Article
Adaptive Compensation Tracking Control for Time-Varying Delay Nonlinear Systems with Unknown Actuator Dead Zone

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Abstract: This paper concerns the problem of adaptive compensation tracking control for a class of time-varying delay nonlinear systems with unknown structures and unknown actuator dead zones where time-varying delays are unknown. First, a variable separation approach is used to overcome the difficulty in dealing with a nonstrict-feedback structure, and multilayer neural networks are used to approximate the unknown nonlinear structures with time-varying delays. On this basis, we designed an adaptive multilayer neural-network compensation controller to reduce the error of multilayer neural networks. Furthermore, for unknown actuator dead zones, this paper separates the controller and adopts multilayer neural networks to deal with unknown actuator dead zones. In order to reduce the error of the dead-zone controller, we designed an adaptive compensation controller for the dead zones. Lastly, this paper proves the stability of the systems with the Lyapunov method, and simulation results demonstrate the effectiveness of the scheme.

Keywords: time-varying delay nonlinear systems; actuator dead zone; nonstrict feedback; multilayer neural networks; adaptive compensation control

1. Introduction

Time-varying delays widely exist in a large number of practical systems, such as material production processes in the chemical field, complex networks with multiple nodes, and mechanical engineering. Time-varying delays often increase the instability of systems, so it is of great significance to study the control of time-varying delay systems. Many researchers have paid much attention to stability analysis and control design for time-varying delay systems [1–6]. In particular, the control problem of time-varying delay nonlinear systems with unknown structures is a considerable research interest.

For nonlinear systems with time-varying delays, various control methods are used, such as impulsive [7,8], robust [9,10], H∞ [11,12], and adaptive [13,14] controls, and the backstepping method [15,16]. Backstepping is a simple and practical controller design method, and its combination with adaptive control has become a common method for the controller design of nonlinear systems with time-varying delays. Recently, there have been many research results on adaptive backstepping control for nonlinear systems with time-varying delays; see, for example, [17,18]. To list a few, the problem of state feedback stabilization for a class of stochastic time-varying delay nonlinear systems was investigated in [17] by using the backstepping design method and adding a power integrator technique. In [18], a predictor feedback controller was designed for discrete-time nonlinear systems to compensate for input delay. However, the systems in these studies are required to be known and cannot contain unknown structures, which greatly limits their applicability in practice. Therefore, in recent years, research on time-varying delay nonlinear systems with unknown structures has been gradually developed. In order to deal with unknown structures, a large number of studies used neural networks [19–25]. Adaptive neural network backstepping control has gradually become a hot topic. An adaptive dynamic
surface control was adopted in [19] for nonlinear systems with unknown nonlinearities and bounded time-varying state delays. The problem of adaptive output-feedback control for uncertain stochastic nonlinear strict-feedback systems with time-varying delays using neural networks was studied in [20]. By using Filippov’s theory, the problem of adaptive neural tracking control was studied in [21] for uncertain nonlinear time-delay systems. The problem of adaptive neural control was considered in [22] for strict-feedback stochastic nonlinear systems with multiple time-varying delays. The problem of adaptive neural control for uncertain stochastic pure-feedback nonlinear systems with time-varying delays was studied in [23]. A prescribed performance adaptive neural tracking control problem was investigated in [24] for strict-feedback Markovian jump nonlinear systems with time-varying delays. An adaptive neural network tracking control method was developed in [25] for uncertain nonlinear strict-feedback systems with time-varying full-state constraints. Most of these control strategies use neural networks to approximate unknown structures, which have errors. However, the control strategies do not compensate the errors, which decrease the control effect in many practical situations.

Moreover, dead-zone characteristics are encountered in many physical components of control systems [26–28]. They are ubiquitous in nonlinear systems, such as power systems, motor systems and biological systems. Affected by the dead zone, the systems become oscillating, and the regulation quality declines. Thus, it is more realistic and reliable to design controllers on the basis of the system model where dead-zone nonlinearities are taken into consideration. In recent years, there have been some research results on dead zones [29–31], but these results are based on other methods. Few results on adaptive neural network backstepping compensation control with dead-zone compensation have been developed for time-varying delay nonlinear systems with unknown structures and unknown dead-zone characteristics.

The above observations show that it is both practically and theoretically significant to investigate the problem of adaptive compensation tracking control for time-varying delay nonlinear systems with unknown structures and unknown dead-zone characteristics; however, this is challenging. This motivated us to carry out the present study. In this paper, a variable separation approach is used to overcome the difficulty in dealing with nonstrict-feedback structures, and the multilayer neural networks method is used to approximate unknown nonlinear structures with time-varying delays. On this basis, for the studied system, a new approach of constructing common virtual control functions is proposed, and a multilayer adaptive compensation controller was designed to reduce the errors of neural networks by using the backstepping method. Furthermore, for unknown actuator dead zones, a compensation controller is proposed by using multiple neural networks. Lastly, the stability of the control system is proven via the Lyapunov method. The contributions of the paper are as follows: (1) our considered time-varying system model is a nonstrict-feedback form; (2) the unknown structures include time-varying delays, which can be completely unknown; (3) the unknown nonsymmetric actuator dead zone is taken into account; (4) the approximation errors of neural networks are considered; (5) the double layer compensation of neural networks is considered instead of single weight compensation; (6) the bounded stability of the resulting control system is analyzed with a new Lyapunov function.

This paper is organized as follows. In Section 2, the problem is formulated. In Section 3, the designed neural-network adaptive compensation controller is outlines. In Section 4, the effectiveness of the proposed scheme is demonstrated through numerical examples. Lastly, the conclusions are drawn in Section 5.

**Notation:** \( \mathcal{R}^n \) denotes the \( n \)-dimensional space, and \( \mathcal{R}^+_n \) is the set of all non-negative real numbers. For a given matrix \( A \) or vector \( B \), \( A^T \) and \( B^T \) denote their transpose, and \( \text{tr} \{ A \} \) denotes their trace. \( \| \cdot \|_F \) is the Frobenius norm, and \( \| \cdot \|_2 \) refers to the 2-norm.
2. Preliminaries

Considering the following time-varying delay nonlinear system in nonstrict-feedback form with unknown structures and unknown dead zones in this paper, the same system can be found in [20,21]:

\[
\begin{align*}
\dot{x}_i(t) &= a_i(t) f_i(x_i(t), x_i(t - \tau_i(t))), \quad i = 1, 2, \ldots, n - 1 \\
\dot{x}_n(t) &= b_n(x_n(t), x_n(t - \tau_n(t))), \quad \tau_n(t) \in J \\
v(t) &= D(u(t)) \\
t_j = h_j(t), \quad j = 1, 2, \ldots, n \\
y &= x_1
\end{align*}
\]

where \( J \) denotes set \([0, \infty)\); \( x_i \in \mathcal{R} \) denotes the system state at the \( i \)th order; \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathcal{R}^n \) denotes the set of system states; \( \tau_i \in \mathcal{R} \) is the time-varying delay; \( u(t) \in \mathcal{R} \) is the system input; \( y \in \mathcal{R} \) is the output; \( f_i : \mathcal{R}^n \times \mathcal{R} \rightarrow \mathcal{R} \) is a nonlinear continuously differentiable function that is known; \( h_j : \mathcal{R}^n \rightarrow [0, \tau_j) \) is a time-varying delay function; \( \tau_j \) is the upper bound of time-varying delay at the \( j \)th order; \( g_i \in \mathcal{R} \) is a continuously differentiable function that is known; \( D(\cdot) \) is a nonsymmetric dead-zone nonlinearity function, which is defined as follows:

\[
v(t) = D(u(t)) = \begin{cases} 
  k_r(u(t) - b_r), & u(t) \geq b_r \\
  0, & -b_l \leq u(t) \leq b_r \\
  k_l(u(t) + b_l), & u(t) \leq -b_l
\end{cases}
\]

where \( k_r > 0 \) and \( k_l > 0 \) stand for the right and the left slopes of the dead-zone characteristic, respectively; \( b_r > 0 \) and \( b_l > 0 \) represent the breakpoints of the input nonlinearity. \( k_r, k_l, b_r, \) and \( b_l \) are unknown in this paper.

In this paper, the neural-network adaptive compensation controller was designed, so that output \( y \) of System (1) could track a time-varying signal \( y_d \).

Assumption 1. \( g_i \) is a known continuously differentiable function that is strictly either positive or negative.

Assumption 2. Time-varying delay function \( h_j \) is unknown, and there exist \( \varnothing \geq 0 \), such that \( h_j \leq \varnothing, j = 1, 2, \ldots, n \) hold for all \( t \geq 0 \), and \( \varnothing = \max\{\tau_1, \ldots, \tau_n\} \).

Assumption 3. Time-varying target \( y_d \) is a continuously differentiable function, and its time derivatives up to the \( n \)-th order \( y_d^{(n)} \) are continuous and bounded, which satisfies \( |y_d^{(n)}| \leq d \), where \( d \geq 0 \) is an a priori known constant.

Assumption 4. For unknown function \( f_i(\cdot), i = 1, \ldots, n \), there exists strictly increasing smooth function \( \phi_i(\cdot) : \mathcal{R}^+ \rightarrow \mathcal{R}^+ \) with \( \phi_i(0) = 0 \) such that for \( i = 1, 2, \ldots, n \)

\[
|f_i(x)| \leq \phi_i(||x||_F).
\]

Remark 1. \( \phi_i(s) \) is a smooth function, and \( \phi_i(0) = 0 \). Therefore, there exists a smooth function \( \eta_i(s) \), such that \( \phi_i(s) = s \cdot \eta_i(s) \), and this can result in

\[
\phi_i \left( \sum_{j=1}^{n} a_j \right) \leq \sum_{j=1}^{n} n a_j \eta_i(n a_j),
\]

where \( a_j \geq 0 \).
In the controller design and stability analysis procedure, multilayer neural networks (MNNs) are used to approximate unknown structures and unknown dead zones. In this paper, multilayer neural networks (MNNs) are described as follows:

\[ k = W^T S(V I), \]

where \( I \in \mathbb{R}^N \) is the input vector of MNNs, and \( N \) is number of inputs; \( V \in \mathbb{R}^{N \times N} \) is the first ideal weight matrix and \( N_w \) is the number of neurons; \( W \in \mathbb{R}^{N \times N} \) is the second ideal weight vector; \( S(\cdot) \) is the activation function, and \( S(x) = \frac{1}{1+\exp(-\lambda x)}, (\lambda > 0) \).

**Lemma 1** ([32]). \( k(\cdot) \) is a continuous function defined on a set \( \Omega \). Then, for a given desired level of accuracy \( \varepsilon > 0 \), there exists a function \( \bar{k} \) to satisfy the following inequality:

\[ \sup_{x \in \Omega} \| k(\cdot) - \bar{k} \| \leq \varepsilon. \]  

**Lemma 2.** For neural networks, ideal weights \( V \) and \( W \), estimated weights \( \hat{V} \) and \( \hat{W} \) define \( \hat{S} = S(\hat{V} I), S = S(V I), \hat{S} = \hat{S}'(\hat{V} I), \) error weight matrix \( \hat{V} = V - \hat{V} \), and error weight vector \( \hat{W} = W - \hat{W} \),

\[ W^T S - \hat{W}^T \hat{S} = \hat{W}^T \hat{S}' \hat{V} I + \hat{W}^T \hat{S}' \hat{V} I + \Phi, \]

where \( \Phi = \hat{W}^T \hat{S}' \hat{V} I + W^T O(\hat{V} I)^2 \), and we have Inequality (8).

\[ |\Phi| \leq \| W \|_2 + \| W \|_F \cdot \| \hat{S}' V I \|_F + \| V \|_F \cdot \| W \|_F. \]  

The proof of Lemma 2 can be found in Appendix A.

**Lemma 3** ([33]). Suppose that there exist a locally Lipschitz continuous functional \( V : \mathbb{R}^+ \times \mathbb{R}^n \to [0, \infty) \), two \( K_\infty \) functions \( \mu_i, i = 1, 2 \), two constants \( c_1 > 0 \) and \( c_2 > 0 \), such that, for all \( t \in J \),

\[ \mu_1(|x(t)|) \leq V(t, x) \leq \mu_2(|x(t)|), \]

\[ \hat{V}(t, x) \leq -c_1 V(t, x) + c_2. \]

Then, System (1) is ISS.

From Inequality (6), Formula (11) can be obtained,

\[ k(\cdot) = \bar{k} + \delta, \]

where \( \delta \) is a constant.

For the approximate ideal function \( \bar{k} \), \( V \) and \( W \) are the ideal approximate weights. The ideal values cannot be reached in the calculation process; therefore, estimated values \( \hat{V} \) and \( \hat{W} \) are taken as the calculated values. The approximate estimated function \( \bar{k} \) is as follows:

\[ \bar{k} = \hat{W}^T S(\hat{V} I). \]

Formula (11) can be rewritten as follows:

\[ k(\cdot) = \bar{k} + \omega + \delta, \]

where \( \omega = \bar{k} - \bar{k} = W^T S - \hat{W}^T \hat{S} \) denotes the error between the ideal and estimated approximate functions.
In order to facilitate analysis and design, it was assumed that the nonsymmetric dead-zone nonlinearity could be reformulated as follows:

\[ v(t) = D(u(t)) = D'(u(t)) + \delta'_D, \]  

where \( v(t) \) is the actual system input, \( D'(u(t)) \) is an unknown smooth function, and \( \delta'_D \) is the bounded error between \( D(u(t)) \) and \( D'(u(t)) \). Moreover, we have

\[ v(t) = u(t) + D'(u(t)) - u(t) + \delta'_D. \]  

By using Formula (11), neural networks \( \bar{D}(u(t)) \) can be used to approximate \( D'(u(t)) - u(t) \), and we have

\[ v(t) = u(t) + \bar{D}(u(t)) + \delta_D, \]  

where \( \bar{D}(u(t)) = W_D^T \hat{S}(V_DI_D); \delta_D \) is the bounded error between \( D(u(t)) \) and \( u(t) + \bar{D}(u(t)). \) Considering Formula (13), we have

\[ v(t) = u(t) + \bar{D}(u(t)) + \omega_D + \delta_D, \]  

where \( \omega_D = \bar{D} - \hat{D} = W_D^T \hat{S}(V_DI_D) - W_D^T \bar{S}(V_DI_D) \) denotes the error between the ideal approximate function and estimated approximate function.

3. Main Results

In this section, a systemic control design and stability analysis procedure is presented that uses the adaptive backstepping technique. Considering System (1), the adaptive control scheme was developed with the following change of coordinates:

\[ e_1 = y - y_d = x_1 - y_d, \]  

\[ e_i = x_i - a_{i-1}, i = 2, \ldots, n, \]  

where \( e_i, i = 1, \ldots, n \) are the virtual errors, and \( a_{i-1}, i = 1, \ldots, n \) are the virtual controllers.

For \( i = 1, \ldots, n - 1 \), a common virtual control function \( \alpha_i \) can be defined as follows.

\[ \alpha_i = -\frac{1}{\delta_i} \left( \lambda_i + \frac{3}{2} e_i + \frac{e_i^3}{2m_i^2} + \frac{e_i^3}{2m_i^2} \cdot \hat{\theta}_{w,i} \cdot \| \hat{S}_i[V_i] \|_2^2 + \frac{e_i^3}{2m_i^2} \cdot \hat{\theta}_{b,i} \cdot \| I_i \hat{S}_i[V_i] \|_2^2 \right), \]  

where \( \lambda_i, a_i, m_i, m_2 > 0 \) are design parameters; \( a_0 = y_d; \hat{k}_i \) is the estimated output of neural networks, which is specified later; \( \hat{\theta}_{w,i} \) and \( \hat{\theta}_{b,i} \) are unknown constants, which are specified later; \( I_i \) is the neural networks’ input; \( I_i = [e, x, e_i(t-t_i), x_i(t-t_i), y_d(i), \hat{\theta}_{w,1}, \ldots, \hat{\theta}_{w,n}]^T \); \( e = [e_1, e_2, \ldots, e_n]; e_i(t-t_i) = [e_1(t-t_i), e_2(t-t_i), \ldots, e_i(t-t_i), e_i(t-t_i), e_i(t-t_i), e_i(t-t_i), \ldots, e_i(t-t_i)] \); \( x_i(t-t_i) = [x_1(t-t_i), x_2(t-t_i), \ldots, x_i(t-t_i), x_i(t-t_i), x_i(t-t_i), \ldots, x_i(t-t_i)] \); \( y_d(i) = [y_d, y_d(i), \ldots, y_d(i)] \).

\( \dot{\alpha}_{i-1} \) can be obtained as follows.

\[ \dot{\alpha}_{i-1} = \sum_{j=1}^{n} \frac{\partial \alpha_{j-1}}{\partial x_j} \dot{x}_j + \sum_{j=0}^{i-1} \frac{\partial \alpha_{j-1}}{\partial y_{d(j)}} \dot{y}_{d(j+1)} + \sum_{j=1}^{n} \frac{\partial \alpha_{j-1}}{\partial e_j} \dot{e}_j + \sum_{j=1}^{i-1} \frac{\partial \alpha_{j-1}}{\partial e_j(t-t_j)} \dot{e}_j(t-t_j) + \sum_{j=1}^{i-2} \frac{\partial \alpha_{j-1}}{\partial e_j(t-t_j+1)} \dot{e}_j(t-t_j+1) + \sum_{j=1}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}_{w,j}} \dot{\hat{\theta}}_{w,j} + \sum_{j=1}^{i-1} \frac{\partial \alpha_{j-1}}{\partial \hat{\theta}_{b,j}} \dot{\hat{\theta}}_{b,j}. \]  

(21)
Consider the following Lyapunov function candidate:

$$ V = \sum_{i=1}^{n} \left[ \frac{1}{4} \varphi_i - \frac{1}{2\beta_{v,i}} \mathbf{W}_i^T \mathbf{W}_i + \frac{1}{2\beta_{v,i}} tr\{\mathbf{V}_i^T \mathbf{V}_i\} + \frac{1}{2P_{v,i}} \theta_{w,i}^2 + \frac{1}{2P_{v,i}} \theta_{v,i}^2 \right] $$

$$ + \frac{1}{2\beta_{w,D}} \mathbf{W}^T_D \mathbf{W}_D + \frac{1}{2P_{v,D}} tr\{\mathbf{V}_D^T \mathbf{V}_D\} + \frac{1}{2P_{v,D}} \theta_{w,D}^2 + \frac{1}{2P_{v,D}} \theta_{v,D}^2, $$

(22)

where $\beta_{w,i}$, $\beta_{v,i}$, $P_{w,i}$, $P_{v,i}$, $\theta_{w,D}$, $\theta_{v,D}$, $P_{v,D}$, $P_{w,D} > 0$ are design parameters; $\bar{\theta}_{w,i} = \theta_{w,i} - \bar{\theta}_{w,i}$, $\bar{\theta}_{v,i} = \theta_{v,i} - \bar{\theta}_{v,i}$ with $\bar{\theta}_{w,i}$ and $\bar{\theta}_{v,i}$ specified later; $\hat{\theta}_{w,i}$ and $\hat{\theta}_{v,i}$ represent the estimation of $\theta_{w,i}$ and $\theta_{v,i}$, respectively; $\hat{\theta}_{w,D} = \theta_{w,D} - \hat{\theta}_{w,D}$, $\hat{\theta}_{v,D} = \theta_{v,D} - \hat{\theta}_{v,D}$ with $\hat{\theta}_{w,D}$ and $\hat{\theta}_{v,D}$ specified later; $\bar{\theta}_{w,D}$ and $\bar{\theta}_{v,D}$ represent the estimation of $\theta_{w,D}$ and $\theta_{v,D}$ respectively.

**Theorem 1.** From the coordinates, the following inequality holds:

$$ \|x, x_i(t - t_i)\|_F \leq \sum_{i=1}^{n} (|e_i| \phi_i) + |e_i(t - t_i)| + \phi_{i-1}(t - t_i)|e_{i-1}(t - t_i)| + d, $$

(23)

where $\phi_i = |a_i|/|e_i| + 1 = |s_i|/(|\lambda_i + \frac{3}{2}| + \frac{c_i^2}{2\lambda_i^2} + \frac{c_i^2}{2\lambda_i^2} \cdot \hat{\theta}_{w,i} \cdot \|\hat{\theta}_{v,i}\|_F^2 + \frac{c_i^2}{2\lambda_i^2} \cdot \hat{\theta}_{v,i} \cdot \|\hat{I_iW_iT_iS_i}\|_F^2 + 1, \phi_{i-1}(t - t_i) = |a_i(t - t_i)|/|e_i(t - t_i)| + 1, \text{ for } j = 1, \cdots, n - 1, \text{ and } \phi_n = 1.$

The proof of Theorem 1 can be found in Appendix B.

By using Formula (22), $V$ can be given by

$$ V = \sum_{i=1}^{n} |e_i| \phi_i + |e_i(t - t_i)| + \phi_{i-1}(t - t_i)|e_{i-1}(t - t_i)| + d, $$

(24)
By using Assumption 2, Theorem 1, and Formulas (19) and (21), $V$ can be given by
\[
V \leq \sum_{i=1}^{n-1} \left[ e_i^2 \gamma_i x_{i+1} + \sum_{j=1}^{n} \left| e_j^2 \cdot \phi_i(\|x, x_j(t-t_i)\|_F) - \sum_{j=1}^{n} e_j^2 \gamma_{i,j-1} + e_{n+1} x_{n+1} \right] + e_{n+1} y_{n+1} \right]
- \sum_{i=1}^{n} \left[ \frac{1}{\beta_{w,i}} \mathbf{W}_i^T \mathbf{W}_i + \frac{1}{\beta_{v,i}} \text{tr} \{ \mathbf{V}_i^T \mathbf{V}_i \} + \frac{1}{\beta_{w,i}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T + \frac{1}{\beta_{v,i}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T \right]
- \frac{1}{\beta_{w,D}} \mathbf{W}_D^T \mathbf{W}_D - \frac{1}{\beta_{v,D}} \text{tr} \{ \mathbf{W}_D^T \mathbf{W}_D \} - \frac{1}{\beta_{w,D}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T - \frac{1}{\beta_{v,D}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T
\leq \sum_{i=1}^{n-1} \left[ e_i^2 \gamma_i x_{i+1} + \sum_{j=1}^{n} \left| e_j^2 \cdot \phi_i(\|x, x_j(t-t_i)\|_F) - \sum_{j=1}^{n} e_j^2 \gamma_{i,j-1} + e_{n+1} x_{n+1} \right] + e_{n+1} y_{n+1} \right]
- \sum_{i=1}^{n} \left[ \frac{1}{\beta_{w,i}} \mathbf{W}_i^T \mathbf{W}_i + \frac{1}{\beta_{v,i}} \text{tr} \{ \mathbf{V}_i^T \mathbf{V}_i \} + \frac{1}{\beta_{w,i}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T + \frac{1}{\beta_{v,i}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T \right]
- \frac{1}{\beta_{w,D}} \mathbf{W}_D^T \mathbf{W}_D - \frac{1}{\beta_{v,D}} \text{tr} \{ \mathbf{W}_D^T \mathbf{W}_D \} - \frac{1}{\beta_{w,D}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T - \frac{1}{\beta_{v,D}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T
\leq \sum_{i=1}^{n-1} \left[ e_i^2 \gamma_i x_{i+1} + \sum_{j=1}^{n} \left| e_j^2 \cdot \phi_i(\|x, x_j(t-t_i)\|_F) - \sum_{j=1}^{n} e_j^2 \gamma_{i,j-1} + e_{n+1} x_{n+1} \right] + e_{n+1} y_{n+1} \right]
- \sum_{i=1}^{n} \left[ \frac{1}{\beta_{w,i}} \mathbf{W}_i^T \mathbf{W}_i + \frac{1}{\beta_{v,i}} \text{tr} \{ \mathbf{V}_i^T \mathbf{V}_i \} + \frac{1}{\beta_{w,i}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T + \frac{1}{\beta_{v,i}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T \right]
- \frac{1}{\beta_{w,D}} \mathbf{W}_D^T \mathbf{W}_D - \frac{1}{\beta_{v,D}} \text{tr} \{ \mathbf{W}_D^T \mathbf{W}_D \} - \frac{1}{\beta_{w,D}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T - \frac{1}{\beta_{v,D}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T
\leq \sum_{i=1}^{n-1} \left[ e_i^2 \gamma_i x_{i+1} + \sum_{j=1}^{n} \left| e_j^2 \cdot \phi_i(\|x, x_j(t-t_i)\|_F) - \sum_{j=1}^{n} e_j^2 \gamma_{i,j-1} + e_{n+1} x_{n+1} \right] + e_{n+1} y_{n+1} \right]
- \sum_{i=1}^{n} \left[ \frac{1}{\beta_{w,i}} \mathbf{W}_i^T \mathbf{W}_i + \frac{1}{\beta_{v,i}} \text{tr} \{ \mathbf{V}_i^T \mathbf{V}_i \} + \frac{1}{\beta_{w,i}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T + \frac{1}{\beta_{v,i}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T \right]
- \frac{1}{\beta_{w,D}} \mathbf{W}_D^T \mathbf{W}_D - \frac{1}{\beta_{v,D}} \text{tr} \{ \mathbf{W}_D^T \mathbf{W}_D \} - \frac{1}{\beta_{w,D}^2} \mathbf{\theta}_w \mathbf{\theta}_w^T - \frac{1}{\beta_{v,D}^2} \mathbf{\theta}_v \mathbf{\theta}_v^T.
\] (25)
 Define \( k_i \) as
\[
k_i = \sum_{j=1}^{n} \phi_i((n+3)|e_j|\varphi_j) + \phi_i((n+3)|e_i(t-t_i)|) + \phi_i((n+3)|d)
\]
\[
+ \phi_i((n+3)|\varphi_{i-1}(t-t_i)|e_{i-1}(t-t_i)|) + \text{sign}(e_i) \cdot |g_i e_{i+1}| - \sum_{j=1}^{n} \frac{\partial \varphi_{i-1}}{\partial x_j} x_j
\]
\[
- \sum_{j=0}^{i} \frac{\partial \varphi_{i+1}}{\partial y_{j}^{d}} y_{(j+1)}^{d} - \sum_{j=1}^{i} \frac{\partial \varphi_{i+1}}{\partial e_j} e_j - \sum_{j=1}^{i-1} \frac{\partial \varphi_{i+1}}{\partial \theta_{w_{i,j}}} \theta_{w_{i,j}} - \sum_{j=1}^{i-1} \frac{\partial \varphi_{i+1}}{\partial \theta_{v_{i,j}}} \theta_{v_{i,j}}
\]
\[
- \sum_{j=1}^{i-2} \frac{\partial \varphi_{i+1}}{\partial e_j(t-t_j)} e_j(t-t_j) - \sum_{j=1}^{i-2} \frac{\partial \varphi_{i+1}}{\partial e_j(t-t_{j+1})} e_j(t-t_{j+1}).
\] (26)

By using neural networks’ approximation ability, \( k_i \) can be approximated by \( \tilde{k}_i \). Considering Formula (13), we can obtain \( \tilde{V} \).

\[
\tilde{V} \leq \sum_{i=1}^{n-1} [e_i^T g_i \varphi_i] + \sum_{i=1}^{n} [e_i^T (\tilde{k}_i + \omega_i + \delta_i)] + e_n^T g_n v - \sum_{i=1}^{n} \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T
\]
\[
+ \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{w_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T
\]
\[
- \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T)
\] (27)

where \( \tilde{k}_i = \tilde{W}_i^T S (\tilde{V}_i I_i), \omega_i = \tilde{W}_i^T S (\tilde{V}_i I_i) - \tilde{W}_i^T S (\tilde{V}_i I_i) \).

According to Formula (26), the neural networks’ input \( I_i \) can be obtained as follows.

\[
I_i = [e, x, e_i(t-t_i), x_i(t-t_i), y_d^{(i)}, \theta_{w_{i,1}}, \ldots, \theta_{w_{i,n}}]^T.
\] (28)

Considering Lemma 2, we have

\[
\tilde{V} \leq \sum_{i=1}^{n-1} [e_i^T g_i \varphi_i] + \sum_{i=1}^{n} [e_i^T (\tilde{k}_i + \omega_i + \delta_i)] + e_n^T g_n v - \sum_{i=1}^{n} \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T
\]
\[
+ \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{w_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T
\]
\[
- \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T)
\]
\[
- \sum_{i=1}^{n-1} e_i^T g_i \varphi_i + \sum_{i=1}^{n} e_i^T \frac{3}{4} e_i + e_n^T \frac{3}{4} e_n + \sum_{i=1}^{n} [e_i^T (\tilde{W}_i^T S_i - \tilde{W}_i^T S_i)]
\]
\[
+ e_n^T g_n v - \sum_{i=1}^{n} \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T + \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{w_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T)
\]
\[
- \frac{1}{\beta_{w_{i}}} \tilde{W}_i^T \tilde{W}_i^T - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{w_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) - \frac{1}{\beta_{v_{i}}} \text{tr} (\tilde{V}_i^T \tilde{V}_i^T) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^T
\]
\[ \leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} |c_i^2| (\tilde{W}_i^T S_i] \\
- \tilde{W}_i^T S_i [\tilde{V}_i \langle 1 \rangle + \tilde{W}_i^T S_i [\tilde{V}_i \langle 1 \rangle + \Phi_i]) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \frac{1}{\beta_{v,j}} \text{tr} (\tilde{V}_i^T \tilde{V}_i] \\
+ \frac{1}{\beta_{w,D}} \theta_{w,D} \beta_{w,D} - \frac{1}{\beta_{v,D}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
+ \frac{1}{\beta_{v,j}} \theta_{v,D} \beta_{v,D} + \sum_{i=1}^{n} \frac{1}{4} \frac{1}{4} \delta_i^j \\
\leq \sum_{i=1}^{n-1} [e_i^3 S_i \langle 1 \rangle + \frac{3}{4} e_i^4 + |c_i|^2 k_i] + e_i^3 S_n v + \frac{3}{4} e_i^4 + |c_i|^2 k_n + \sum_{i=1}^{n} (\tilde{W}_i^T [|c_i^2| (\tilde{S}_i] \\
- \tilde{S}_i^T \langle 1 \rangle) - \frac{1}{\beta_{w,j} \beta_{v,j}} \tilde{W}_j^T \tilde{W}_j + \tilde{V}_i [|c_i|^2 \tilde{S}_i^T \langle 1 \rangle - \frac{1}{\beta_{v,j}} \tilde{V}_i]) + \sum_{i=1}^{n} \frac{1}{4} \delta_i^j \\
\[
- \frac{1}{\beta_{w,D}} \mathbf{W}_D^T \mathbf{W}_D' - \frac{1}{\beta_{v,D}} \operatorname{tr}(\mathbf{V}_D^T \mathbf{V}_D') - \frac{1}{\beta_{w,D}} \hat{\theta}_{w,D} \hat{\theta}_{w,D}' - \frac{1}{\beta_{v,D}} \hat{\theta}_{v,D} \hat{\theta}_{v,D}' \\
+ \sum_{i=1}^{n} \left[ \frac{a^2_i}{2} + \frac{m^2_i}{2} + \frac{m^2_i}{2} + \frac{1}{4} \mathbf{w}_i^T \mathbf{w}_i + \frac{1}{4} \delta_i^4 \right],
\]

where \(a_i = 1, \cdots, n\), \(m_1, m_2 > 0\) are design parameters; \(\theta_{w,i} = ||\mathbf{W}_i^T||^2\) and \(\theta_{v,i} = ||\mathbf{V}_i||^2\).

The virtual control function was designed as follows.

\[
\alpha_i = -\frac{1}{\delta_i} \left[ \left( \lambda_i + \frac{3}{2} \right) e_i + \frac{e_3^2}{2a_i^2} \right] \cdot \hat{\theta}_{w,i} \cdot \|S'_i \mathbf{v}_i \mathbf{I}_i\|^2 + \frac{e_3^3}{2m_i^2} \cdot \hat{\theta}_{v,i} \cdot \|\mathbf{I}_i \mathbf{W}_i^T S'_i \|^2 \right],
\]

By substituting (30) into (29), \(\mathbf{V}\) can be given by

\[
\mathbf{V} \leq \sum_{i=1}^{n-1} \left[ -\lambda_i \epsilon_i^4 \right] + \frac{e_3^3 \delta_i^4}{2a_i^2} \cdot \epsilon_i^4 + \frac{e_6^3}{2m_i^2} \cdot \epsilon_i^2 + \sum_{i=1}^{n} \left[ \mathbf{W}_i^T [\mathbf{e}_i^2] (\tilde{S}_i - \tilde{S}'_i \mathbf{v}_i \mathbf{I}_i) - \frac{1}{\beta_{w,D}} \mathbf{W}_i^T \right] \\
+ \mathbf{V}_i [\mathbf{e}_i^2] \mathbf{W}_i^T \mathbf{S}'_i - \frac{1}{\beta_{v,D}} \mathbf{V}_i' + \frac{e_6^3}{2m_i^2} \cdot \epsilon_i^2 \cdot \|\mathbf{S}_i \mathbf{v}_i \mathbf{I}_i\|^2 + \frac{e_6^3}{2m_i^2} \cdot \epsilon_i \cdot \|\mathbf{V}_i \mathbf{W}_i^T \mathbf{S}'_i \|^2 \\
+ \sum_{i=1}^{n-1} \left[ \frac{a_i^2}{2} + \frac{m_i^2}{2} + \frac{m_i^2}{2} + \frac{1}{4} \mathbf{w}_i^T \mathbf{w}_i + \frac{1}{4} \delta_i^4 \right] \\
- \sum_{i=1}^{n-1} \left[ -\lambda_i \epsilon_i^4 \right] + \sum_{i=1}^{n} \left[ \mathbf{W}_i^T [\mathbf{e}_i^2] (\tilde{S}_i - \tilde{S}'_i \mathbf{v}_i \mathbf{I}_i) - \frac{1}{\beta_{w,D}} \mathbf{W}_i^T + \mathbf{V}_i [\mathbf{e}_i^2] \mathbf{W}_i^T \mathbf{S}'_i - \frac{1}{\beta_{v,D}} \mathbf{V}_i' \right] \\
+ \sum_{i=1}^{n} \left[ \frac{e_6^3 \delta_i^4}{2a_i^2} \cdot \epsilon_i^2 + \frac{e_6^3}{2m_i^2} \cdot \epsilon_i^2 \cdot \|\mathbf{S}_i \mathbf{v}_i \mathbf{I}_i\|^2 + \frac{e_6^3}{2m_i^2} \cdot \epsilon_i \cdot \|\mathbf{V}_i \mathbf{W}_i^T \mathbf{S}'_i \|^2 \\
- \frac{1}{\beta_{w,D}} \hat{\theta}_{w,D} \hat{\theta}_{w,D}' - \frac{1}{\beta_{v,D}} \hat{\theta}_{v,D} \hat{\theta}_{v,D}' + \sum_{i=1}^{n} \left[ \frac{a_i^2}{2} + \frac{m_i^2}{2} + \frac{m_i^2}{2} + \frac{1}{4} \mathbf{w}_i^T \mathbf{w}_i + \frac{1}{4} \delta_i^4 \right].
\]

By using Formula (17), controller \(u(t)\) is divided into two parts, \(u_c(t)\) and \(u_d(t)\), and we have

\[
u(t) = u_c(t) + u_d(t) + D(u(t)) + \omega_D + \delta_D.
\]

where \(u_c(t)\) is a main controller of System (1), and \(u_d(t)\) is the compensator controller of dead-zone nonlinearity.

The main control function was designed as follows.

\[
u_c = -\frac{1}{\delta_n} \left[ \left( \lambda_n + \frac{3}{2} \right) e_n + \frac{e_3^3}{2a_n^2} \cdot \epsilon_n^2 + \frac{e_3^3}{2m_n^2} \cdot \epsilon_n \cdot \|S'_n \mathbf{v}_n \mathbf{I}_n\|^2 + \frac{e_3^3}{2m_n^2} \cdot \epsilon_n \cdot \|\mathbf{I}_n \mathbf{W}_n^T S'_n \|^2 \right].
\]
By substituting (33) and (32) into (31), \( \dot{V} \) can be given by
\[
\dot{V} \leq \sum_{i=1}^{n-1} \left[ -\lambda_i \hat{e}_i^4 + \sum_{i=1}^{n} \{ \hat{W}_i^T [|e_i|^2](\dot{S}_i - \hat{S}_i^T \dot{V}_i) - \frac{1}{\hat{p}_{w,i}^j} \hat{W}_i^T \hat{W}_i \} + \tilde{V}_i[|e_i|^2]I_1[I_1^T S_i^j - \frac{1}{\hat{p}_{v,i}^j} \tilde{V}_i]\right]
\]
\[
+ \sum_{i=1}^{n} \{ \hat{\theta}_{w,i} \frac{e_i^6}{2m_i^2} \cdot |S_i^j|^2 \hat{W}_i \} + \hat{\theta}_{v,i} \frac{e_i^6}{2m_i^2} \cdot |I_1[I_1^T S_i^j]| \}
\]
\[
+ e_n \theta_n u + e_n \theta_n \dot{D}(u) + e_n \theta_n \dot{g}_D + e_n \theta_n \delta_D - \frac{1}{\hat{p}_{w,D}^2} \tilde{W}_D^4 \tilde{W}_D - \frac{1}{\hat{p}_{v,D}^2} \tilde{V}_D^4 \tilde{V}_D
\]
\[
- \frac{1}{\hat{p}_{w,D}^2} \hat{w}_D^2 \hat{e}_D^2 + \frac{n}{\hat{p}_{w,D}^2} \hat{w}_D^2 \hat{e}_D^2 + \frac{1}{4} \| \hat{W}_T^2 \|^2 + \frac{1}{4} \delta_i^4
\]
\[
\leq \sum_{i=1}^{n} \left[ -\lambda_i \hat{e}_i^4 + \sum_{i=1}^{n} \{ \hat{W}_i^T [|e_i|^2](\dot{S}_i - \hat{S}_i^T \dot{V}_i) - \frac{1}{\hat{p}_{w,i}^j} \hat{W}_i^T \hat{W}_i \} + \tilde{V}_i[|e_i|^2]I_1[I_1^T S_i^j - \frac{1}{\hat{p}_{v,i}^j} \tilde{V}_i]\right]
\]
\[
+ \sum_{i=1}^{n} \{ \hat{\theta}_{w,i} \frac{e_i^6}{2m_i^2} \cdot |S_i^j|^2 \hat{W}_i \} + \hat{\theta}_{v,i} \frac{e_i^6}{2m_i^2} \cdot |I_1[I_1^T S_i^j]| \}
\]
\[
+ e_n \theta_n u + e_n \theta_n \dot{D}(u) + e_n \theta_n \dot{g}_D + e_n \theta_n \delta_D - \frac{1}{\hat{p}_{w,D}^2} \tilde{W}_D^4 \tilde{W}_D - \frac{1}{\hat{p}_{v,D}^2} \tilde{V}_D^4 \tilde{V}_D
\]
\[
- \frac{1}{\hat{p}_{w,D}^2} \hat{w}_D^2 \hat{e}_D^2 + \frac{n}{\hat{p}_{w,D}^2} \hat{w}_D^2 \hat{e}_D^2 + \frac{1}{4} \| \hat{W}_T^2 \|^2 + \frac{1}{4} \delta_i^4
\]
\[ V \leq \sum_{i=1}^{n} [-\lambda_i e_i^2] + \sum_{i=1}^{n} \left\{} (\mathbf{W}_i^T [c_i^e] (\mathbf{S}_i - \mathbf{S}_i' \mathbf{V}_i \mathbf{I}_i) - \frac{1}{\beta_{v_i}} \mathbf{W}_i^T \right\} + \mathbf{V}_i [c_i^e] [I_i] [\mathbf{W}_i^T S_i']^2 \\
- \frac{1}{\beta_{v_i}} \dot{\mathbf{V}}_i' + \sum_{i=1}^{n} \left\{ (\theta_{w_i} [e_i^0] 2m_1^2 \cdot ||S_i' \mathbf{V}_i \mathbf{I}_i||^2_2 - \frac{1}{\beta_{v_i}} \theta_{w_i} \dot{e}_i^0 2m_2^2 \cdot ||I_i \mathbf{W}_i^T S_i'||^2_2 \\
- \frac{1}{\beta_{v_i}} \theta_{w_i} \dot{e}_i^0 \right\} + \sum_{i=1}^{n} \left\{ (\theta_{v_i} \dot{e}_i^0 2m_1^2 \cdot ||S_i' \mathbf{V}_i \mathbf{I}_i||^2_2 - \frac{1}{\beta_{v_i}} \theta_{v_i} \dot{e}_i^0 2m_2^2 \cdot ||I_i \mathbf{W}_i^T S_i'||^2_2 \\
- \frac{1}{\beta_{v_i}} \theta_{v_i} \dot{e}_i^0 \right\} - \lambda_D e_n^2 + \mathbf{W}_D [e_n^3 S_n (S_D - S_D' \mathbf{V}_D \mathbf{I}_D)] - \frac{1}{\beta_{v_i}} \mathbf{W}_D^T \right\} + \mathbf{V}_D [e_n^3 S_n] \\
= \frac{e_n^3 S_n}{2m_2^2} \cdot \theta_{v_i} \cdot ||I_i \mathbf{W}_D^T S_i'||^2_2 \right\}. \tag{36} \]

By substituting (36) into (35), \( V \) can be given by

\[ V \leq \sum_{i=1}^{n} [-\lambda_i e_i^2] + \sum_{i=1}^{n} \left\{ (\mathbf{W}_i^T [c_i^e] (\mathbf{S}_i - \mathbf{S}_i' \mathbf{V}_i \mathbf{I}_i) - \frac{1}{\beta_{v_i}} \mathbf{W}_i^T \right\} + \mathbf{V}_i [c_i^e] [I_i] [\mathbf{W}_i^T S_i']^2 \\
- \frac{1}{\beta_{v_i}} \dot{\mathbf{V}}_i' + \sum_{i=1}^{n} \left\{ (\theta_{w_i} [e_i^0] 2m_1^2 \cdot ||S_i' \mathbf{V}_i \mathbf{I}_i||^2_2 - \frac{1}{\beta_{v_i}} \theta_{w_i} \dot{e}_i^0 2m_2^2 \cdot ||I_i \mathbf{W}_i^T S_i'||^2_2 \\
- \frac{1}{\beta_{v_i}} \theta_{w_i} \dot{e}_i^0 \right\} + \sum_{i=1}^{n} \left\{ (\theta_{v_i} \dot{e}_i^0 2m_1^2 \cdot ||S_i' \mathbf{V}_i \mathbf{I}_i||^2_2 - \frac{1}{\beta_{v_i}} \theta_{v_i} \dot{e}_i^0 2m_2^2 \cdot ||I_i \mathbf{W}_i^T S_i'||^2_2 \\
- \frac{1}{\beta_{v_i}} \theta_{v_i} \dot{e}_i^0 \right\} - \lambda_D e_n^2 + \mathbf{W}_D [e_n^3 S_n (S_D - S_D' \mathbf{V}_D \mathbf{I}_D)] - \frac{1}{\beta_{v_i}} \mathbf{W}_D^T \right\} + \mathbf{V}_D [e_n^3 S_n] \\
= \frac{e_n^3 S_n}{2m_2^2} \cdot \theta_{v_i} \cdot ||I_i \mathbf{W}_D^T S_i'||^2_2 \right\}. \tag{37} \]
The adaptive main control laws are updated with Equation (38).

\[ \dot{W}_i' = \beta_{w,i} e_i r^3 (\dot{S}_i - \dot{S}_i \dot{V}_i) - \mu_i \dot{W}_i \]

\[ \dot{V}_i' = \beta_{v,i} e_i r^3 (\dot{1}\dot{W}_i^T S_i' - \mu_i \dot{V}_i) \]

\[ \delta_{w,i} = \mathcal{P}_{w,i} e_i \cdot \frac{e_i}{2m_i^2} \cdot \| \dot{S}_i \dot{V}_i \|_F^2 - \mu_i \delta_{w,i} \]

\[ \delta_{v,i} = \mathcal{P}_{v,i} e_i \cdot \frac{e_i}{2m_i^2} \cdot \| \dot{1}\dot{W}_i^T S_i' \|_F^2 - \mu_i \delta_{v,i} \]

The adaptive compensator control laws of dead-zone are updated with Equation (39).

\[ \dot{W}_D' = \beta_{w,D} e_i r^3 (\dot{S}_D - \dot{S}_D \dot{V}_D) - r_4 \dot{W}_D \]

\[ \dot{V}_D' = \beta_{v,D} e_i r^3 (\dot{1}\dot{W}_D^T S_D' - r_2 \dot{V}_D) \]

\[ \delta_{w,D} = \mathcal{P}_{w,D} e_i \cdot \frac{e_i}{2m_{2D}} \cdot \| \dot{S}_D \dot{V}_D \|_F^2 - r_3 \delta_{w,D} \]

\[ \delta_{v,D} = \mathcal{P}_{v,D} e_i \cdot \frac{e_i}{2m_{2D}} \cdot \| \dot{1}\dot{W}_D^T S_D' \|_F^2 - r_4 \delta_{v,D} \]

where \( \mu_1, \mu_2, \mu_3, \mu_4, r_1, r_2, r_3, r_4 > 0 \) are design parameters. By substituting (38) and (39) into (37), \( \dot{V} \) can be given by

\[ \dot{V} \leq \sum_{i=1}^{n} \left[ -\lambda_i e_i^4 + \sum_{i=1}^{n} \left[ \frac{\mu_1}{\beta_{w,i}} \dot{W}_i^T \dot{W}_i + \frac{\mu_2}{\beta_{v,i}} \text{tr} \{ \dot{V}_i^T \dot{V}_i \} + \frac{\mu_3}{\beta_{w,D}} \delta_{w,D} + \frac{\mu_4}{\beta_{v,D}} \delta_{v,D} \right] \right] \]

\[ - \delta_{D}^4 \]

\[ \leq \sum_{i=1}^{n} \left[ -\lambda_i e_i^4 + \sum_{i=1}^{n} \left[ \frac{\mu_1}{\beta_{w,i}} \dot{W}_i^T \dot{W}_i + \frac{\mu_2}{\beta_{v,i}} \text{tr} \{ \dot{V}_i^T \dot{V}_i \} + \frac{\mu_3}{\beta_{w,D}} \delta_{w,D} + \frac{\mu_4}{\beta_{v,D}} \delta_{v,D} \right] \right] \]

\[ - \delta_{D}^4 \]
Then, from Equations (22) and (41), we have

\[ V_n(t) \leq Ce^{-pt} + \frac{q}{p}. \]  

(41)

Then, from Equations (22) and (41), we have

\[ \frac{1}{4} \varepsilon_1^2 \leq V_n(t) \leq Ce^{-pt} + \frac{q}{p}. \]  

(42)

Considering Lemma 3 ensures that the signals \( e_i, W_i, \dot{V}_i, \hat{\theta}_{v,i}, \hat{\theta}_{v,\varepsilon} \) \( W_D, \dot{V}_D, \hat{\theta}_{v,D} \) and \( \hat{\theta}_{v,D} \) of the closed-loop system are bounded with bounded initial conditions.

**Remark 2.** From (30), the expressions of each virtual controller are given in our results. From (33) and (36), two controller expressions are given. From (38), the adaptive compensation laws of each order are given. From (39), the compensation control laws for the dead zone are given.

**Remark 3.** In this paper, multilayer neural networks were used to approximate the unknown structures and unknown actuator dead zones of nonlinear systems with time-varying delays. This approximation has its limitations, which are described in [34].

### 4. Simulation Analysis

In this section, two numerical examples are presented to verify the effectiveness of the proposed neural-network adaptive compensation controller based on the backstepping method for time-varying delay nonlinear systems with unknown structures and unknown actuator dead zones. An industrial application case of the chemical cycle is presented to verify the effectiveness of the scheme.

#### 4.1. Nonlinear System with an Unknown Actuator Dead Zone in Known Time-Varying Delay Form

Considering the time-varying delay nonlinear system with an unknown actuator dead zone and unknown structures in which the time-varying delays are known, the system can be described as follows:

\[
\begin{align*}
\dot{x}_1(t) &= (1 + x_1^2(t)) \cdot x_2(t) + x_1(t) \cdot (\sin(x_1(t)) + x_2(t) + x_3(t)) \cdot x_1(t - t_d(t)) \\
\dot{x}_2(t) &= (1 + x_1^2(t) + x_2^2(t)) \cdot x_1(t) + x_1(t) \cdot x_2(t) \cdot x_3(t - t_b(t)) \\
\dot{x}_3(t) &= (1 + x_1(t) \cdot x_2(t) \cdot x_3(t)) \cdot x_1(t + x_2(t)) - (x_1(t) + x_2(t))^2 \cdot \sin(x_3(t)) \cdot x_3(t - t_c(t)) \\
y(t) &= x_1(t) \quad t_d(t) = |2 \cdot \sin(2\pi f_{td} \cdot t)| \quad t_b(t) = |2 \cdot \sin(2\pi f_{tb} \cdot t)| \quad t_c(t) = |\frac{1}{2} \cdot \sin(2\pi f_{tc} \cdot t)|
\end{align*}
\]

where \( t_d(t) \) is the time-varying delay of \( x_1(t) \); \( f_{td} \) is the frequency of time-varying delay \( t_d(t) \); \( t_b(t) \) is the time-varying delay of \( x_2(t) \); \( f_{tb} \) is the frequency of time-varying
The input of $\hat{\theta}_1$ is $I_1$, as follows:

$$ I_1 = [e_1, e_2, e_3, x_1, x_2, x_3, e_1(t-t_a), x_1(t-t_a), y_d, \dot{y}_d, 1]^T. $$

The input of $\hat{\dot{\theta}}$ is $I_2$, as follows:

$$ I_2 = \begin{bmatrix} e_1, e_2, e_3, x_1, x_2, x_3, e_1(t-t_a), e_2(t-t_b), e_3(t-t_b), \\
\dot{x}_1(t-t_a), \dot{x}_2(t-t_b), x_1(t-t_b), y_d, y_d^{(1)}, y_d^{(2)}, \dot{\theta}_{w,1}, \dot{\theta}_{w,2} \end{bmatrix}^T. $$

The input of $\hat{\ddot{\theta}}$ is $I_3$, as follows:

$$ I_3 = \begin{bmatrix} e_1, e_2, e_3, x_1, x_2, x_3, e_1(t-t_a), e_2(t-t_b), e_3(t-t_b), \\
\dot{e}_1(t-t_a), \dot{e}_2(t-t_b), \dot{x}_1(t-t_b), \dot{x}_2(t-t_b), \dot{x}_3(t-t_b), \\
\dot{x}_1(t-t_b), \dot{x}_2(t-t_b), y_d, y_d^{(1)}, y_d^{(2)}, y_d^{(3)}, \dot{\theta}_{w,1}, \dot{\theta}_{w,2}, \dot{\theta}_{w,2} \end{bmatrix}^T. $$

The input of $\hat{\dot{\theta}}(u(t))$ is $I_D$, as follows:

$$ I_D = [u(t), v(t) - u(t) - D(u(t)), 1]^T. $$

We set the order parameters, constants $\lambda_1 = \lambda_2 = \lambda_3 = 7.5$, $\lambda_D = 40$, initial state $x_1(0) = 0.01$, $x_2(0) = -0.02$ and $x_3(0) = 0.01$, coefficients $a_1 = a_2 = a_3 = a_4 = 1$ and $m_1 = m_2 = m_{1D} = m_{2D} = 0.1$, and compensation coefficients $P_{w,1} = P_{w,2} = P_{w,3} = P_{w,D} = 1$ and $P_{v,1} = P_{v,2} = P_{v,3} = P_{v,D} = 1$.

The simulation results were shown in Figures 1–6, where Figure 1 presents system output $y$ and target signal $y_d$. Figure 2 shows the tracking error, Figure 3 demonstrates the controller output (with dead-zone compensation controller) and the system input (with dead-zone compensation controller). Figure 4 illustrates the operating status of the system. Figure 5 presents the adaptive compensation laws. In order to verify the validity of the dead-zone compensator, Figure 6 shows the tracking error without a dead-zone compensation controller.
Figure 1. Output and target curves in known time-varying delay form (with dead-zone compensation controller).

Figure 2. Tracking error curve in known time-varying delay form (with dead-zone compensation controller).

Figure 3. Controller output curve and system input curve in known time-varying delay form (with dead-zone compensation controller).
Figure 4. State curves in known time-varying delay form (with dead-zone compensation controller).

Figure 5. Adaptive compensation laws in known time-varying delay form (with dead-zone compensation controller).

Figure 6. Tracking error curve in known time-varying delay form (without dead-zone compensation controller).

4.2. Nonlinear System with an Unknown Actuator Dead-Zone in Unknown Time-Varying Delay Form

In order to verify the effectiveness of the controller for the time-varying delay nonlinear system with an unknown actuator dead-zone and unknown structures in which the
time-varying delays are unknown, a comparative test was added. In the last numerical experiment, \( t_a(t) \), \( t_b(t) \) and \( t_c(t) \) were known, and \( I_1 \), \( I_2 \) and \( I_3 \) are given. Since \( t_a(t) \), \( t_b(t) \) and \( t_c(t) \) were unknown in this case, the \( I_1 \), \( I_2 \) and \( I_3 \) could be redefined as follows.

\[
I_1 = [e_1, e_2, e_3, x_1, x_2, x_3, y_d, y_d^{(1)}, 1]^T \\
I_2 = [e_1, e_2, e_3, x_1, x_2, x_3, y_d, y_d^{(1)}, y_d^{(2)}, \hat{\theta}_{w,1}, \hat{\theta}_{v,1}, 1]^T \\
I_3 = [e_1, e_2, e_3, x_1, x_2, x_3, y_d, y_d^{(1)}, y_d^{(2)}, y_d^{(3)}, \hat{\theta}_{w,1}, \hat{\theta}_{v,1}, \hat{\theta}_{w,2}, \hat{\theta}_{v,2}, 1]^T.
\]

In order to compare the tracking effect while considering the same system and the same parameters, the comparative simulation results in the case of unknown \( t_a(t) \), \( t_b(t) \), and \( t_c(t) \) are shown in Figures 7–12.

**Figure 7.** Output and target curves in unknown time-varying delay form (with dead-zone compensation controller).

**Figure 8.** Tracking error curve in unknown time-varying delay form (with dead-zone compensation controller).
Figure 9. Controller output curve and system input curve in unknown time-varying delay form (with dead-zone compensation controller).

Figure 10. State curves in unknown time-varying delay form (with dead-zone compensation controller).

Figure 11. Adaptive compensation laws in unknown time-varying delay form (with dead-zone compensation controller).
4.3. Industrial Application Case of a Chemical Cycle

In the process of chemical production, the chemical reaction is often incomplete. For example, in Process A to B, the conversion from A into B cannot reach 100% in a limited amount of time. In this case, the circular reaction method is generally adopted, that is, the unreacted A in the reactants is transferred to the reaction kettle through the pipeline. A cascade chemical reactor system with Reactors A and B is shown in Figure 13.

\[ \dot{x}_1 = (-K_A - \frac{1}{\theta_A + \Delta \theta_A})x_1(t) + \left(\frac{1}{V_A} - R_B\right)x_2(t) - \frac{1}{\theta_A + \Delta \theta_A}x_1(t - d_1(t)) \]
\[ \dot{x}_2 = \frac{F}{V_B}v(t) - K_Bx_2(t) - \frac{1}{\theta_B + \Delta \theta_B}x_2(t) - \frac{2C^*_B}{\theta_B + \Delta \theta_B}x_2(t + d_2(t)) + \frac{R_B}{V_B}x_2(t - d_2(t)) \]
\[ y(t) = x_1(t), \]

where \( x_1 = C_A - C^*_A \) and \( x_2 = C_B - C^*_B \) are the state variables of this system. \( C_A \) and \( C_B \) are compositions of produced streams from the reactors. \( C^*_A \) and \( C^*_B \) are the equilibrium points of the system. \( R_B \) is the recycle flow rate. \( \theta_A \) and \( \theta_B \) are the reactor residence times. \( \Delta \theta_A \) and \( \Delta \theta_B \) are the fluctuation ranges of reactor residence time \( K_A \) and \( K_B \) are the reaction constants. \( F \) is the feed rate, and \( V_A \) and \( V_B \) are the reactor volumes.

The actuator dead-zone function is as follows:

\[ v(t) = D(u(t)) = \begin{cases} 
   u(t) - 1, & u(t) \geq 1 \\
   0, & -1 \leq u(t) \leq 1 \\
   u(t) + 1, & u(t) \leq -1 
\end{cases} \]
In practice, time-varying delays $d_1(t)$ and $d_2(t)$ are difficult to measure. $\Delta \theta_A$ and $\Delta \theta_B$ are also very unpredictable. Considering System 1, we have

$$\dot{x}_1 = (-K_A - \frac{1}{\theta_A})x_1(t) + (\frac{1 - R_B}{V_A})x_2(t) + f_1,$$

$$\dot{x}_2 = \frac{F}{V_B}v(t) - (K_B + \frac{1}{\theta_B})x_2(t) + \frac{2C^*_B}{\theta_B}x_2(t) + f_2,$$

where $f_1 = \frac{\Delta \theta_A}{\theta_A(\theta_A + \Delta \theta_A)} x_1(t) - \frac{1}{\theta_A(\theta_A + \Delta \theta_A)} x_1(t - d_1(t))$, $f_2 = \frac{\Delta \theta_B}{\theta_B(\theta_B + \Delta \theta_B)} x_2(t) + \frac{2C^*_B}{\theta_B(\theta_B + \Delta \theta_B)} x_2(t) + \frac{R_B}{V_B} x_2(t - d_2(t)).$

The reactor parameters were $K_A = K_B = 0.5$, $\theta_A = \theta_B = 2$, $R_B = 0.5$, $V_A = V_B = 0.5$, and $C^*_B = \frac{7}{5}$. In this simulation, the initial conditions were: $x_1(0) = 0.4$, $x_2(0) = -0.3$ and $d_1(t) = d_2(t) = 0.3(1 + \cos(t))$. The reference signal was set to $y_d = 0.5 \sin(0.5t) + 0.5 \sin(t)$. We set order parameters, constants $\lambda_1 = \lambda_2 = 8$, $\lambda_D = 1$, coefficients $a_1 = a_2 = 1$ and $m_1 = m_2 = m_{1D} = m_{2D} = 1$, compensation coefficients $P_{w,1} = P_{w,2} = P_{w,D} = 1$, and $P_{c,1} = P_{c,2} = P_{c,D} = 1$.

The sampling time was $T = 100$ s, and the sampling period was $\Delta T = 0.001$ s.

The simulation results are shown in Figures 14–16, where Figure 14 presents system output $y$ and target signal $y_d$. Figure 15 shows the tracking error. Figure 16 demonstrates the controller output and the system input (with dead-zone compensation controller).

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**Figure 14.** Output and target curves in the case of a chemical cycle.

**Figure 15.** Tracking error curve in the case of a chemical cycle.
4.4. Description of Simulation Results

Figures 1 and 2 show that output $y$ could track target signal $y_d$ within a small bounded error. On the other hand, Figure 6 verifies that the dead-zone nonlinearity could be compensated by $u_d$. The comparison of Figures 1 and 7 shows that the tracking performance of the controller was not dependent on whether the time-varying delay was known or unknown. The controller could achieve good tracking performance even when the time-varying delay was unknown. Figures 2 and 8 show that the tracking error remained below 0.1. Figures 3 and 9 show that the fluctuation of the controller output with known time-varying delay was same as the unknown time-varying delay. Figures 5 and 11 show similar compensation law curves. Simulation results indicate that the neural-network adaptive compensation controller had stronger adaptability and achieved good control performance for nonlinear systems with an unknown actuator dead-zone and unknown structures in unknown time-varying delay form. The adaptive compensation controller for an unknown actuator dead-zone reduced the system tracking error. Figures 14–16 prove the effectiveness of the scheme in applications of the cascade chemical industry.

5. Conclusions

This paper investigated the tracking control problem for a class of time-varying delay nonlinear systems with unknown structures and unknown dead zones where time-varying delays are unknown. Adaptive multilayer neural-network compensation controllers were designed for the considered systems by using the backstepping method. The target signal could be almost surely tracked by the system output within a small bounded error. In future works, we will pay attention to the control problems of time-varying delay nonlinear systems with unknown structures by using more advanced neural networks.

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Appendix A. Proof of Lemma 2

Proof. The Taylor expansion of \( W^T S - \hat{W}^T S \) at the point \( \hat{V} \) is as follows:

\[
W^T S - \hat{W}^T S = W^T (S + S'(\hat{V} - \hat{V})) + O(\hat{V})^2 - \hat{W}^T S
\]

\[
= W^T S - \hat{W}^T S + (\hat{W}^T + \hat{W}^T)S'(\hat{V} - \hat{V}) + W^T O(\hat{V})^2
\]

\[
= W^T S + \hat{W}^T S'(\hat{V} - \hat{V}) + \hat{W}^T S'(\hat{V} - \hat{V}) + W^T O(\hat{V})^2
\]

\[
= \hat{W}^T S - \hat{W}^T S' \hat{V} + \hat{W}^T S' \hat{V} + \hat{W}^T S' \hat{V} + W^T O(\hat{V})^2.
\]

The first part of Lemma 2 can be proved, and from the Equation (7) we have

\[
\Phi = W^T S - \hat{W}^T S - \hat{W}^T S' \hat{V} - \hat{W}^T S' \hat{V}
\]

\[
= W^T (S - \hat{S}) + \hat{W}^T S' \hat{V} - \hat{W}^T S' \hat{V}.
\]

The function \( S(\cdot) \) is a bounded function, and \( 0 \leq S(\cdot) \leq 1 \), then

\[
\|W^T (S - \hat{S})\| \leq \|W^T\|_2,
\]

\[
\|\hat{W}^T S' \hat{V} = \text{tr}(W^T S' \hat{V}) \leq \|W^T\|_F \cdot \|S' \hat{V}\|_F.
\]

\[
\|\hat{W}^T S' \hat{V} = \text{tr}(W^T S' \hat{V}) \leq \|W\|_F \cdot \|\hat{W} S'\|_F.
\]

Then \( |\Phi| \leq \|W\|_2 + \|W\|_F \cdot \|S' \hat{V}\|_F + \|W\|_F \cdot \|\hat{W} S'\|_F. \)

Therefore, the second part of Lemma 2 can be proven. □

Appendix B. Proof of Theorem 1

Proof.

\[
\|x, x_i(t - t_i)\|_F \leq \sum_{j=1}^n |x_j| + |x_i(t - t_i)|
\]

\[
\leq \sum_{j=1}^n (|x_j| + |x_{j-1}|) + |e_i(t - t_i)| + |a_i(t - t_i)|
\]

\[
\leq \sum_{j=1}^n |e_j| + \sum_{j=1}^{n-1} |x_j| + |y_d| + |e_i(t - t_i)| + |a_i(t - t_i)|
\]

Take \( \varphi_j = |x_j|/|e_j| + 1 \), \( \varphi_j(t - t_i) = |a_i(t - t_i)|/|e_i(t - t_i)| + 1 \), for \( j = 1, \ldots, n \), and \( \varphi_n = 1 \), then

\[
\|x, x_i(t - t_i)\|_F \leq \sum_{j=1}^n |e_j| + \sum_{j=1}^{n-1} (\varphi_j - 1)|e_i| + |y_d| + |e_i(t - t_i)| + |a_i(t - t_i)|
\]

\[
(\varphi_j - 1(t - t_i) - 1)|e_{j-1}(t - t_i)|
\]

\[
\leq \sum_{j=1}^{n-1} (|e_j|/\varphi_j) + |e_{n}| + |y_d| + |e_i(t - t_i)| + \varphi_{j-1}(t - t_i)|e_{j-1}(t - t_i)|
\]

\[
\leq \sum_{j=1}^{n} (|e_j|/\varphi_j) + |e_i(t - t_i)| + \varphi_{j-1}(t - t_i)|e_{j-1}(t - t_i)| + d.
\]

The proof of Theorem 1 is completed. □
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