3 - \dot{d}_5.$$

(62)

The virtual control variables $\phi_3$ is designed as

$$\phi_3 = g_5^{-1} (K_6 \dot{d}_6 + \dot{x}_{6d} - \dot{d}_5)$$

(63)

where $K_6 \in \mathbb{R}^{2 \times 2}$ are positive definite matrices. The control input $\phi_3$ represents the input torque and input force along the z-axis with respect to the body frame of the quadrotor UAV in which the saturation constraints cannot be implemented directly. The design of the input saturation is introduced in the next step.

4.3. Constraint Design for Actuator System

The input saturation is related to the thrust force generated by each rotor of the quadrotor UAV. The virtual control variables $\phi_2$ and $\phi_3$ represent the input torque along three axes and the input force along the z-axis, which cannot implement saturation constraints directly. Thus, a switch matrix and an auxiliary system are proposed to fulfill the actuator saturation constraints.

Define the final real control variable $\phi_4$ as

$$\phi_4 = \Phi^{-1} \left( \frac{\phi_2}{\phi_3} - K_{\phi_2} \phi_2 \right)$$

(64)

where $\Phi$ is the matrix in (5) and $K_{\phi_2} \in \mathbb{R}^4$ is a variable from following auxiliary system for compensating the effect of input saturation.

$$\xi_2 = \begin{cases} -K_2 \xi_2 - \frac{\|G^T \dot{\phi}_2 + \frac{1}{2} \dot{\xi}_2^T \Phi^T \dot{\phi}_2 \|_2^2}{\| \xi_2 \|} \dot{\xi}_2 + \Phi \xi_2, & \text{if } \| \xi_2 \| \geq \delta_2 \\ 0, & \text{if } \| \xi_2 \| < \delta_2 \end{cases}$$

(65)
where \( G^* = [e_1^T g_3 e_2^T g_5] \), \( e_2 = \varphi_4 - u(\varphi_4) \) is the error of the input saturation, \( \kappa_2 \in \mathbb{R}^{4 \times 4} \) is a positive definite matrix and \( \delta_2 \) is a small positive constant.

**Remark 6.** As for the variable \( \zeta_2 \) in (64), which is defined in (65), its value is dependent on \( \varepsilon_2 \). If the derivative of \( \zeta_2 \) is not equal to zero in the auxiliary system, the result of \( \zeta_2 \) might render the real control \( \varphi_4 \) smaller and closed to \( u(\varphi_4) \). Therefore, the saturation error can be compensated by \( \zeta_2 \). When the derivative of \( \zeta_2 \) is equal to zero, the result of \( \zeta_2 \) is a small constant value, and it may affect the virtual control \( \varphi_4 \) slightly since \( \delta_2 \) is a small constant.

Consider the Lyapunov function candidate \( V_{46} = V_4 + V_6 = \frac{1}{2} e_4^T e_4 + \frac{1}{2} e_6^T e_6 + \frac{1}{2} \zeta_2^T \zeta_2 \), when \( \| \zeta_2 \| \geq \delta_2 \). Its derivative with respect to time is

\[
\dot{V}_{46} = e_4^T \left( -K_4 e_4 + K_4 \dot{e}_4 - \ddot{d} \dot{\varphi}_2 \right) + e_6^T \left( -K_6 e_6 + K_6 \dot{e}_6 - \ddot{d} \dot{\varphi}_2 \right) \\
+ G^* \zeta_2 \dot{\zeta}_2 + G^* \Phi e_2 + \frac{1}{2} \zeta_2^T \zeta_2 \left( -\kappa_2 \zeta_2 - \frac{|G^* \Phi e_2| + \frac{1}{2} e_2^T \Phi^T \Phi e_2}{\| \zeta_2 \|^2} \right) \zeta_2 + \Phi e_2 \tag{66}
\]

Define \( g_{3m} = \max \left\{ \frac{1}{2}, \frac{1}{1} \right\} \), \( g_{5m} = \max \left\{ \frac{1}{2}, \frac{\varepsilon_2}{m} \right\} \). Invoking Young’s inequality, the \( \dot{V}_{46} \) can be further expressed as

\[
\dot{V}_{46} \leq -e_4^T K_4 e_4 - e_6^T K_6 e_6 - \zeta_2^T \left( \kappa_2 - \frac{1}{2} \right) \zeta_2 + \lambda_m \left( K_{\delta_2} \right) \left( g_{3m} \| e_4 \| + g_{5m} \| e_6 \| \right) \| \zeta_2 \| \\
+ \frac{w_1^2 \lambda_m(K_4 J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_3 \| + \frac{w_3^2 \lambda_m(K_6 J_3)}{\lambda_n(J_2 J_4)} \| e_6 \| \| e_5 \| + \frac{w_4^2 \lambda_m(J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_2 \| \tag{67}
\]

When \( \| \zeta_2 \| < \delta_2 \), the last term is \( \zeta_2^T \zeta_2 = 0 \), thus, its derivative with respect to time is

\[
\dot{V}_{46} = e_4^T \left( -K_4 e_4 + K_4 \dot{e}_4 - \ddot{d} \dot{\varphi}_2 \right) + e_6^T \left( -K_6 e_6 + K_6 \dot{e}_6 - \ddot{d} \dot{\varphi}_2 \right) + G^* \zeta_2 \dot{\zeta}_2 + G^* \Phi e_2 \\
\leq -e_4^T \left( K_4 - g_{3m}^2 I \right) e_4 - e_6^T \left( K_6 - g_{5m}^2 I \right) e_6 + \lambda_m \left( K_{\delta_2} \right) \left( g_{3m} \| e_4 \| + g_{5m} \| e_6 \| \right) \| \zeta_2 \| \\
+ \frac{w_1^2 \lambda_m(K_4 J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_3 \| + \frac{w_3^2 \lambda_m(K_6 J_3)}{\lambda_n(J_2 J_4)} \| e_6 \| \| e_5 \| + \frac{w_4^2 \lambda_m(J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_2 \| \tag{68}
\]

Synthesizing (67) and (68), the inequality becomes

\[
\dot{V}_{46} \leq -e_4^T \left( K_4 - g_{3m}^2 I \right) e_4 - e_6^T \left( K_6 - g_{5m}^2 I \right) e_6 - \zeta_2^T \left( \kappa_2 - \frac{1}{2} \right) \zeta_2 + \frac{1}{2} \zeta_2^T \Phi^T \Phi e_2 \\
+ \lambda_m \left( K_{\delta_2} \right) \left( g_{3m} \| e_4 \| + g_{5m} \| e_6 \| \right) \| \zeta_2 \| + \frac{w_1^2 \lambda_m(K_4 J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_3 \| \\
+ \frac{w_3^2 \lambda_m(K_6 J_3)}{\lambda_n(J_2 J_4)} \| e_6 \| \| e_5 \| + \frac{w_4^2 \lambda_m(J_3)}{\lambda_n(J_2 J_4)} \| e_4 \| \| e_2 \| \tag{69}
\]

According the above procedures, the structure of the proposed controller is demonstrated in Figure 3.
Figure 3. The block diagram of the proposed control scheme.

4.4. Stability Analysis

For clarity, define vectors 
\[ s = \begin{bmatrix} \alpha_1^T, \epsilon_2^T, \epsilon_3^T, \epsilon_4^T, \alpha_5^T, \epsilon_6^T \end{bmatrix}^T, z = \begin{bmatrix} z_1^T, z_2^T, z_3^T, z_4^T \end{bmatrix}^T, \zeta = \begin{bmatrix} \epsilon_1^T, \epsilon_2^T \end{bmatrix}^T \]

The whole Lyapunov function candidate is considered as

\[ V = \frac{1}{2} \sum_{i=1}^{6} V_i + \frac{1}{2} \epsilon^TP^\star \epsilon \]  

(70)

where \( P^\star \) is a positive definite matrix satisfying \( \bar{A}^T P^\star + P^\star \bar{A}^* = -I \) and \( A^* = \text{diag}(\bar{A}_1, \bar{A}_2) \).

For the low-pass filter error dynamics \( z_i, i = 1 \ldots 4 \), according to the bound properties in Assumptions 1, 2, 3 and 4, there exist positive constants \( O_{im}, i = 1 \ldots 4 \) such that \( \|O_i\| \leq O_{im}, i = 1 \ldots 4 \) [14,24].

According to (17), (18) and (30) that \( \|f_i\| = 1 \), the maximum value of \( \|f_3\| \) is 1, and the minimum values of \( \|f_2\|, \|I_4\| \) and \( \|f_5\| \) are related to the inclination angles of the quadrotor UAV, which are predefined within valid ranges as Assumption 3. Therefore, define \( J_m \) as the minimum value of \( \|f_i\| \) where \( i = 2, 4, 5 \).

Define a set of known constants as

\[ \sigma_1 = 1, \sigma_2 = g_1m I, \sigma_3 = 1, \sigma_4 = g_4m I, \sigma_5 = \lambda_m(K_{z_1})g_1m I, \sigma_6 = \lambda_m(K_{z_2})g_3m I, \]

\[ \sigma_7 = \lambda_m(K_{z_2})g_5m I, \sigma_8 = I_2 I, \sigma_9 = \omega_m(K_2) I, \sigma_{10} = \frac{\omega_2^2 \lambda_m(K_3)}{J_{2n}} I, \]

\[ \sigma_{11} = \frac{\omega_1^2 \lambda_m(K_4)}{J_{2n} J_{4n}} I, \sigma_{12} = \frac{\omega_1^2 \lambda_m(K_4)}{J_{4n}} I, \sigma_{13} = \frac{\omega_2^2 \lambda_m(K_3)}{J_{5n}} I, \sigma_{14} = \frac{\omega_2^2 \lambda_m(K_5)}{J_{5n}} I, \]

\[ \sigma_{15} = \omega_1^2 I, \sigma_{16} = \frac{\omega_1^2 \lambda_1}{J_{2n}} I, \sigma_{17} = \frac{\omega_1^2 \lambda_2}{J_{4n}} I, \sigma_{18} = \frac{\omega_1^2 \lambda_3}{J_{4n}} I, \sigma_{19} = w_1 I, \sigma_{20} = \frac{w_2^2}{J_{5n}} I, \sigma_{21} = w_1 - \frac{1}{2\omega_2^2}(w_2 \lambda_4 + \lambda_5), \sigma_{22} = w_2 - \frac{1}{2\omega_2^2}(w_2 \lambda_4 + \lambda_5), \]

\[ \zeta = \frac{1}{2} \sum_{i=1}^{4} \xi^2_i m + \frac{1}{2} \epsilon_1^T \epsilon_1 + \frac{1}{2} \epsilon_2^T \Phi^T \Phi \epsilon_2 + \frac{\lambda_1}{w_1^2} \xi_1^2 + \frac{\lambda_2}{w_1^2} \xi_2^2 + \frac{\lambda_3}{w_1^2} \xi_3^2 + \frac{\lambda_4}{w_2^4} \xi_4^2 + \frac{\lambda_5}{w_2^2} \xi_5^2. \]

Theorem 2. Considering the quadrotor UAV system in the UF frame as (2), (3), under the Assumptions 1, 2 and 3, the ESO designed with proposed feedback matrices in (22), (23), the controller designed in (34), (40), (49), (55), (57), (63), (64), with low-pass filter designed in (35), (42), (50), (58), the auxiliary systems designed in (41), (65), and properly selecting gains \( K_i (i = 1 \ldots 6, \xi_1, \xi_2), \tau_i (i = 1 \ldots 4), K_i (i = 1, 2) \) and bandwidth \( w_1, w_2 \) such that the matrix \( \Lambda \) in the following is pos-
itive definite, error dynamics \{s,z,\zeta,\epsilon\} are uniformly ultimately bounded, and the prescribed performance of state \(x_1\) and \(x_5\) are satisfied.

\[
\Lambda = \begin{bmatrix}
\Lambda_1 & -\Lambda_5 & -\Lambda_6 & -\Lambda_7 \\
-\Lambda_5^T & \Lambda_2 & 0 & 0 \\
-\Lambda_6 & 0 & \Lambda_3 & 0 \\
-\Lambda_7 & 0 & 0 & \Lambda_4
\end{bmatrix}
\] (72)

where

\[
\begin{align*}
\Lambda_1 &= \text{diag}\left( K_1 - \frac{1}{2}I, K_2 - \frac{2\tilde{g}_1}{2} + \frac{1}{2}I, K_3 - I, K_4 - \frac{2\tilde{g}_2}{2} + \frac{1}{2}I \right), \\
\Lambda_2 &= \text{diag}\left( \tau_1^{-1} - \frac{1}{2}I, \tau_2^{-1} - \frac{1}{2}I, \tau_3^{-1} - \frac{1}{2}I, \tau_4^{-1} - \frac{1}{2}I \right), \\
\Lambda_3 &= \text{diag}\left( k_1 - \frac{1}{2}I, k_2 - \frac{1}{2}I \right), \\
\Lambda_4 &= \text{diag}\left(\sigma_{21}^{12 \times 12}, \sigma_{22}^{10 \times 10} \right), \\
\Lambda_5 &= \begin{bmatrix}
\frac{\sigma_1}{2} & 0 & 0 & 0 \\
0 & \frac{\sigma_2}{2} & 0 & 0 \\
0 & 0 & \frac{\sigma_3}{2} & 0 \\
0 & 0 & 0 & \frac{\sigma_4}{2}
\end{bmatrix}, \\
\Lambda_6 &= \begin{bmatrix}
0 & 0 \\
0 & \frac{\sigma_5}{2} & 0 \\
0 & 0 & \frac{\sigma_6}{2} \\
0 & 0 & 0 & \frac{\sigma_7}{2}
\end{bmatrix}, \\
\Lambda_7 &= \begin{bmatrix}
\frac{\sigma_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sigma_9}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\sigma_{10}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\sigma_{11}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\sigma_{12}}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\sigma_{13}}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{14}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{15}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{16}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma_{17}}{2}
\end{bmatrix}
\end{align*}
\] (73)

**Proof of Theorem 2.** Substituting corresponding \(V\) and observer error dynamics (27), (28) into the derivative of (70) results in

\[
\dot{V} \leq -\omega^T \Lambda \omega + \zeta
\] (74)

where \(\omega = [s^T, z^T, \xi_1^T, \epsilon_1^T]^T\). Since the matrix \(\Lambda\) defined in (72) is positive definite, thus

\[
\begin{align*}
\dot{V} &\leq -\lambda_n(\Lambda) \left( \|s'\|^2 + \|z\|^2 + \|\xi\|^2 + \frac{\alpha_1^2}{\gamma_1} + \frac{\alpha_2^2}{\gamma_2} + \frac{\epsilon_1^T P^s \epsilon}{\lambda_m(P^s)} \right) + \zeta \\
&\leq -\theta V + \zeta
\end{align*}
\] (75)

where \(s' = [\epsilon_1^T, \epsilon_1^T, \epsilon_1^T, \epsilon_1^T]^T\) and \(\theta = 2\lambda_n(\Lambda)\min(1, \gamma_1, \gamma_2, \frac{1}{\lambda_m(P^s)})\). Thus, the right-hand side of (70) is upper bounded as

\[
V(t) \leq V(0)\exp(-\theta t) + \frac{\zeta}{\theta} (1 - \exp(-\theta t)).
\] (76)

Therefore, (70) and (76) indicate that \(s, z, \zeta\) and \(\epsilon\) are uniformly ultimately bounded. According to Lemma 1, the tracking error \(e_1\) and \(e_5\) remains within the prescribed bounds defined in (8). By appropriately choosing the constants \(\tilde{\sigma}, \beta, \rho_0\), the states \(\{x(t), y(t), z(t), \psi(t)\}\), which is \(x_1\) and \(x_5\), can track the desired trajectory \(\{x_d(t), y_d(t), z_d(t), \psi_d(t)\}\) with guaranteed errors. The proof is finished. \(\square\)
5. Simulation Results

In this section, simulations are carried out to demonstrate the effectiveness of the prescribed performance with the attitude and input saturation controller (PPAISC) for the quadrotor UAV. Additionally, the performance of the proposed nonlinear ESO based DSC is also illustrated.

The dynamics of a typical quadrotor UAV is governed by Equation (1) with the physical parameters shown in Table 1 from a small-size quadrotor UAV platform [52]. All the physical parameters were obtained through experiment parameter identification, specifications and theoretical analysis.

Table 1. Physical parameters of the quadrotor UAV.

<table>
<thead>
<tr>
<th>Description</th>
<th>Values</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Body mass</td>
<td>0.7735</td>
<td>Kg</td>
</tr>
<tr>
<td>Body inertia</td>
<td>( J_x = J_y = 5.196 \times 10^{-3}, J_z = 10.272 \times 10^{-3} )</td>
<td>Kg·m²</td>
</tr>
<tr>
<td>Rotor arm</td>
<td>( l = 0.165 )</td>
<td>m</td>
</tr>
<tr>
<td>Rotor thrust drag coefficient</td>
<td>( c = 0.0046 )</td>
<td>NA</td>
</tr>
<tr>
<td>Rotor thrust range</td>
<td>( T_{\text{max}} = 6.1049, T_{\text{min}} = 0.06 )</td>
<td>N</td>
</tr>
<tr>
<td>Air drag coefficient</td>
<td>( K_x = K_y = K_z = 0.01, K_p = K_q = K_r = 0.01 )</td>
<td>Nms²</td>
</tr>
</tbody>
</table>

The control parameters are selected, satisfying Theorem 2, as the following values:

\[
\begin{align*}
K_1 &= \text{diag}(5,5), K_2 = \text{diag}(2,2), K_3 = \text{diag}(20,20), \\
K_4 &= \text{diag}(5,5), K_5 = \text{diag}(20,20), K_6 = \text{diag}(5,5), \\
K_{\zeta_1} &= \text{diag}(0.5,0.5), K_{\zeta_2} = \text{diag}(0.5,0.5), \\
\kappa_1 &= \text{diag}(5,5), \kappa_1 = \text{diag}(20,20), \\
\tau_1 &= \tau_2 = \tau_3 = \tau_4 = \text{diag}(0.01,0.01), \\
w_1 &= w_2 = 5. \\
\end{align*}
\] (77)

In order to show the effectiveness of the proposed control scheme under attitude and input saturation, the prescribed performance control using the traditional DSC approach (PPDSC) without taking account of saturation constraints, the active disturbance rejection control (ADRC) strategies in the work of Zhang et al. [15], and, the most widely used in open source and real application, cascade-PID (CPID) control strategies [2] are also simulated as a comparison. To this end, the CPID controller is designed as follows:

\[
\begin{align*}
e_i &= K_{F,i}(x_{d,i} - x_i) - \dot{x}_i \\
U_j &= K_{P,i}e_i + K_{I,i} \int e_idt - K_{D,i}\ddot{x}_i
\end{align*}
\] (78)

where \( x_i, U_j, i = x, y, z, \phi, \theta, \psi \) are the states and control inputs of quadrotor UAV. Additionally, the CPID controller needs a conversion algorithm between position control output \( \{U_x, U_y, U_z\} \) and attitude control input \( \{U_1, x_{d,\phi}, x_{d,\theta}\} \) as

\[
\begin{align*}
U_1 &= \sqrt{U_x^2 + U_y^2 + U_z^2} \\
x_{d,\phi} &= \tan^{-1}\left(\frac{U_y S_{\psi} - U_x C_{\psi}}{U_z}\right) \\
x_{d,\theta} &= \tan^{-1}\left(\frac{U_z C_{\phi} + U_y S_{\phi}}{U_z}\right)
\end{align*}
\] (79)

For fair comparison, the control parameters of the CPID and ADRC are arrived at by trial and error such that these four controllers have nearly identical convergence rates, and the parameters of the PPDSC are chosen to be the same as that of the PPAISC. The original
is at zero, and the target position is given at $[1, -1, 1]^T$ with the heading angle remained at zero.

The disturbances can be caused by the external wind, air drag, blade flapping and installation error of four rotors, etc., and will cause both extra force and torque on the quadrotor UAV. Therefore, two kinds of disturbances are considered here: (1) external forces $d_\xi$ on the translational dynamics, which are unmatched disturbances for UF subsystem (2), and (2) external torques $d_\eta$ on rotational dynamics, which are matched disturbances for the UF subsystem (2). These two variables are defined as follows:

$$d_\xi = [0.6, 0.5, 0.2]^T$$
$$d_\eta = [1.0, 1.2, 0.5]^T$$  \hspace{1cm} (80)

The comparative results are shown from Figures 4–7. The time response of positions is depicted in Figure 4a. With the prescribed performance approach, PPDSC and PPAISC are within the required bounds during the transit convergence and the steady state error. The response errors for all controllers are converging to a small neighborhood of the target position; here, we take the x-axis as representative as shown in Figure 4b. The steady-state errors for PPDSC and PPAISC are largely reduced at about 4.5 s compared to CPID and ADRC as shown in the zoomed section in Figure 4b. Considering PPDSC and PPAISC in the zoomed section of Figure 4b, one can conclude that the steady-state errors are slightly affected by the augmented two auxiliary systems for attitude and the input saturation constraints cause bounded variables $\epsilon_1$ and $\epsilon_2$.

Although the backstepping-typed control scheme shows a strong capability in stabilization nonlinear systems, the obvious drawback is the steep variation of control variables at start. For controlling a quadrotor UAV, this phenomenon reflects as a large inclination angle of body as shown in Figure 5. Under the same convergence rates, our proposed attitude saturation method effectively constrains the inclination angle, $\sin^{-1}(\|\phi_1\|)$ in (4), within the predefined requirement (15 degrees).

Figure 6 shows the simulation results of disturbance estimation for ADRC, PPDSC and PPAISC. All three controllers integrated with ESO are successfully estimates $\Delta d_\xi$ and $\Delta d_\eta$ at a steady state. The main difference is that, in ADRC, the six ESOs are independent to each other inside each control channel, and only single measurement are considered for the observation feedback. For the proposed nonlinear ESO in PPAISC, the translational and rotational dynamics are considered together in one high order ESO. Thus, using multiple observation feedback, a uniform convergence speed of estimation can be reached. The simulation results exhibited a small variation during response and fast tracking of disturbance estimation compared to the ADRC method. In Figure 6b, the simulation of disturbance estimation for PPDSC during 0 to 0.3 second appears as large deviations, which is caused by the unmodeled limitation of input variables as shown in Figure 7. Compared to the proposed PPAISC with input saturation, the out-ranged control variables are dominated by the output of the auxiliary systems result as constrained control variables for the estimation of disturbances in ESO. Furthermore, the system stability is also enhanced.

To quantitatively compare the response performance of various controllers, six performance indices are used as following:

1. Integral squared errors (ISE) of position are defined as [31]

$$ISE = \int_0^T \left( x_e^2(t) + y_e^2(t) + z_e^2(t) \right) dt$$

where $x_e(t)$, $y_e(t)$ and $z_e(t)$ are position errors, thus the controller with lower ISE index reflects a fast convergence speed.

2. Integral time-multiplied absolute errors (ITAE) of position are defined as [31]

$$ITAE = \int_0^T t \left( |x_e|(t) + |y_e|(t) + |z_e|(t) \right) dt.$$
Different from ISE, ITAE considers the steady-state error rather than the initial response, thus the controller with lower ITAE index reflects a smaller steady-state errors.

(3) Maximum inclination angle (MIA) is $\sin^{-1}(\|\phi_1\|)$.

(4) Variance of thrust force (VTF). The controller with a lower VTF index reflects a smooth output and less aggressive maneuvers.

(5) Root mean square error of estimated force disturbances (RMSEEFD). The observer with lower RMSEEFD index means a faster convergence speed and fewer oscillations during estimation of the force disturbances on the translational dynamics.

(6) Root mean square error of estimated torque disturbances (RMSEETD). The observer with lower RMSEETD index means a faster convergence speed and less oscillations during estimation of the force disturbances on the rotational dynamics.

The quantitative results of performance indices are collected in Table 2. Since there is no observer in the CPID method, RMSEEFD and RMSEETD items are neglected. The control method of PPDSC is the same as PPAISC except attitude and input saturation constraints. Therefore, the lowest values in the term ISE and ITAE reflect faster convergence character and less steady-state errors of PPAISC and PPDSC compared with ADRC and CPID. The effectiveness of our proposed attitude and input saturation constraints are shown in the values of MIA and VTF. The values in RMSEEFD and RMSEETD show that the proposed ESO design for quadrotor UAV system has faster convergence and fewer oscillations, compared to conventional ESOs.

![Graphs of x, y, z positions](image)

**Figure 4.** Time responses of positions along $x$-axis, $y$-axis and $z$-axis under difference controllers. (a) Positions with upper and lower bounds of prescribed performance function. (b) Position errors.
Figure 5. Time responses of inclination angle \( \sin^{-1}(\|\varphi_1\|) \) under difference controllers.

Figure 6. Time responses of disturbance estimation under difference controllers. (a) External force disturbances \( d_x \). (b) External torque disturbances \( d_\eta \).

Figure 7. Time responses of thrust force generated by four rotors under different controllers.
Table 2. The performance comparisons of different control scheme.

<table>
<thead>
<tr>
<th>Controller</th>
<th>ISE</th>
<th>ITAE</th>
<th>MIA</th>
<th>VTF</th>
<th>RMSEEFD</th>
<th>RMSEETD</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPID</td>
<td>2.3334</td>
<td>2.4060</td>
<td>19.5259</td>
<td>3.3635</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ADRC</td>
<td>1.8498</td>
<td>1.9539</td>
<td>23.8298</td>
<td>3.0931</td>
<td>1.8140</td>
<td>4.0141</td>
</tr>
<tr>
<td>PPDSC</td>
<td><strong>1.6102</strong></td>
<td><strong>1.2179</strong></td>
<td>22.3933</td>
<td>2.9834</td>
<td>1.5764</td>
<td>3.2063</td>
</tr>
<tr>
<td>PPAISC</td>
<td>1.69476</td>
<td>1.2330</td>
<td><strong>15.0335</strong></td>
<td>2.2569</td>
<td>0.6319</td>
<td>2.1411</td>
</tr>
</tbody>
</table>

6. Conclusions

Prescribed performance control for quadrotor UAV under limited attitude and input saturation constraints are addressed in this paper. Based on the underactuated subsystem of the quadrotor UAV, a nonlinear extended state observer is proposed to tackle with the pure-feedback system, and multiple observations are designed to estimate the matched and unmatched disturbances on translational and rotational dynamics. In addition to the important input saturation, the safe attitude range for the quadrotor UAV is also considered as a saturation constraint in the control scheme with a compensation auxiliary system. In order to implement attitude saturation constraints, dynamic surface control is adopted with prescribed performance to guarantee the convergence. As verified by employing the Lyapunov technique, all the errors are uniformly and ultimately bounded. Finally, comparison simulations are demonstrated to verify the effectiveness of the proposed control strategy.

In future research, aiming for flying safety for the quadrotor UAV in a limited space, the fault tolerant control will be taken into account with more complex input constraints, such as loss of effectiveness, input bias and input delay [31]. Moreover, the finite-time convergence technique for ESO [53] will be considered to further improve the estimate performance. For practical implementations, the measurement noise is also a challenge issue for designing an ESO for the quadrotor UAV system [54].

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Appendix A

The variables \( \{J_1, J_2, J_3, J_4, J_5\} \) are computed as

\[
J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
J_2 = \begin{bmatrix} C_\phi S_\theta - C_\theta S_\phi S_\theta & C_\phi C_\theta \\ -C_\phi C_\theta & S_\phi S_\theta \end{bmatrix}
\]

\[
J_3 = \begin{bmatrix} J_{3,1} & J_{3,2} \\ J_{3,3} & J_{3,4} \end{bmatrix}
\]

\[
J_4 = \begin{bmatrix} (C_\phi S_\theta - C_\theta S_\phi S_\theta) & (C_\phi^2 C_\theta - C_\phi S^2_\phi S_\theta + C_\phi S_\theta S_\phi S_\theta)/C_\theta \\ -(C_\phi C_\theta + S_\phi S_\theta) & -(C_\phi^2 S_\phi^2 S_\theta^2 + C_\phi S_\theta S_\phi S_\theta)/C_\theta \end{bmatrix}
\]

\[
J_5 = \begin{bmatrix} C_\phi/C_\theta & 0 \\ 0 & 1 \end{bmatrix}
\]

where

\[
J_{3,1} = -(qS_\phi S_\theta + 2qC_\phi C_\theta S_\phi + pC_\phi S_\theta S_\phi - 2qC_\phi^2 S_\phi S_\theta + pC_\phi C_\phi S_\theta)/C_\theta
\]

\[
J_{3,2} = -qC_\phi^2 S_\theta - C_\phi S_\phi (pC_\phi + qS_\phi S_\theta) + qS_\phi (C_\phi S_\phi - C_\phi S_\theta S_\phi)/(T_\phi^2 + 1)
\]

\[
J_{3,3} = -(qC_\phi S_\theta - 2qC_\phi C_\theta S_\phi - pC_\phi C_\phi S_\theta + 2qC_\phi^2 C_\theta S_\phi + pC_\phi C_\phi S_\phi S_\theta)/C_\theta
\]

\[
J_{3,4} = -qC_\phi^2 S_\theta S_\phi + S_\phi S_\theta (pC_\phi + qS_\phi S_\theta) - qS_\phi (C_\phi C_\phi + S_\phi S_\phi S_\theta)/(T_\phi^2 + 1)
\]

References


52. Ma, T.; Wong, S. Trajectory tracking control for quadrotor UAV. In Proceedings of the 2017 IEEE International Conference on Robotics and Biomimetics (ROBIO), Macau, Macao, 5–8 December 2017; pp. 1751–1756. [CrossRef]
