

Article

A Novel Parallel Exoskeleton for Wrist Rehabilitation: Conceptual Design, Kinematics, and Singularity Analysis

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Abstract

Wrist rehabilitation requires high precision, haptic transparency, and accurate alignment with the human joint's physiological center of rotation. Conventional robotic systems often suffer from high moving inertia or joint misalignment. This paper presents the design and kinematic validation of a novel 3-DOF spherical parallel exoskeleton featuring base-fixed actuators. By mounting all actuators to a fixed base, the proposed architecture significantly reduces moving mass, achieving a low-inertia response critical for safe patient–robot interaction. The “virtual center” concept eliminates physical central joints, enabling a compact design completed by the user's anatomy. To perform the kinematic and singularity analyses of the manipulator, two distinct models were used: Rotated Frame Based (RFB) and Initial Frame Based (IFB). Performance metrics, namely the manipulability index and condition number, are evaluated to assess dexterity and isotropy. Comparative kinematic analysis of Rotated (RFB) and Initial Frame Based (IFB) models confirm ideal central isotropy ($\kappa=1.0$). While RFB yields 94.14% high-dexterity ($\kappa<5.0$) and 99.78% usable ($\kappa<10.0$) workspace coverage, IFB achieves 78.80% high-dexterity ($\kappa<5.0$) and 92.18% usable ($\kappa<10.0$) workspace coverage. Analytical manipulability metrics validate singularity-free motion throughout the anatomical range. Additionally, the use of exponential rotational matrices in this paper provides systematic derivations of equations in compact form.

Keywords: wrist rehabilitation; spherical parallel mechanism; virtual center; low-inertia; manipulability analysis; workspace coverage

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1. Introduction

When analyzing global health data, stroke remains one of the leading causes of long-term disability and mortality worldwide [1]. According to current 2025/2026 data from the American Heart Association (AHA), approximately 11.9 million new stroke cases occur annually, and over 93.8 million people globally continue to live with neurological and motor impairments following a stroke. The rise in neurological disorders and stroke cases worldwide is driving a daily increase in the demand for upper extremity rehabilitation. This situation constitutes a massive burden on the global economy, leading to an annual loss of 160 million Disability-Adjusted Life Years (DALYs) [2] and highlighting the labor-intensive nature of the rehabilitation process for these cases [3,4].

The literature emphasizes that the wrist joint must possess a specific range of motion (ROM) and stability to perform several fundamental functions, such as feeding, hygiene, and object manipulation [5,6]. The wrist joint plays a central role in the rehabilitation process as it provides spatial orientation for the hand and optimizes grasping abilities critical for Activities of Daily Living (ADLs) [7–9]. However, robotic systems designed for wrist rehabilitation are classified into two main groups based on their mechanical structures: serial and parallel joint architectures. Although serial exoskeleton systems offer ease of design, they face challenges such as high inertia and weight due to the placement of motors on the links, which negatively impacts the system's haptic transparency [7,10]. On the other hand, while parallel robots offer high stiffness and precision, they possess ergonomic limitations due to harmful parasitic forces that arise when the robot joint does not align with the human joint (misalignment) [10,11]. Nevertheless, the fact that traditional manual therapy methods are labor-intensive and insufficient to meet the needs of an increasing patient population has heightened the demand for robot-assisted rehabilitation systems. Indeed, the rehabilitation robots market is projected to grow from 1.8 billion USD in 2025 to 7.5 billion USD by 2034 [12]. Furthermore, it is emphasized that existing devices still encounter significant barriers in both clinical and home-based use [13,14]. The most recent review articles in the literature have identified indispensable criteria for next-generation rehabilitation devices [7]. Chief among these criteria are (i) low mechanical inertia, (ii) virtual center alignment that is fully compatible with human anatomy, and (iii) a wide workspace coverage encompassing functional movements. When existing commercial and research-oriented devices are examined [8,13], it is observed that the number of systems capable of optimizing these three criteria simultaneously is quite limited. In particular, the complex structure of parallel mechanisms often leads to a narrowing of the workspace or axis shifts that compromise user comfort [11,15].

An examination of current parallel wrist rehabilitation systems in the literature reveals that the range of motion (ROM) offered by the proposed mechanisms generally remains far from physiological limits and Activities of Daily Living (ADLs) requirements. In many existing studies, it has been observed that movement capacity in specific axes cannot fully cover the rehabilitation spectrum and remains below anatomical boundaries [16–18]. Specifically, recent research on spherical parallel manipulators or metamorphic structures has reported that the workspace is restricted to certain degrees, which limits functional training efficiency [17–19]. Furthermore, manipulators focusing on hemispherical designs or those with restricted degrees of freedom are seen to be insufficient in holistically supporting the complex 3-axis movements of the wrist [16,20]. In addition to the range of motion, the problem of misalignment between the robot and human joints persists as a fundamental challenge to be overcome in the literature [21]. Although some studies attempt to address this issue by using hybrid structures or additional mechanical compensation mechanisms, or by proposing advanced control strategies such as gravity compensation [22–24], these approaches complicate clinical applicability by increasing the system's mechanical complexity and inertia, while also increasing dependence on complex software algorithms [22,24]. On the other hand, parallel robots that demonstrate theoretical success during the kinematic modeling phase are found to struggle with achieving full compatibility with user ergonomics during the practical application and design stages [19,23]. Specifically, in modular systems using active impedance control, high-cost hardware is required to mask the system's inertia [24]. In parallel exoskeleton designs, as the complexity of the kinematic chain increases, coinciding the joint centers becomes more difficult, which directly reduces the haptic transparency of the system [21,25]. Existing parallel wrist exoskeletons in the literature generally must

choose between a wide range of motion (ROM) and perfect alignment [25].

In order to mimic the complex biomechanical structure of the wrist joint, there is a significant trend in the literature toward virtual center-oriented and high degree-of-freedom (3-DOF) designs. Accordingly, 3-DOF exoskeleton systems such as “Biomech-Wrist” [26] and new concepts developed through systematic design approaches [27] aim to maximize the alignment between the robotic joints and the anatomical axes of the wrist. On the other hand, cable-driven 3-DOF exoskeletons proposed to reduce mechanical weight and inertia offer advantages in terms of lightness and flexibility; however, they bring along operational disadvantages such as cable tension control, maintenance difficulties, and concerns regarding the overall rigidity of the system [28]. Particularly, parallel exoskeleton structures designed for home-based rehabilitation focus on providing a user-friendly and safe architecture to enable patients to work independently in non-clinical environments [29]. Yet, in most of these systems, maintaining the balance between the “mechanical simplicity” required for home use and a “high range of motion (ROM)” remains a challenge to be overcome [28,29]. Many of these innovative solutions in the literature are hindered either by the mechanical uncertainties of cable systems [29] or by kinematic constraints arising in wide workspaces [26,27]. The kinematic comparison of the rehabilitation robots mentioned in this section above is summarized in the table below (Table 1).

Table 1. Kinematic comparison of the rehabilitation robots mentioned in this work.

Name	Structure	DOFs	% ROM Coverage	Singularity Free ROM	Reference
Hemispherical cable driven	Parallel	3	~%100	Not cited	[16]
3UPS/S	Parallel	3	~%41	Not fully singularity free	[17]
Metamorphic 3RPUPc-UPS	Parallel	3–4 (according to mode)	~%100	Not fully singularity free	[18]
PWRR	Parallel	2	~%100	Not cited	[19]
Kim et al.	Parallel	2	~%100	Not cited	[20]
Liu et al.	Hybrid	4	~%40	Not cited	[21]
Tucan et al.	Parallel	2	Not cited	Not fully singularity free	[23]
Ghaffari et al.	Serial	3	~%100	Not cited	[24]
Valayil and Tanev	Parallel	2	~%41	Not cited	[25]
Biomech-Wrist	Serial	3	~%100	Not cited	[26]
Saeedi et al.	Serial	3	~%98	Not cited	[27]
W-EXOS	Serial	3	~%93	Not cited	[28]
He et al.	Parallel	6	~%52	Not cited	[29]

Robotic devices for upper limb rehabilitation are broadly categorized into two main architectural paradigms based on their physical interaction with the human limb: end-effector-based and exoskeleton-based systems [30]. End-effector-based robots interact with the patient exclusively at a single distal point (e.g., a hand grip or handle) to apply forces at the endpoint without explicitly constraining or matching individual anatomical joint axes. Conversely, exoskeleton-based robots feature a kinematic structure that mirrors or directly aligns with the physiological joint axes of the human limb, enabling joint-specific trajectory tracking and torque transmission across isolated degrees of freedom

(DOFs). Although wearable, body-mounted mechanisms represent the traditional embodiment of exoskeletons, stationary grounded (table- or frame-mounted) devices that interface with the user through a combined forearm cuff and hand-grip assembly are extensively classified as wrist exoskeletons in the rehabilitation literature [31–34]. Because the proposed parallel manipulator in this study enforces isolated physiological rotations about a virtual center matching the human wrist joint while constraining both the forearm and hand, it fundamentally operates as a grounded wrist exoskeleton rather than a pure end-effector system. Throughout this manuscript, the terms exoskeleton, parallel mechanism, and rehabilitation device refer respectively to its anatomical alignment strategy, its underlying kinematic architecture, and its functional application domain.

In the design presented in this study, unlike traditional serial and some parallel architectures, the moving mass has been minimized thanks to an architecture where all motors are fixed to the base. This situation reduces the mechanical inertia of the system to the lowest level, and thereby has the potential to increase haptic transparency and dynamic response speed, which are of critical importance in the rehabilitation process. The proposed spherical parallel structure coincides the rotation center of the robot with the approximate anatomical wrist center of the patient, thanks to its virtual center geometry. This feature mechanically eliminates the parasitic forces seen in many devices in the literature that threaten joint health, without the need for any additional compensation mechanisms. Furthermore, mathematical analyses performed on the workspace of the mechanism analytically confirm that the proposed design covers 100% of the anatomical movement limits of the human wrist.

2. Design of the Spherical Parallel Exoskeleton (İzmir, Türkiye)

The proposed mechanism is a 3-DOF spherical parallel exoskeleton designed for wrist rehabilitation (Figure 1). In spherical mechanisms, joint axes intersect at one point which is the center of the mechanism. Unlike traditional hybrid or serial architectures, this design utilizes a base-fixed actuation strategy where all three motors are mounted on the stationary frame. This configuration fundamentally minimizes the moving mass of the system, leading to a significant reduction in mechanical inertia. In the context of robotic rehabilitation, this low-inertia property is vital for achieving high haptic transparency, allowing the patient to interact with the device without perceiving the weight or resistive inertia of the actuators. The exoskeleton provides three anatomical degrees of freedom: flexion/extension (α), radial/ulnar deviation (β), and pronation/supination (γ).

The mechanical chain consists of three independent Revolute (R) actuated joints located at the base. Actuator 1 is aligned with the longitudinal z-axis to drive the pronation/supination motion, while Actuator 2 and Actuator 3 are positioned along the x and y axes, respectively, to actuate the slotted link mechanism (Figure 2).

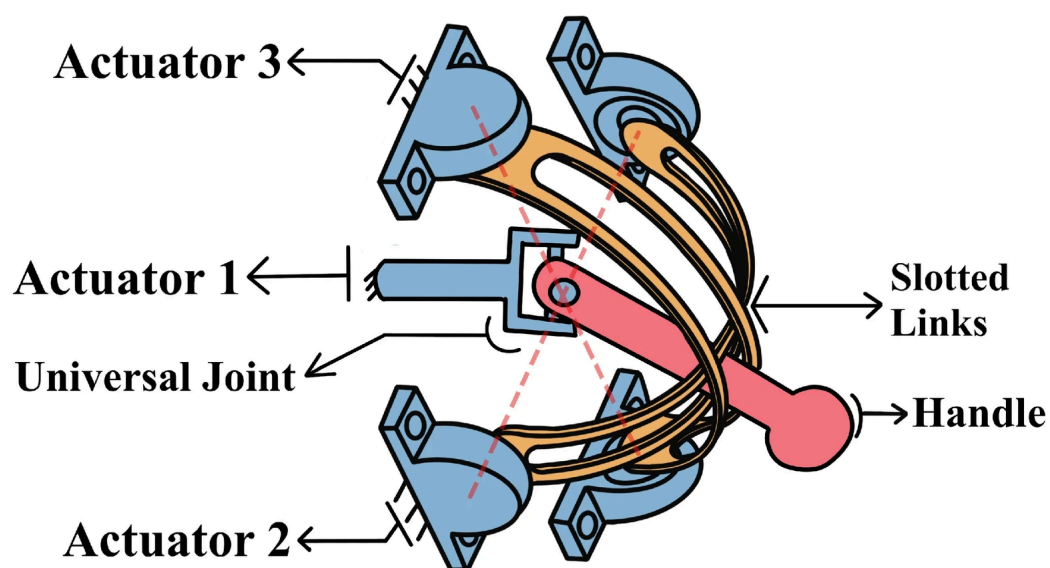


Figure 1. Preliminary visual design of the parallel structure, excluding the human arm.

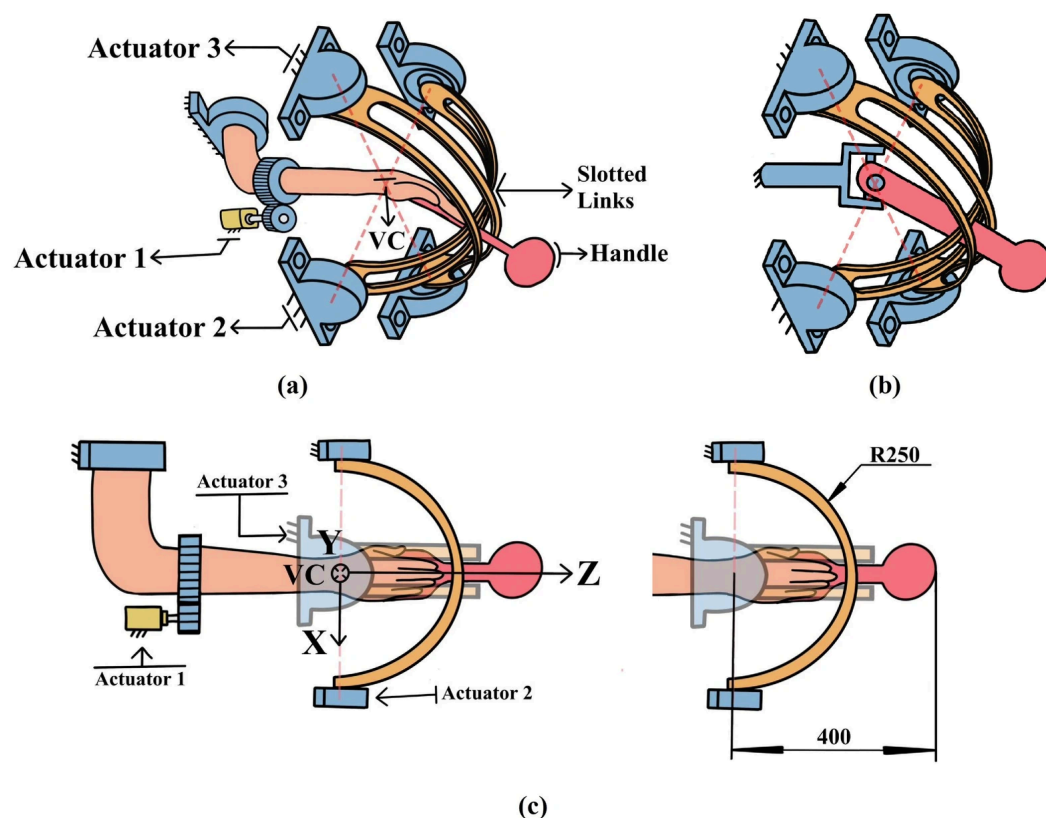


Figure 2. Placing the virtual axis rotation of the mechanism on the human wrist joint (a), preliminary visual design of the parallel structure (b), view of the mechanism with the forearm and wrist joint in the neutral position (left) and approximate dimensions of the mechanism in mm (right) (c).

The defining feature of this design is the virtual center (VC), which eliminates the need for a physical universal joint at the center of rotation (Figure 2). In this architecture, the kinematic loop is closed not by a mechanical component, but by the user's biological wrist (Figure 2). By aligning the approximate physiological rotation center of the wrist with the intersection point VC of the robotic axes, a concentric motion is achieved. This approach reduces the overall volume of the device, eliminates mechanical friction at the

center, and simplifies the sterilization process—a critical factor for clinical applications. To transform the base-fixed rotations into spherical handle movements, a dual slotted link (arc) mechanism is employed. Two circular arcs, driven by Actuator 2 and Actuator 3, intersect at a perpendicular orientation to guide the handle's orientation. This slotted link geometry provides a compact structure while maintaining a wide range of motion (ROM) for flexion/extension and radial/ulnar deviation. The handle, extending from the virtual center along the z-axis, is constrained by the intersection of these two arcs.

3. Notation

Before proceeding with the formulation of the proposed spherical exoskeleton, it is necessary to introduce the notation used in the paper together with the exponential rotation matrices. In this study, the reference frame attached to body i is denoted by F_i . The reference frame F_i can also be defined as one of the frames in a rotation sequence. The unit basis vectors of F_i along its x , y and z axes are denoted by $\vec{u}_1^{(i)}$, $\vec{u}_2^{(i)}$ and $\vec{u}_3^{(i)}$, respectively. Hence, vector $\vec{r}$ in frame F_i is decomposed as

$$\vec{r} = r_1 \vec{u}_1^{(i)} + r_2 \vec{u}_2^{(i)} + r_3 \vec{u}_3^{(i)} \quad (1)$$

The components of $\vec{r}$ in frame F_i can be expressed as a column matrix as

$$\vec{r}^{(i)} = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}^t \quad (2)$$

where the symbol t represents the matrix transpose operation. In Equation (3), the k^{th} unit basis vector of frame F_i is expressed in F_i as a column matrix.

$$\vec{u}_k^{(i/i)} = \vec{u}_k \quad (3)$$

Following this definition, the basic column matrices which are the same for all frames, are written as

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (4)$$

When a vector $\vec{p}$ is rotated around the unit vector $\vec{n}$ by an angle θ , the resulting vector becomes $\vec{r}$. As a result of this rotation, $\vec{p}$ is related to $\vec{r}$ as given by the Rodrigues' rotation formula in Equation (5) [35,36]. In this paper, the sine and cosine functions are de-

noted by the letters s and c in short form, respectively.

$$\vec{r} = \vec{p}c\theta + (\vec{n} \times \vec{p})s\theta + (\vec{p} \cdot \vec{n})\vec{n}(1 - c\theta) \quad (5)$$

When all vectors in Equation (5) are expressed in the same frame, the matrix equation given in Equation (6) is obtained where $\widehat{R}$ is the associated rotation matrix whose expanded form is written in Equation (7).

$$\vec{r} = \widehat{R}\vec{p} \quad (6)$$

$$\widehat{R} = \widehat{I}c\theta + \widetilde{n}s\theta + \overline{n}\overline{n}^t(1 - c\theta) \quad (7)$$

$\widehat{I}$ is the (3×3) identity matrix. The rotation matrix $\widehat{R}$ can be expressed in short form by the exponential rotation matrix $e\widetilde{n}\theta$ as given by Equation (8) [36], where $\widetilde{n}$ is the cross-product matrix [35] whose elements are formed by the components of $\vec{n}$ as shown in Equation (9). $\overline{n}$ is the column matrix representation of the unit vector $\vec{n}$ which is the axis of rotation.

$$\widehat{R} = e\widetilde{n}\theta = \widehat{I} + \widetilde{n}s\theta + \widetilde{n}^2(1 - c\theta) \quad (8)$$

$$\overline{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad \widetilde{n} = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \quad (9)$$

When frame F_a rotates to become frame F_b , the unit basis vectors of F_a are rotated to become the unit basis vectors of F_b . For a given vector $\vec{v}$, the mathematical relation between the components of $\vec{v}$ in F_a and F_b is given in Equation (10) where the matrix $\widehat{C}^{(a,b)}$ represents the transformation matrix from F_a to F_b .

$$\vec{v}^{(a)} = \widehat{C}^{(a,b)}\vec{v}^{(b)} \quad (10)$$

$$\vec{v}^{(b)} = \widehat{C}^{(b,a)}\vec{v}^{(a)} \Rightarrow \vec{v}^{(b)} = \left[\widehat{C}^{(a,b)} \right]^{-1} \vec{v}^{(a)} \Rightarrow \widehat{C}^{(b,a)} = \left[\widehat{C}^{(a,b)} \right]^{-1} \quad (11)$$

Transformation matrix is an orthonormal matrix with its transpose equal to its inverse; hence,

$$\widehat{C}^{(a,b)t} = \widehat{C}^{(b,a)} = \left[\widehat{C}^{(a,b)} \right]^{-1} \quad (12)$$

In Equation (13), the left-hand side of the equation shows the k^{th} unit basis vector of frame F_b as expressed in frame F_a and the right-hand side of the equation involves the transformation matrix $\widehat{C}^{(a,b)}$ multiplied by the k^{th} unit basis vector of frame F_b as expressed in frame F_b .

$$\widehat{u}_k^{(b/a)} = \widehat{C}^{(a,b)} \widehat{u}_k^{(b/b)} = \widehat{C}^{(a,b)} \widehat{u}_k \quad (13)$$

Equation (14) shows the rotation of the k^{th} unit basis vector of frame F_a to the k^{th} unit basis vector of frame F_b when F_a rotates to become F_b . The vector rotation described by Equation (14) is expressed in frame F_a . Equations (13) and (14) prove that the transformation matrix $\widehat{C}^{(a,b)}$ is indeed equal to the rotation matrix $\widehat{R}_{ab}^{(a)}$ which represents the rotation from F_a to F_b . The superscript in $\widehat{R}_{ab}^{(a)}$ indicates that the rotation matrix $\widehat{R}_{ab}$ is expressed in frame F_a .

$$\widehat{u}_k^{(b/a)} = \widehat{C}^{(a,b)} \widehat{u}_k^{(b/b)} = \widehat{C}^{(a,b)} \widehat{u}_k^{(a/a)} = \widehat{R}_{ab}^{(a)} \widehat{u}_k^{(a/a)} \quad (14)$$

$\widehat{R}_{ab}^{(a)}$ is also the matrix representation of the rotation dyadic $\widetilde{R}_{ab}$ as expressed in frame F_a . The matrix representation of the rotation dyadic $\widetilde{R}_{ab}$ in another frame F_m is given by the following dyadic (second order tensor) transformation [35,36].

$$\widehat{R}_{ab}^{(m)} = \widehat{C}^{(m,a)} \widehat{R}_{ab}^{(a)} \widehat{C}^{(a,m)} \quad (15)$$

Following Equation (15), it can be shown that $\widehat{R}_{ab}^{(a)} = \widehat{R}_{ab}^{(b)} = \widehat{C}^{(a,b)}$ as follows:

$$\widehat{R}_{ab}^{(b)} = \widehat{C}^{(b,a)} \widehat{R}_{ab}^{(a)} \widehat{C}^{(a,b)} = \widehat{C}^{(b,a)} \widehat{C}^{(a,b)} \widehat{C}^{(a,b)} = \widehat{I} \widehat{C}^{(a,b)} = \widehat{C}^{(a,b)} \quad (16)$$

Consider any given frame F_a : if frame F_a is rotated around its k^{th} unit basis vector by an angle θ and the resulting rotation matrix is also expressed in F_a , then this resulting rotation matrix can be represented by the basic exponential rotation matrix $e^{\tilde{u}_k \theta}$ as written in Equations (17)–(19).

$$e^{\tilde{u}_1 \theta} = e^{\tilde{u}_1^{(a/a)} \theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix} = \widehat{R}_1(\theta) \quad (17)$$

$$e^{\tilde{u}_2 \theta} = e^{\tilde{u}_2^{(a/a)} \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} = \widehat{R}_2(\theta) \quad (18)$$

$$e^{\tilde{u}_3 \theta} = e^{\tilde{u}_3^{(a/a)} \theta} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \widehat{R}_3(\theta) \quad (19)$$

$\widehat{R}_1(\theta)$, $\widehat{R}_2(\theta)$ and $\widehat{R}_3(\theta)$ can be called the basic rotation matrices.

Transformation and rotation matrices are orthonormal matrices and this property is shown in Equation (20). The inverse of an orthonormal matrix is equal to its transpose. In Equations (21) and (22), the mathematical operations frequently used in this paper are written which can be derived by using the Rodrigues' rotation formula.

$$\left(e^{\tilde{n}\theta}\right)^{-1} = \left(e^{\tilde{n}\theta}\right)^t = e^{-\tilde{n}\theta} \quad (20)$$

$$e^{\tilde{u}_i \theta} \bar{u}_j = \bar{u}_j c\theta + \left(\tilde{u}_i \bar{u}_j\right) s\theta, \quad i \neq j \quad (21)$$

$$e^{\tilde{u}_i \theta} \bar{u}_i = \bar{u}_i \quad (22)$$

4. Kinematic Analysis of the Spherical Parallel Exoskeleton

In this section, the kinematic model of the proposed 3-DOF parallel exoskeleton is established using both Initial Frame Based (IFB) and Rotated Frame Based (RFB) formulations that comply with the design of the mechanism. IFB formulation defines spatial orientation through successive rotations about the unit basis vectors of a fixed, world (inertial) reference frame (F_0), corresponding to extrinsic (space-fixed) coordinate transformations [37,38]. RFB formulation defines spatial orientation through successive rota-

tions about moving, body-fixed coordinate axes (F_1, F_2, F_3), corresponding to body-fixed coordinate transformations [39,40]. IFB and RFB formulations indicate two different ways of defining the wrist anatomical rotations either with respect to the world frame or with respect to the body-fixed frames in the kinematic chain, respectively.

4.1. Kinematic Analysis for Initial Frame Based Formulation

The origin of the world coordinate frame F_0 (alternatively denoted as $\{W\}$) is rooted at the geometric center VC of the virtual universal joint, which coincides with the approximate rotation center of the human wrist. The orthogonal coordinate axes of the reference frames are defined as follows (Figures 3 and 4):

- $\vec{u}_3^{(0)}$ -axis: Corresponds to the rotation axis of Actuator 1. Aligned with the longitudinal axis of the forearm and the neutral position of the exoskeleton handle, serving as the primary axis for pronation/supination (P/S) movement. The unit vector along the z-axis is $\vec{u}_3^{(0)}$. The unit vector aligned with the handle is $\vec{u}$. The anatomical wrist angle for pronation/supination is denoted by γ . In pronation and supination motions, the forearm rotates medially and laterally, respectively.
- $\vec{u}_1^{(0)}$ -axis: Corresponds to the rotation axis of Actuator 2, governing flexion/extension (F/E) movement. The unit vector along the x-axis is $\vec{u}_1^{(0)}$. The anatomical wrist angle for flexion/extension is denoted by α .
- $\vec{u}_2^{(0)}$ -axis: Corresponds to the rotation axis of Actuator 3, governing radial/ulnar (R/U) deviation. The unit vector along the y-axis is $\vec{u}_2^{(0)}$. The anatomical wrist angle for radial/ulnar deviation is denoted by β .

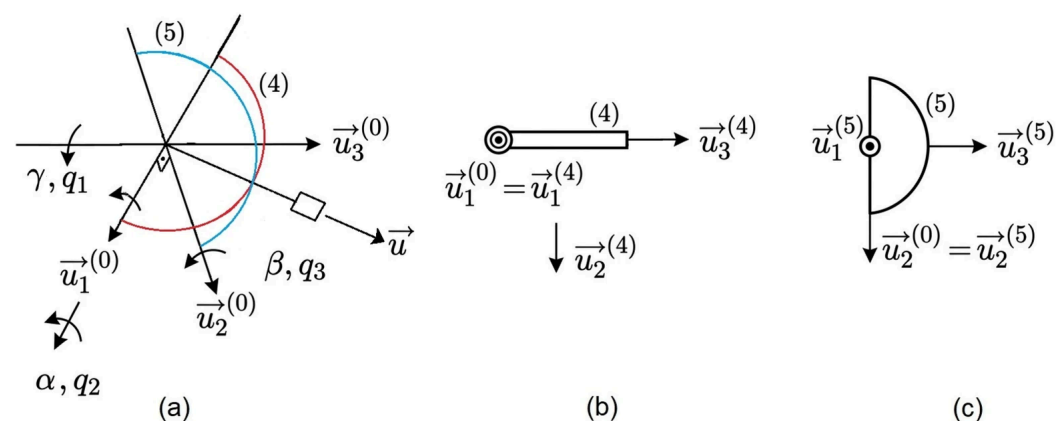


Figure 3. Kinematical representation of the proposed mechanism in IFB: (a) isometric view, (b)

front view of slotted arc 1 (denoted as body 4), and (c) front view of slotted arc 2 (denoted as body 5).

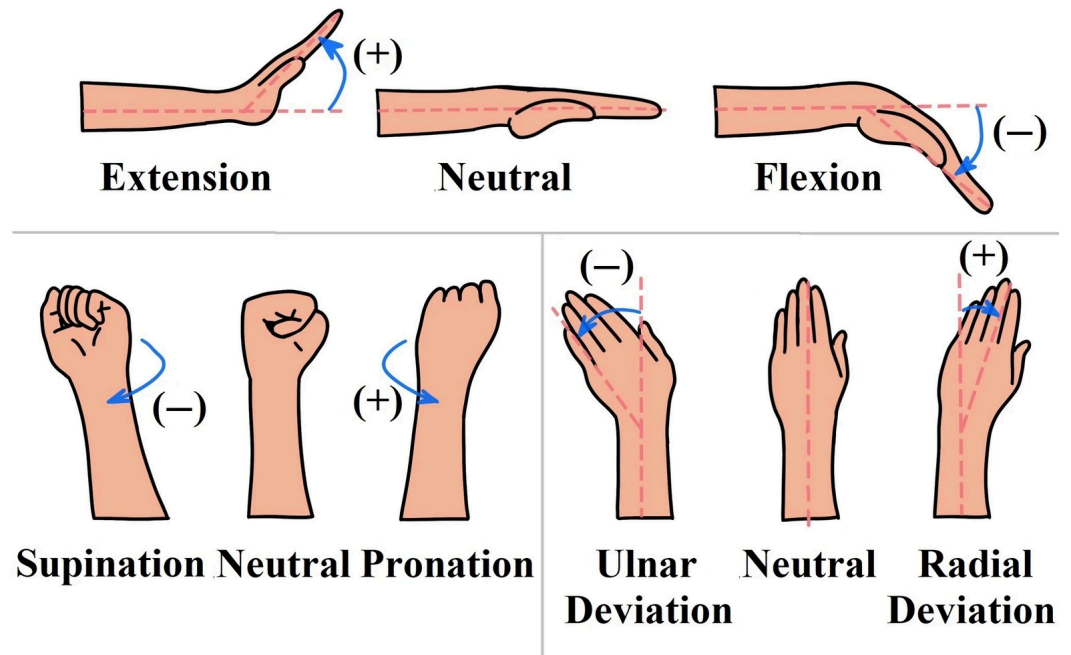


Figure 4. Movement of the human wrist around rotation axes.

The total spatial orientation of the wrist is mathematically represented by a sequence of successive rotations defined relative to the initial frame F_0 . The spherical parallel exoskeleton rotates the wrist always along the unit basis vectors of F_0 (world frame) with the aid of the three actuators. Therefore, an Initial Frame Based rotation sequence can be defined as follows involving the intermediate frames F_m and F_n .

$$F_0 \xrightarrow{\vec{u}_1^{(0)}} F_m \xrightarrow{\vec{u}_2^{(0)}} F_n \xrightarrow{\vec{u}_3^{(0)}} F_b \quad (23)$$

The total spatial orientation of the wrist is given by $\widehat{C}^{(0,b)}$ whose derivation is shown in Equations (24)–(34). The derivation involves the use of second order tensor transformations to obtain $\widehat{C}^{(0,b)}$ as a multiplication of basic exponential matrices. The general Euler angle rotation sequence matrix $\widehat{C}^{(0,b)}$ representing the transformations through intermediate frames is finally written in Equation (34).

$$\widehat{C}^{(0,b)} = \widehat{C}^{(0,m)} \widehat{C}^{(m,n)} \widehat{C}^{(n,b)} \quad (24)$$

$$\widehat{C}^{(0,m)} = e^{\vec{u}_1^{(0/0)} \alpha} = e^{\vec{u}_1 \alpha} = \widehat{R}_1(\alpha) \quad (25)$$

$$\widehat{C}^{(m,n)} = \widehat{R}_{mn}^{(m)} = \widehat{R}_{mn}^{(n)} = e^{\widetilde{u}_2^{(0/m)}\beta} \quad (26)$$

$$\widehat{C}^{(n,b)} = e^{\widetilde{u}_3^{(0/n)}\gamma} \widehat{R}_{nb}^{(n)} = \widehat{R}_{nb}^{(b)} \quad (27)$$

$$\widehat{R}_{mn}^{(0)} = e^{\widetilde{u}_2^{(0/0)}\beta} = e^{\widetilde{u}_2\beta} = \widehat{R}_2(\beta) = \widehat{C}^{(0,m)} \widehat{R}_{mn}^{(m)} \widehat{C}^{(m,0)} \quad (28)$$

$$\widehat{R}_{mn}^{(m)} = \widehat{C}^{(m,0)} \widehat{R}_2(\beta) \widehat{C}^{(0,m)} = \widehat{C}^{(m,n)} \quad (29)$$

$$\widehat{R}_{nb}^{(0)} = e^{\widetilde{u}_3^{(0/0)}\gamma} = e^{\widetilde{u}_3\gamma} = \widehat{R}_3(\gamma) = \widehat{C}^{(0,n)} \widehat{R}_{nb}^{(n)} \widehat{C}^{(n,0)} \quad (30)$$

$$\widehat{R}_{nb}^{(n)} = \widehat{C}^{(n,0)} \widehat{R}_3(\gamma) \widehat{C}^{(0,n)} = \widehat{C}^{(n,b)} = \widehat{C}^{(n,m)} \widehat{C}^{(m,0)} e^{\widetilde{u}_3\gamma} \widehat{C}^{(0,m)} \widehat{C}^{(m,n)} \quad (31)$$

$$\widehat{C}^{(0,m)} \widehat{C}^{(m,n)} = e^{\widetilde{u}_1\alpha} e^{-\widetilde{u}_1\alpha} e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} = e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} \quad (32)$$

$$\widehat{C}^{(0,m)} \widehat{C}^{(m,n)} \widehat{C}^{(n,b)} = \left(e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} \right) \left(e^{-\widetilde{u}_1\alpha} e^{-\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} \right) e^{-\widetilde{u}_1\alpha} e^{\widetilde{u}_3\gamma} e^{\widetilde{u}_1\alpha} \left(e^{-\widetilde{u}_1\alpha} e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} \right) \quad (33)$$

$$\widehat{C}^{(0,b)} = e^{\widetilde{u}_3\gamma} e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} = \widehat{R}_3(\gamma) \widehat{R}_2(\beta) \widehat{R}_1(\alpha) \quad (34)$$

The rotation between two successive frames can also be expressed by the rotation of the first frame's basis vectors to the second frame's basis vectors. Hence,

$$\vec{u}_k^{(0)} \xrightarrow{\widehat{R}_1^{(0)}} \vec{u}_k^{(m)} \xrightarrow{\widehat{R}_2^{(m)}} \vec{u}_k^{(n)} \xrightarrow{\widehat{R}_3^{(n)}} \vec{u}_k^{(b)} \quad (35)$$

Equations (36)–(40) describe the alternative derivation of the total spatial orientation of the wrist as follows.

$$\vec{u}_k^{(m/0)} = \widehat{R}_1^{(0)} \vec{u}_k^{(0/0)} \quad (36)$$

$$\vec{u}_k^{(n/0)} = \widehat{R}_2^{(0)} \vec{u}_k^{(m/0)} \quad (37)$$

$$\vec{u}_k^{(b/0)} = \widehat{R}_3^{(0)} \vec{u}_k^{(n/0)} \quad (38)$$

$$\vec{u}_k^{(b/0)} = \widehat{R}_3^{(0)} \widehat{R}_2^{(0)} \widehat{R}_1^{(0)} \vec{u}_k^{(0/0)} = \widehat{R}_3^{(0)} \widehat{R}_2^{(0)} \widehat{R}_1^{(0)} \vec{u}_k^{(b/b)} = \widehat{C}^{(0,b)} \vec{u}_k^{(b/b)} \quad (39)$$

$$\widehat{C}^{(0,b)} = \widehat{R}_3^{(0)} \widehat{R}_2^{(0)} \widehat{R}_1^{(0)} = e^{\widetilde{u}_3\gamma} e^{\widetilde{u}_2\beta} e^{\widetilde{u}_1\alpha} = \widehat{R}^{(0)} \quad (40)$$

It can be seen that the order of the basic exponential rotation matrices depends on the frame of expression of the rotation dyadics.

The initial pointing vector of the exoskeleton handle is defined at its home position along the $\vec{u}_3^{(0)}$ axis as (see Figure 3)

$$\vec{u}_0^{(0)} = \vec{u}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^t \quad (41)$$

At any given instance during multi-axis therapeutic motion, the instantaneous orientation vector of the handle $\vec{u}^{(0)}$ in the world frame F_0 is determined by the composite rotation matrix $\widehat{R}^{(0)}$ parameterized by the anatomical wrist angles as shown below.

$$\vec{u}^{(0)} = \widehat{R}^{(0)} \vec{u}_0^{(0)} = e^{\tilde{u}_3 \gamma} e^{\tilde{u}_2 \beta} e^{\tilde{u}_1 \alpha} \vec{u}_3 \quad (42)$$

Using Equation (42), the transformation under the first anatomical rotation (α) yields

$$e^{\tilde{u}_1 \alpha} \vec{u}_3 = \vec{u}_3 c\alpha - \vec{u}_2 s\alpha \quad (43)$$

Compounding this result with the subsequent rotation $e^{\tilde{u}_2 \beta}$,

$$\vec{u}^{(0)} = e^{\tilde{u}_3 \gamma} \left(c\alpha \quad c\beta \quad \vec{u}_3 + c\alpha \quad s\beta \quad \vec{u}_1 - s\alpha \quad \vec{u}_2 \right) \quad (44)$$

Expanding the remaining trigonometric terms yields the definitive forward kinematic equation for the handle's spatial tracking vector in terms of task-space coordinates:

$$\vec{u}^{(0)} = \begin{bmatrix} c\alpha \quad s\beta c\gamma + s\alpha \quad s\gamma \\ c\alpha \quad s\beta \quad s\gamma - s\alpha \quad c\gamma \\ c\alpha \quad c\beta \end{bmatrix} \quad (45)$$

The expression for $\vec{u}^{(0)}$ can be used to determine the space required to install the exoskeleton setup. To establish the inverse mapping to the actuator joint space $\vec{q} = [q_1 \quad q_2 \quad q_3]^t$ under the IFB formulation, geometric constraints based on the surface normal vectors of circular slotted arcs (4) and (5) are utilized. Let the components of the spatial tracking vector in Equation (45) be denoted as $\vec{u}^{(0)} = [m_1 \quad m_2 \quad m_3]^t$, where

The normal vectors to arc (4) (rotating about $\vec{u}_1^{(0)}$ by q_2) and arc (5) (rotating about

$\vec{u}_2^{(0)}$ by q_3), expressed in the inertial base frame, are given by

$$\vec{n}_4^{(0)} = e^{\tilde{u}_1 q_2} \vec{u}_2^{(0/0)} = \begin{bmatrix} 0 \\ \cos q_2 \\ \sin q_2 \end{bmatrix}, \quad \vec{n}_5^{(0)} = e^{\tilde{u}_2 q_3} \vec{u}_1^{(0/0)} = \begin{bmatrix} \cos q_3 \\ 0 \\ -\sin q_3 \end{bmatrix} \quad (46)$$

Since the handle vector $\vec{u}$ must remain constrained within the circular guides, it satisfies orthogonality conditions with respect to both normal vectors ($\vec{u} \cdot \vec{n}_4 = 0$ and $\vec{u} \cdot \vec{n}_5 = 0$), yielding two fundamental constraint equations:

$$m_2 \cos q_2 + m_3 \sin q_2 = 0, m_1 \cos q_3 - m_3 \sin q_3 = 0 \quad (47)$$

Additionally, the first actuator variable directly tracks forearm pronation/supination:

$$q_1 = \gamma \quad (48)$$

Solving Equations (47) and (48) provides the explicit closed-form inverse kinematic expressions for the remaining actuator angles:

$$q_2 = \text{atan2}(-\sigma_2 m_2, \sigma_2 m_3) \quad (49)$$

$$q_3 = \text{atan2}(\sigma_3 m_1, \sigma_3 m_3) \quad (50)$$

where $\sigma_2, \sigma_3 = \mp 1$ represent the configuration branch parameters of the mechanism.

When the wrist anatomical rotations are defined in agreement with the IFB formulation, the mechanism must be used by adhering to the Initial Frame Based rotation sequence specified in Equations (23) and (34). During rehabilitation, only one anatomical rotation can be desired and this can be achieved by driving only one actuator while fixing the other two. A complex rotation involving a combination of all anatomical rotations must still follow this specified order of Initial Frame Based rotations. This specified order of Initial Frame Based rotations preserves the anatomical ranges of motion of the wrist (as depicted in Figure 4) in combinations of rotations as well.

4.2. Kinematic Analysis for Rotated Frame Based Formulation

The origin of the world coordinate frame $\mathbf{F}_0$ (alternatively denoted as base frame) is rooted at the geometric center VC of the virtual universal joint, which coincides with the approximate rotation center of the human wrist. The orthogonal coordinate axes of the rotating reference frames in the Rotated Frame Based (RFB) model are defined as follows (Figures 5–7) where bodies (1), (2) and (3) form the mechanical equivalent of the forearm, the wrist and the handle attached to the hand (Frames $\mathbf{F}_0$, $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ are at-

tached to the base, body (1), body (2) and body (3), respectively):

- $\vec{u}_1^{(0)}$ - axis: Corresponds to the rotation axis of Actuator 1, governing pronation/supination (P/S) movement. It is aligned with the longitudinal axis of the forearm (1) and the neutral position of the exoskeleton handle, serving as the primary reference axis for P/S rotation. The unit vector along this axis in the base frame is $\vec{u}_1^{(0)}$, which remains aligned with $\vec{u}_1^{(1)}$ of the first rotating frame F_1 . The unit vector aligned with the handle (3) in frame F_3 is given by $\vec{u}_1^{(3)}$.
- $\vec{u}_2^{(0)}$ - axis which is the rotation axis of Actuator 2, corresponds to the rotation axis of the slotted arc (4). The flexion/extension (F/E) movement is defined as the rotation along $\vec{u}_2^{(1)}$ (coinciding with $\vec{u}_2^{(2)}$) by an angle of θ_2 .
- $\vec{u}_3^{(0)}$ - axis which is the rotation axis of Actuator 3, corresponds to the rotation axis of the slotted arc (5). The radial/ulnar (R/U) deviation is defined as the rotation along $\vec{u}_3^{(2)}$ (coinciding with $\vec{u}_3^{(3)}$) by an angle θ_3 .

In the Rotated Frame Based (RFB) formulation, rotation transformations are applied sequentially about the axes of the rotated body frames F_1, F_2 , and F_3 , yielding the frame rotation sequence:

$$F_0 \xrightarrow{\vec{u}_1^{(0)}, \theta_1} F_1 \xrightarrow{\vec{u}_2^{(1)}, \theta_2} F_2 \xrightarrow{\vec{u}_3^{(2)}, \theta_3} F_3 \quad (51)$$

where $\theta_1 = \gamma$ (pronation/supination), $\theta_2 = \alpha$ (flexion/extension), and $\theta_3 = \beta$ (radial/ulnar

deviation). The composite orientation matrix $\widehat{C}^{(0,3)}$ mapping the body-fixed frame F_3

back to the world reference frame F_0 is evaluated by compounding the basic exponential rotation matrices about the successive body axes:

$$\widehat{C}^{(0,3)} = \widehat{C}^{(0,1)} \widehat{C}^{(1,2)} \widehat{C}^{(2,3)} \quad (52)$$

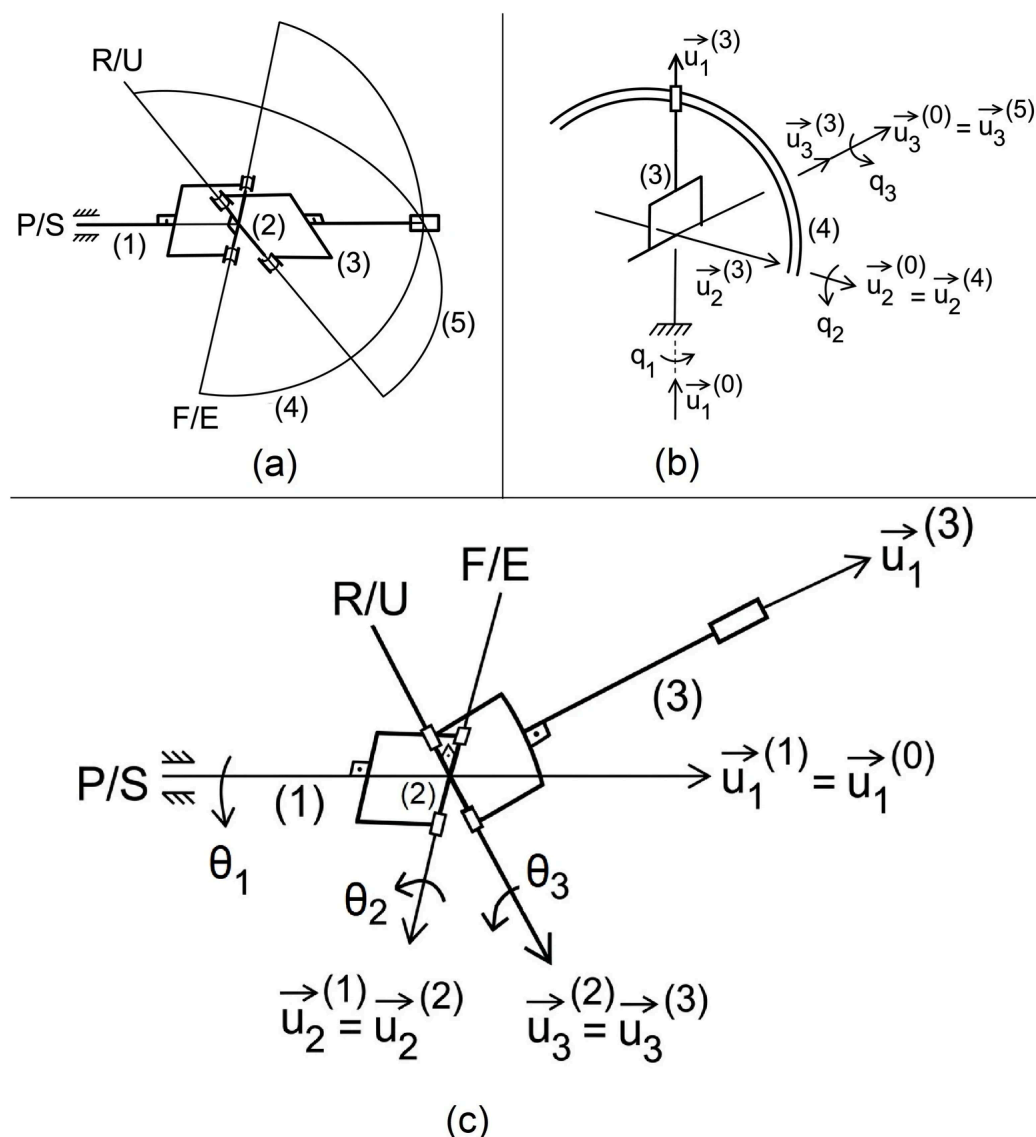


Figure 5. Kinematic schematic and frame assignment for the Rotated Frame Based (RFB) model: (a) Isometric schematic view of the 3-DOF wrist exoskeleton mechanism showing bodies (1)–(5) and joint motion axes for pronation/supination (P/S), flexion/extension (F/E), and radial/ulnar deviation (R/U); (b) 3D isometric structural layout of the slotted arc (4) and active joint angles (q_1, q_2, q_3); (c) side/frontal schematic view of the equivalent serial kinematic chain showing body-fixed coordinate frames (F_0, F_1, F_2, F_3) and wrist anatomical rotations ($\theta_1, \theta_2, \theta_3$).

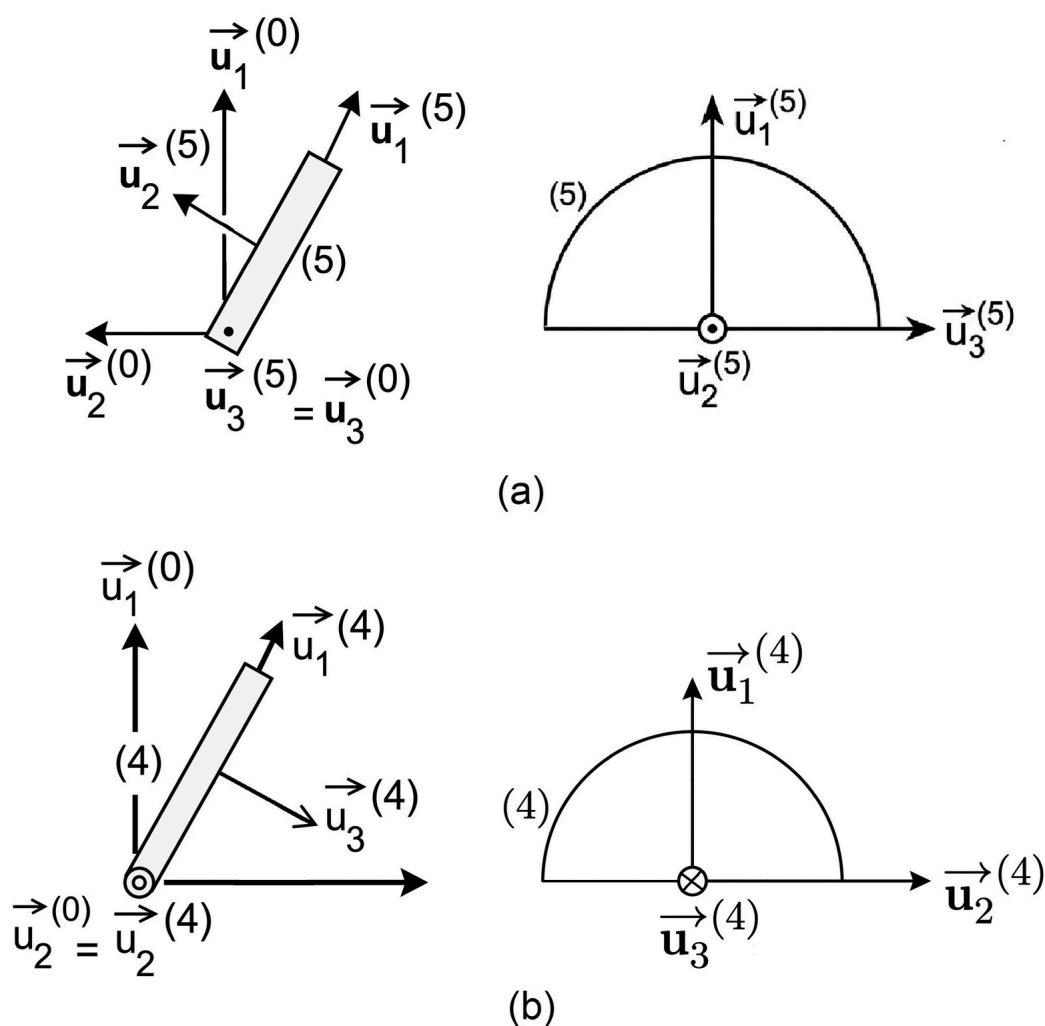


Figure 6. (a) Side and top views of the slotted arc (5) with the unit basis vectors of its body-fixed frame; (b) side and top views of the slotted arc (4) with the unit basis vectors of its body-fixed frame.

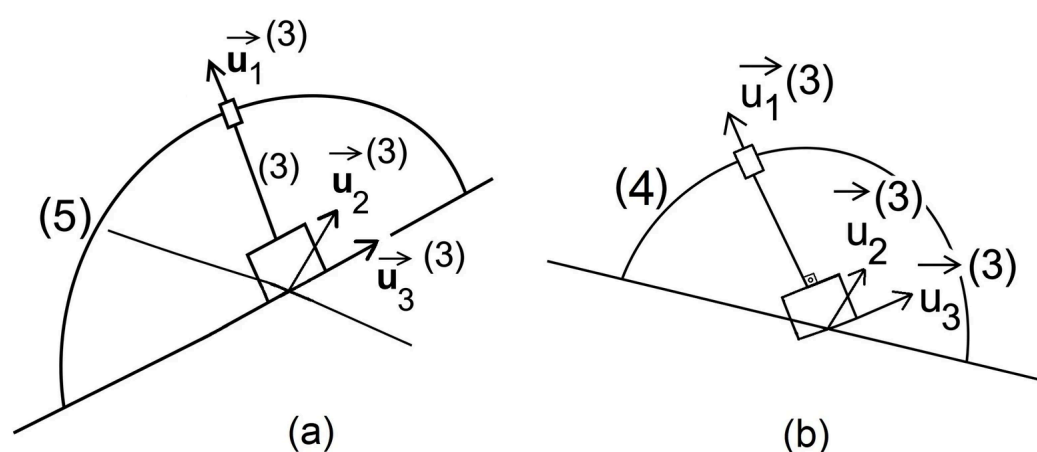


Figure 7. Spatial orientation and motion tracking of the handle (3): (a) along slotted arc (5); (b) along slotted arc (4).

The individual elemental rotation matrices associated with each body frame transformation are defined as

$$\widehat{C}^{(0,1)} = e^{\widetilde{u}_1 \theta_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{bmatrix} \quad (53)$$

$$\widehat{C}^{(1,2)} = e^{\widetilde{u}_2 \theta_2} = \begin{bmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{bmatrix} \quad (54)$$

$$\widehat{C}^{(2,3)} = e^{\widetilde{u}_3 \theta_3} = \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 & 0 \\ \sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (55)$$

In the local body frame F_3 , the unit pointing vector of the exoskeleton handle is aligned along its primary geometric axis as $\bar{u}^{(3)} = [1 \ 0 \ 0]^t$. This unit vector is expressed in frame F_0 by using the following transformation:

$$\bar{u}^{(0)} = \widehat{C}^{(0,3)} \bar{u}^{(3)} \quad (56)$$

$$\bar{u}^{(0)} = \begin{bmatrix} \cos \theta_2 \cos \theta_3 \\ \cos \theta_1 \sin \theta_3 + \sin \theta_1 \sin \theta_2 \cos \theta_3 \\ \sin \theta_1 \sin \theta_3 - \cos \theta_1 \sin \theta_2 \cos \theta_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (57)$$

Substituting the anatomical wrist angles ($\theta_1 = \gamma, \theta_2 = \alpha, \theta_3 = \beta$) directly into Equation (57) establishes the explicit task-space components r_1, r_2 , and r_3 :

$$r_1 = \cos \alpha \cos \beta, \quad r_2 = \cos \gamma \sin \beta + \sin \gamma \sin \alpha \cos \beta, \quad r_3 = \sin \gamma \sin \beta - \cos \gamma \sin \alpha \cos \beta \quad (58)$$

To establish the inverse kinematic mapping to the actuator joint space $q = [q_1, q_2, q_3]^t$ under the RFB formulation, geometric constraints based on the surface normal vectors of slotted arcs (4) and (5) are employed. The unit normal vector $\vec{n}_4$ to the plane of arc (4) and the unit normal vector $\vec{n}_5$ to the plane of arc (5), expressed in the world frame F_0 , are defined by

$$\begin{matrix} \vec{n}_4^{(0)} \\ =\vec{u}_3^{(4/0)} \\ =\widehat{C}^{(0,4)} \\ \vec{u}_3^{(4/4)} \\ =e^{\tilde{u}_2 q_2} \vec{u}_3 \end{matrix} = \begin{bmatrix} \sin q_2 \\ 0 \\ \cos q_2 \end{bmatrix} \quad (59)$$

$$\begin{matrix} \vec{n}_5^{(0)} \\ =\vec{u}_2^{(5/0)} \\ =\widehat{C}^{(0,5)} \\ \vec{u}_2^{(5/5)} \\ =e^{\tilde{u}_3 q_3} \vec{u}_2 \end{matrix} = \begin{bmatrix} -\sin q_3 \\ \cos q_3 \\ 0 \end{bmatrix} \quad (60)$$

Because the handle remains constrained to travel along both circular guide arcs simultaneously, the handle vector $\vec{u}$ must satisfy strict orthogonality conditions with respect to both surface normals ($\vec{u} \cdot \vec{n}_4 = 0$ and $\vec{u} \cdot \vec{n}_5 = 0$):

$$\vec{u}^{(0)t} \vec{n}_4^{(0)} = r_1 \sin q_2 + r_3 \cos q_2 = 0 \implies \tan q_2 = -\frac{r_3}{r_1} \quad (61)$$

$$\vec{u}^{(0)t} \vec{n}_5^{(0)} = -r_1 \sin q_3 + r_2 \cos q_3 = 0 \implies \tan q_3 = \frac{r_2}{r_1} \quad (62)$$

Furthermore, the first active joint parameter directly tracks the forearm pronation/supination angle:

$$q_1 = \theta_1 = \gamma \quad (63)$$

Solving Equations (61) and (62) using the standard two-argument arctangent function yields the explicit, closed-form inverse kinematic equations for actuator angles q_2 and q_3 in terms of the anatomical wrist joint angles with $\sigma_2, \sigma_3 = \mp 1$:

$$q_2 = \text{atan2} \left(\sigma_2 (\cos \gamma |\cos \beta| \sin \alpha| - |\sin \gamma| \sin \beta), \sigma_2 \cos \alpha \cos \beta \right) \quad (64)$$

$$q_3 = \text{atan2} \left(\sigma_3 (\cos \gamma |\sin \beta| + |\sin \gamma| \sin \alpha| \cos \beta), \sigma_3 \cos \alpha \cos \beta \right) \quad (65)$$

5. Singularity Analysis of the Spherical Parallel Exoskeleton

In this section, the kinematic performance of the proposed spherical parallel mechanism is evaluated through analytical singularity identification, manipulability measures, and human-machine workspace coverage verification. The fundamental relationship governing the joint velocities and the operational task-space angular velocities of the exoskeleton handle is established via the analytical Jacobian matrix ($\hat{J}$). The rates of the

wrist anatomical angles $\dot{x}$ are directly mapped to the active joint rates $\dot{q}$ by the Jacobian

matrices $\hat{J}_q$ and $\hat{J}_x$, and are expressed in matrix form as

$$\hat{J}_q \dot{\bar{q}} = \hat{J}_x \dot{\bar{x}} \quad (66)$$

$$\dot{\bar{q}} = \hat{J}_q^{-1} \hat{J}_x \dot{\bar{x}} \quad (67)$$

For Initial Frame Based (IFB) and Rotated Frame Based (RFB) models, Equations (48)–(50) and Equations (63)–(65) are differentiated to obtain $\hat{J}_q$ and $\hat{J}_x$ where $\hat{J}_q = \hat{I}$ (identity matrix) for both. In the IFB model, $\dot{\bar{x}} = \begin{bmatrix} \dot{\alpha} & \dot{\beta} & \dot{\gamma} \end{bmatrix}^t$, whereas in the RFB model,

$\dot{\bar{x}} = \begin{bmatrix} \dot{\gamma} & \dot{\alpha} & \dot{\beta} \end{bmatrix}^t$. For both models $\dot{\bar{q}} = \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 \end{bmatrix}^t$ denotes the active joint velocity vector

corresponding to the individual actuator rotations.

$$\hat{J}_x = \begin{bmatrix} \frac{\partial q_1}{\partial \alpha} & \frac{\partial q_1}{\partial \beta} & \frac{\partial q_1}{\partial \gamma} \\ \frac{\partial q_2}{\partial \alpha} & \frac{\partial q_2}{\partial \beta} & \frac{\partial q_2}{\partial \gamma} \\ \frac{\partial q_3}{\partial \alpha} & \frac{\partial q_3}{\partial \beta} & \frac{\partial q_3}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \quad (IFB), \quad \hat{J}_x = \begin{bmatrix} \frac{\partial q_1}{\partial \gamma} & \frac{\partial q_1}{\partial \alpha} & \frac{\partial q_1}{\partial \beta} \\ \frac{\partial q_2}{\partial \gamma} & \frac{\partial q_2}{\partial \alpha} & \frac{\partial q_2}{\partial \beta} \\ \frac{\partial q_3}{\partial \gamma} & \frac{\partial q_3}{\partial \alpha} & \frac{\partial q_3}{\partial \beta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} \quad (RFB) \quad (68)$$

The differentiation of two-argument arctangent actuator functions $q_i = \text{atan2}(N_i, D)$

with respect to task-space variable $x_k \in \{\alpha, \beta, \gamma\}$ follows the general quotient differential identity:

$$\frac{\partial}{\partial x_k} [\text{atan2} | (N_i, D)] = \frac{D \frac{\partial N_i}{\partial x_k} - N_i \frac{\partial D}{\partial x_k}}{D^2 + N_i^2} \quad (69)$$

where $D = \cos\alpha\cos\beta$ represents the shared denominator term across both IFB and RFB formulations.

5.1. Singularity Analysis for Initial Frame Based Formulation

In the Initial Frame Based (IFB) formulation, the numerators of the arctangent functions for q_2 and q_3 are defined as $N_2 = \sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma$ and $N_3 = \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma$, respectively. Differentiating q_1, q_2 , and q_3 yields the individual Jacobian elements:

$$J_{21} = \frac{\partial q_2}{\partial \alpha} = \frac{\cos \beta \cos \gamma}{\cos 2\alpha \cos 2\beta + (\sin \alpha |\cos \gamma| - |\cos \alpha \sin \beta| \sin \gamma)^2} = \frac{\cos \beta \cos \gamma}{D^2 + N_2^2} \quad (70)$$

$$J_{22} = \frac{\partial q_2}{\partial \beta} = \frac{\cos \alpha (\sin \alpha |\sin \beta| \cos \gamma| - |\cos \alpha \sin \gamma|)}{D^2 + N_2^2} \quad (71)$$

$$J_{23} = \frac{\partial q_2}{\partial \gamma} = \frac{-\cos \alpha \cos \beta (\sin \alpha |\sin \gamma| + |\cos \alpha \sin \beta| \cos \gamma)}{D^2 + N_2^2} \quad (72)$$

$$J_{31} = \frac{\partial q_3}{\partial \alpha} = \frac{\cos \beta \sin \gamma}{D^2 + N_3^2} \quad (73)$$

$$J_{32} = \frac{\partial q_3}{\partial \beta} = \frac{\cos \alpha (\cos \alpha |\cos \gamma| + |\sin \alpha \sin \beta| \sin \gamma)}{D^2 + N_3^2} \quad (74)$$

$$J_{33} = \frac{\partial q_3}{\partial \gamma} = \frac{\cos \alpha \cos \beta (\sin \alpha |\cos \gamma| - |\cos \alpha \sin \beta| \sin \gamma)}{D^2 + N_3^2} \quad (75)$$

Expanding the determinant along the first row of $\hat{J}_x$ in Equation (68) yields

$$\det(\hat{J}_x) = J_{21}J_{32} - J_{22}J_{31} = \frac{\cos 2\alpha \cos \beta}{\left(D^2 + N_2^2\right) \left(D^2 + N_3^2\right)} \quad (76)$$

5.2. Singularity Analysis for Rotated Frame Based Formulation

In the Rotated Frame Based (RFB) formulation, the numerator terms for q_2 and q_3 are given by $N_2 = \cos \gamma \cos \beta \sin \alpha - \sin \gamma \sin \beta$ and $N_3 = \cos \gamma \sin \beta + \cos \beta \sin \gamma \sin \alpha$, respectively. Applying Equation (69) yields the corresponding Jacobian matrix entries:

$$J_{21} = \frac{\partial q_2}{\partial \gamma} = \frac{-\cos \alpha \cos \beta (\sin \gamma |\cos \beta| \sin \alpha| + |\cos \gamma| \sin \beta)}{D^2 + N_2^2} \quad (77)$$

$$J_{22} = \frac{\partial q_2}{\partial \alpha} = \frac{\cos \beta (\cos \gamma |\cos \beta| - |\sin \gamma| \sin \beta| \sin \alpha)}{D^2 + N_2^2} \quad (78)$$

$$J_{23} = \frac{\partial q_2}{\partial \beta} = \frac{-\cos \alpha \sin \gamma}{D^2 + N_2^2} \quad (79)$$

$$J_{31} = \frac{\partial q_3}{\partial \gamma} = \frac{\cos \alpha \cos \beta (\cos \gamma |\cos \beta| \sin \alpha| - |\sin \gamma| \sin \beta)}{D^2 + N_3^2} \quad (80)$$

$$J_{32} = \frac{\partial q_3}{\partial \alpha} = \frac{\cos \beta (\sin \gamma |\cos \beta| + |\cos \gamma| \sin \beta \sin \alpha)}{D^2 + N_3^2} \quad (81)$$

$$J_{33} = \frac{\partial q_3}{\partial \beta} = \frac{\cos \alpha \cos \gamma}{D^2 + N_3^2} \quad (82)$$

Evaluating the determinant of the Jacobian matrix $\hat{J}_x$ via $\det(\hat{J}_x) = J_{21}J_{32} - J_{22}J_{31}$ results in an exceptionally simplified expression due to internal trigonometric identity cancelations:

$$\det(\hat{J}_x) = J_{22}J_{33} - J_{23}J_{32} = \frac{\cos \alpha \cos 2\beta}{(D^2 + N_2^2)(D^2 + N_3^2)} \quad (83)$$

5.3. Singularities, Manipulability Index and Condition Number

The position equations for the IFB model indicate that singularities occur when the handle unit vector $\vec{u}$ is on the plane defined by the axes $\vec{u}_1^{(0)}$ and $\vec{u}_2^{(0)}$ or when $\vec{u}$ is particularly on the $\vec{u}_1^{(0)}$ or $\vec{u}_2^{(0)}$ axes. These singular configurations correspond to an extreme radial/ulnar deviation of $\beta = \pm 90^\circ$ and/or flexion/extension of $\alpha = \pm 90^\circ$. Similarly, the position equations for the RFB model show that singularities occur when the handle unit vector $\vec{u}_1^{(3)}$ is on the plane defined by the axes $\vec{u}_2^{(0)}$ and $\vec{u}_3^{(0)}$ or when $\vec{u}_1^{(3)}$ is particularly on the $\vec{u}_2^{(0)}$ or $\vec{u}_3^{(0)}$ axes. These singular configurations again correspond to an extreme radial/ulnar deviation of $\beta = \pm 90^\circ$ and/or flexion/extension of $\alpha = \pm 90^\circ$. In these singular configurations, the actuators lose their ability to transmit motion to the wrist in all directions. The figures in Section 6 can be consulted to visualize the motion of the mechanism at the singularities.

The overall Jacobian matrix of the system is $\hat{J} = \hat{J}_x^{-1} \hat{J}_q$ with $\dot{\vec{x}} = \hat{J}_x^{-1} \hat{J}_q \dot{q}$; hence, singularities of this mechanism occur when $\det(\hat{J}_x)$ becomes zero and these represent critical configurations corresponding to an extreme radial/ulnar deviation of $\beta = \pm 90^\circ$ and/or flexion/extension of $\alpha = \pm 90^\circ$. In these singular configurations, the wrist motion is not controllable by the actuators. However, the physiological range of motion (ROM) of the human wrist strictly limits this deviation to a much narrower envelope, specifically $-20^\circ \leq \beta \leq 30^\circ$ for radial/ulnar deviation and $-70^\circ \leq \alpha \leq 80^\circ$ for flexion/extension [41]. Because the physical joint limits of the patient are far removed from the theoretical singularity threshold, the determinant never approaches zero within the active workspace.

Consequently, the mechanism is mathematically proven to be singularity-free throughout any multi-axis therapeutic routine.

To quantify the mechanism's fluid motion transmission capability and ease of maneuvering in distinct anatomical directions in terms of kinematics, the manipulability index (w) is defined directly via the geometric formulation of the Jacobian system [42] as

$$w = 1 / \left| \det \left(\hat{J}_x \right) \right| \quad (84)$$

$$\text{In IFB} \rightarrow w = \left| \frac{\left[D^2 + N_2^2 \right] \left[D^2 + N_3^2 \right]}{\cos 2\alpha \cos \beta} \right| \quad (85)$$

$$\text{In RFB} \rightarrow w = \left| \frac{\left[D^2 + N_2^2 \right] \left[D^2 + N_3^2 \right]}{\cos \alpha \cos^2 \beta} \right| \quad (86)$$

A manipulability value of $w = 1$ at the neutral posture signifies kinematic efficiency, indicating its potential for higher trajectory tracking where the patient requires central motor recovery.

Furthermore, the directional uniformity of haptic interaction and force feedback transparency is evaluated using the condition number (κ) of the overall Jacobian matrix, defined as

$$\kappa(J) = \left\| \hat{J} \right\| \cdot \left\| \hat{J}^{-1} \right\| = \frac{\sigma_{\max}}{\sigma_{\min}} \quad (87)$$

where $\sigma_{\max}$ and $\sigma_{\min}$ represent the maximum and minimum singular values of the Jacobian matrix, respectively. For physical rehabilitation applications, an isotropic state ($\kappa \rightarrow 1$) is highly desired, as it ensures that the patient perceives therapeutic resistance or automated guidance uniformly across all physical movement axes without artificial, axis-dependent distortion.

The clinical utility of the developed parallel exoskeleton can be shown by its ability to fully encompass the natural anatomical range of motion (ROM) of the human wrist [41], which is governed by the following constraints (Figure 4):

- Flexion/extension (α): $-70^\circ \leq \alpha \leq 80^\circ$
- Radial/ulnar deviation (β): $-20^\circ \leq \beta \leq 30^\circ$
- Pronation/supination (γ): $-80^\circ \leq \gamma \leq 80^\circ$

By integrating these performance indices over the entire human joint space, the Serviceable Workspace is formally defined as the region satisfying standard safety and operational haptic thresholds ($0.5 < w < 2.0$ and $\kappa < 5.0$) [43–45]. In this study, the total anatomical range of motion (ROM) area is defined by the angular limits of wrist flexion/extension and radial/ulnar deviation. If the hand is considered as a straight rigid linkage, the area it sweeps between these two limits of motion forms an inverted dome

shape. The product of the angular ranges for flexion/extension and radial/ulnar deviation directly yields the angular area of this inverted dome. The quantitative efficiency of the system is evaluated using the Coverage Percentage metric:

$$\text{Coverage \%} = \frac{\text{Serviceable Workspace Area}}{\text{Total Anatomical ROM Area}} \times 100 \quad (88)$$

The mechanism achieves 100% reachability, fully encompassing the targeted anatomical range of motion (ROM). However, full geometric reachability does not imply uniform kinematic performance throughout the workspace. Evaluated via the manipulability index (w) and condition number (κ), the manipulator exhibits optimal dexterity and isotropy around neutral configurations, whereas performance deteriorates toward the workspace boundaries ($\alpha, \beta \rightarrow \pm 90^\circ$).

6. Verification of Kinematic Analysis

To validate the theoretical kinematic formulations derived for both the Initial Frame Based (IFB) and Rotated Frame Based (RFB) models, a 3D parametric simulation environment was developed in Wolfram Mathematica 10.0. The simulation interactively computes the exact spatial configuration of the 3-DOF spherical parallel exoskeleton, circular slotted arcs, and handle end-effector for any arbitrary set of anatomical wrist joint angles (α, β, γ). The actuator joint angles (q_1, q_2, q_3) computed by the Mathematica simulation engine match the explicit analytical closed-form inverse kinematic equations with 100% precision across all workspace poses, thereby verifying the mathematical consistency of both formulations.

The kinematic response was first evaluated at the neutral (home) pose and along the boundary limits of individual single-axis anatomical motions. Figure 8 illustrates the manipulator at the home orientation ($\alpha=0^\circ, \beta=0^\circ, \gamma=0^\circ$). In both IFB and RFB modes, the analytical inverse kinematic equations yield $q_1=0.0^\circ, q_2=0.0^\circ$, and $q_3=0.0^\circ$, confirming that both reference frames share an identical initial baseline configuration. Figure 9 demonstrates single-axis pitch motion across maximum anatomical limits ($\alpha=-70^\circ$ for flexion and $\alpha=80^\circ$ for extension) with $\beta=\gamma=0^\circ$. For both IFB and RFB formulations, Actuator 2 tracks the anatomical input directly ($q_2=-70.0^\circ$ and $q_2=80.0^\circ$, respectively), while $q_1=q_3=0.0^\circ$. Figure 10 presents single-axis yaw motion across limits ($\beta=-20^\circ$ for ulnar deviation and $\beta=30^\circ$ for radial deviation) with $\alpha=\gamma=0^\circ$. Actuator 3 tracks the deviation directly ($q_3=-20.0^\circ$ and $q_3=30.0^\circ$), while $q_1=q_2=0.0^\circ$ for both IFB and RFB modes. These uncoupled tests confirm that during pure single-axis rotations along the principal axes, both IFB and RFB models exhibit decoupled primary joint behavior and yield identical actuator displacements.

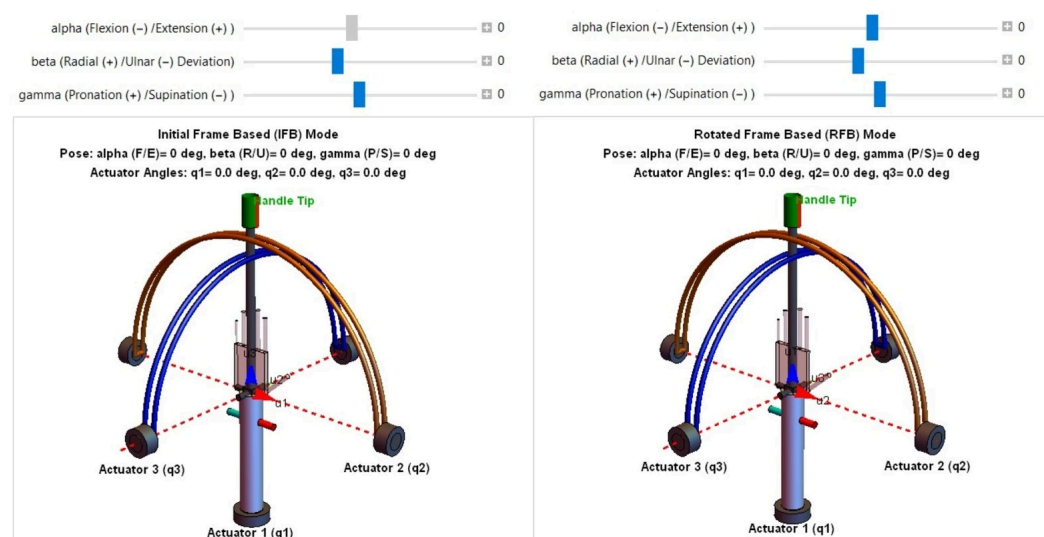


Figure 8. 3D Mathematica kinematic simulation at the neutral/home pose ($\alpha=0^\circ$, $\beta=0^\circ$, $\gamma=0^\circ$): Initial Frame Based (IFB) mode (left picture) and Rotated Frame Based (RFB) mode (right picture).

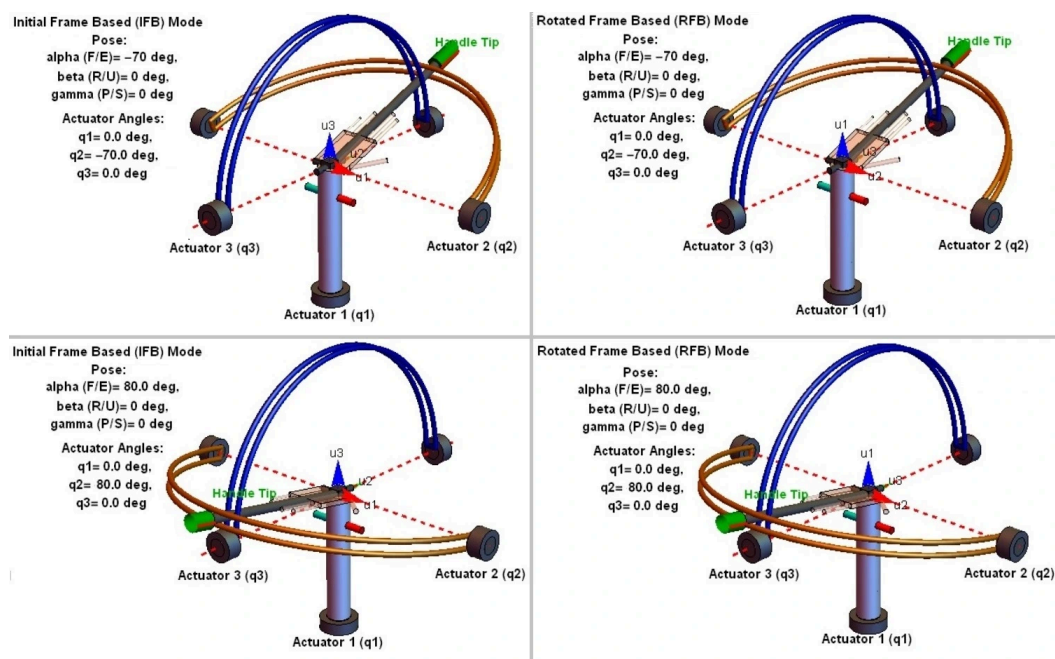


Figure 9. 3D kinematic verification during single-axis flexion/extension (F/E) motion limits comparing IFB (left) and RFB (right) modes: Maximum flexion pose (top); maximum extension pose (bottom).

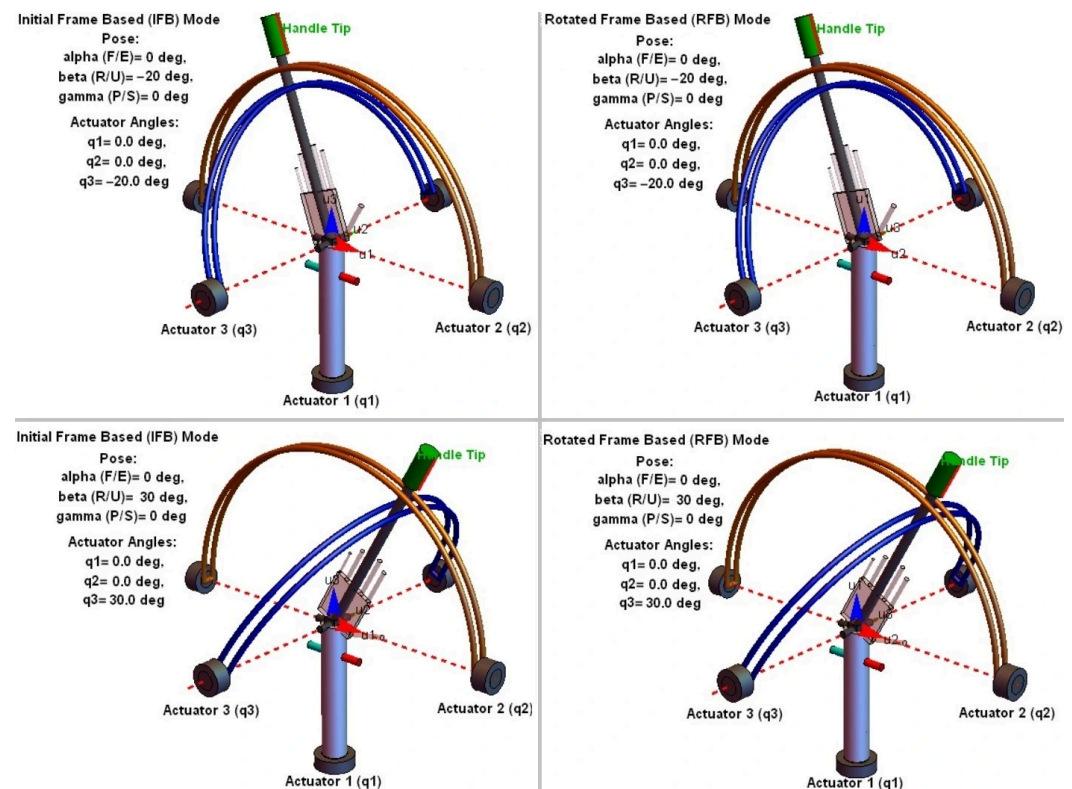


Figure 10. 3D kinematic verification during single-axis radial/ulnar (R/U) deviation motion limits comparing IFB (left) and RFB (right) modes: Maximum ulnar deviation pose (top); maximum radial deviation pose (bottom).

To verify the analytical models during complex coupled therapeutic trajectories, a multi-axis combined task-space pose was evaluated at $\alpha=49^\circ$ (extension), $\beta=-12^\circ$ (ulnar deviation), and $\gamma=34^\circ$ (pronation).

As shown in Figure 11, the resulting actuator values computed in the simulation environment precisely mirror the closed-form theoretical equations:

- Evaluating the IFB closed-form equations (Equations (48)–(50)) yields

$$q_1=34.0^\circ, \quad q_2=47.6^\circ, \quad q_3=25.7^\circ$$

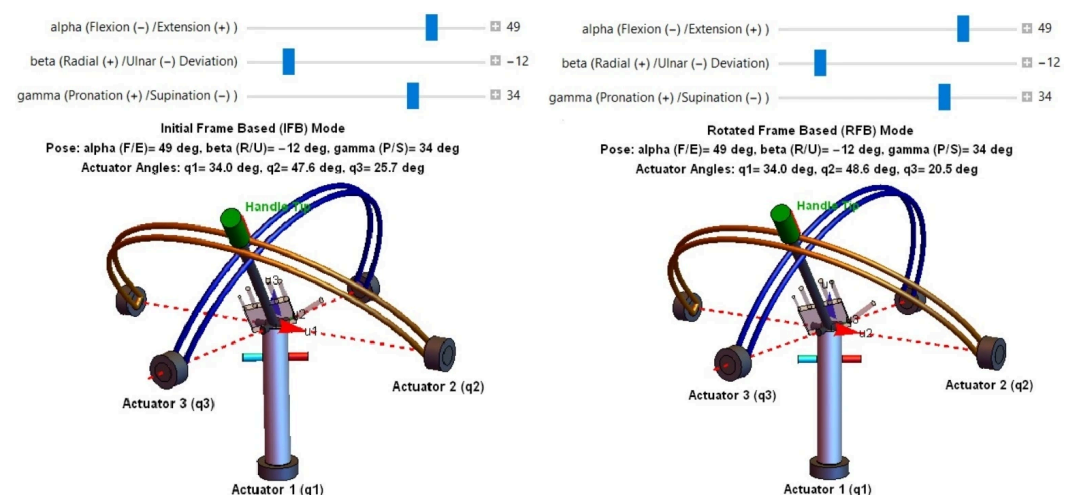


Figure 11. 3D kinematic simulation and actuator angle validation for a multi-axis combined task-space pose ($\alpha=49^\circ$, $\beta=-12^\circ$, $\gamma=34^\circ$): IFB mode (left) resulting in $q_1=34.0^\circ$, $q_2=47.6^\circ$, $q_3=25.7^\circ$ and RFB mode (right) resulting in $q_1=34.0^\circ$, $q_2=48.6^\circ$, $q_3=20.5^\circ$.

- Evaluating the RFB closed-form equations (Equations (63)–(65)) yields

$$q_1=34.0^\circ, \quad q_2=48.6^\circ, \quad q_3=20.5^\circ$$

While the pronation actuator ($q_1=\gamma=34.0^\circ$) remains identical in both formulations, q_2 and q_3 exhibit sequence-dependent variations ($\Delta q_2=1.0^\circ$, $\Delta q_3=5.2^\circ$) due to the fundamental transformation difference between fixed-world (IFB) and rotated-body (RFB) coordinates. The exact 1-to-1 numerical match between the analytical formulas and the 3D Mathematica simulation across all joint variables confirms the validity of the derived kinematic models.

7. Results and Discussion

The objective of the analysis conducted on the proposed design is to demonstrate the mechanism's kinematic performance using Jacobian analysis and to indicate its clinical application workspace. In this context, the manipulability index (w) is calculated via the analytical Jacobian matrix of the mechanism. Furthermore, the directional uniformity of motion transmission was evaluated using the condition number (κ) distribution across the workspace. In robotic design, a condition number close to unity ($\kappa \rightarrow 1$) signifies an isotropic motion transmission, which is crucial for ensuring haptic transparency, allowing the patient to perceive therapeutic resistances homogeneously without axis-dependent distortion. Conversely, a condition number approaching infinity indicates a kinematic singularity where the mechanism locks or loses control. In this section, the results are also examined under separate subheadings for the Initial Based Frame (IBF) and the Rotated Based Frame (RBF).

7.1. Kinematic Performance Analysis Under IFB Formulation

Under the IFB formulation, the task-space manipulability index w_{IFB} (Figures 12 and 13) and condition number κ (Figure 14) are coupled functions of all three anatomical joint angles, α , β , and γ , reflecting the extrinsic coordinate transformation relative to the base reference frame. The 2D contour maps of the manipulability index w_{IFB} across the flexion/extension ($\alpha \in [-70^\circ, 80^\circ]$) and radial/ulnar deviation ($\beta \in [-20^\circ, 30^\circ]$) axes are evaluated at five discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$), as depicted in Figure 12. At the neutral pronation pose (

$\gamma=0^\circ$), the mechanism exhibits superior kinematic dexterity across almost the entire operational domain, maintaining $w_{\text{IFB}} \approx 1.0$ throughout the central region. As the forearm pronation/supination angle increases ($|\gamma| \rightarrow 40^\circ, 80^\circ$), spatial cross-axis coupling causes a sharp rise in the manipulability index near the boundaries of flexion/extension ($\alpha \rightarrow -70^\circ$ and $\alpha \rightarrow 80^\circ$), reflecting the approach toward representation singularities ($w_{\text{IFB}} \rightarrow \infty$). Nevertheless, within the high-use clinical rehabilitation zone (centered around $\alpha, \beta \approx 0^\circ$), the manipulability index maintains smooth, continuous contours without abrupt gradient spikes and the mechanism has the potential to provide stable velocity transmission without control jitter during therapeutic exercises.

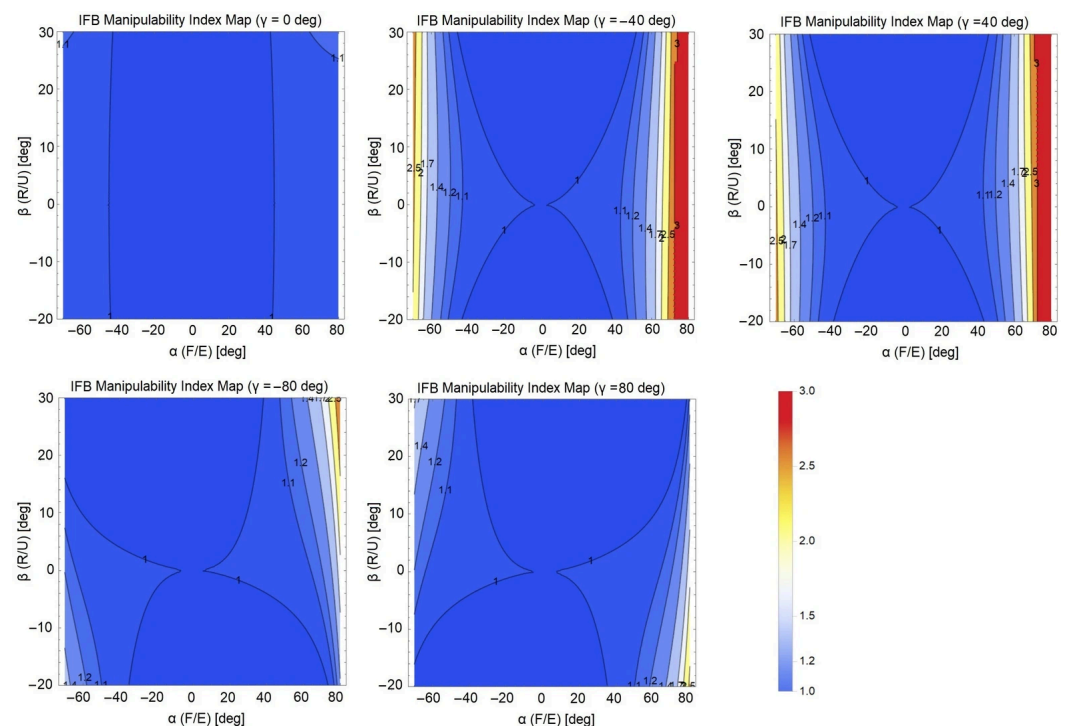


Figure 12. Manipulability index (w_{IFB}) contour maps across the anatomical flexion/extension (α) and radial/ulnar deviation (β) axes evaluated at discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$).

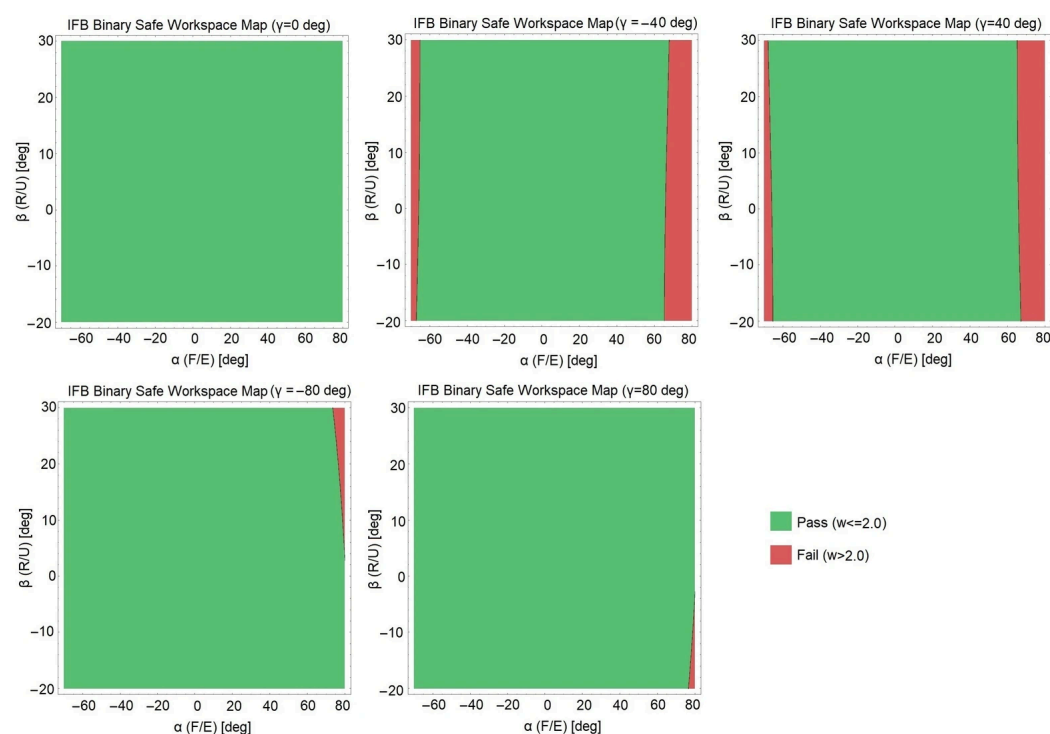


Figure 13. IFB binary safe workspace maps evaluating operational safety against the threshold criterion $w \leq 2.0$ (green = Pass, red = Fail) across discrete pronation/supination orientations.

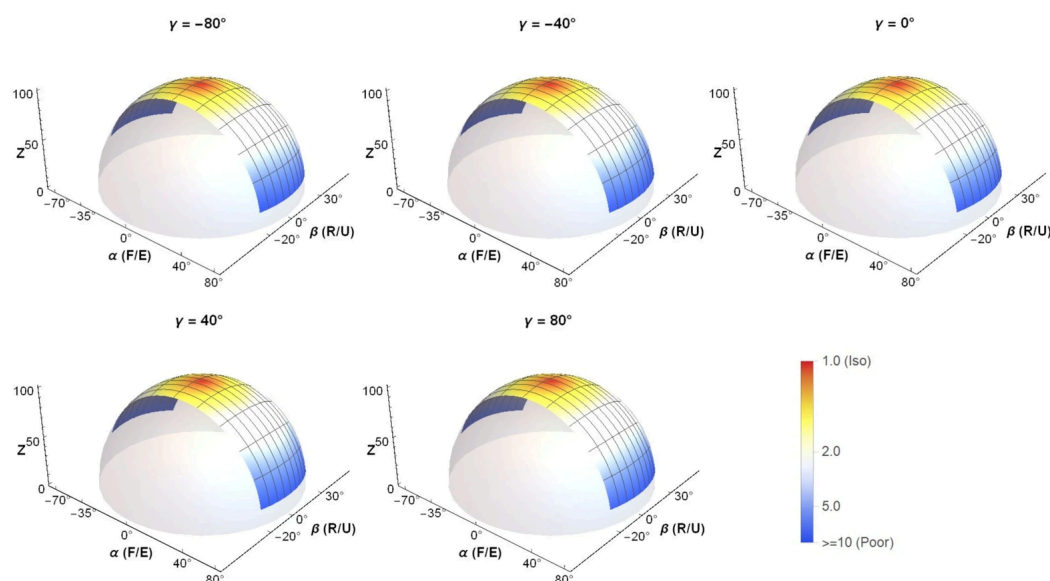


Figure 14. 3D hemispherical workspace heatmaps of the condition number (κ) distribution under IFB formulation for varying pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$), depicting local kinematic isotropy.

To quantify workspace reliability and identify potential operational vulnerabilities, a binary safety threshold of $w_{th} = 2.0$ was established. Figure 13 presents the resulting Binary Safe Workspace Maps for the IFB model. The green regions represent the Pass

Zone ($w \leq 2.0$), indicating safe kinematic states suitable for active/passive therapy. At $\gamma = 0^\circ$, 100% of the anatomical ROM falls squarely within the safe zone. The red regions represent the Fail Zone ($w > 2.0$), indicating boundary zones where velocity sensitivity degrades rapidly due to extrinsic frame representation singularities as $\alpha, \beta \rightarrow \pm 90^\circ$. At moderate to extreme pronation/supination orientations ($|\gamma| \geq 40^\circ$), boundary regions near extreme extension ($\alpha > 65^\circ$) and flexion exceed the safety threshold. Overall, the vast majority of the functional workspace complies with the safety criterion, verifying that the manipulator provides sufficient dexterity for standard daily living and clinical wrist exercises.

Kinematic isotropy measures the uniformity of force and velocity transmission in all task-space directions, which is critical for achieving haptic transparency during patient–robot interactions. The condition number $\kappa = \|J\| \|J^{-1}\|$ serves as the primary indicator, where $\kappa = 1.0$ indicates perfect isotropic behavior and $\kappa \rightarrow \infty$ signifies proximity to a kinematic singularity. Figure 14 illustrates the 3D hemispherical spatial distribution of the condition number mapped directly onto the spherical workspace shell for varying pronation/supination angles (γ). At around the neutral pose for α, β , the condition number approaches an ideal state ($\kappa \approx 1.0$, shown in deep red/orange), indicating isotropic force transparency. In this posture, the patient experiences uniform resistance regardless of the direction of motion. Moving toward the spatial workspace boundaries, κ increases progressively (shifting toward yellow and blue shades). The highest condition numbers are confined strictly to extreme workspace limits. This spatial visualization confirms that performance degradation under IFB is localized to outer bounds, leaving the primary therapeutic working envelope highly isotropic.

7.2. Kinematic Performance Analysis Under RFB Formulation

Under the RFB formulation, the task-space manipulability index w_{RFB} (Figures 15 and 16) and condition number κ (Figure 17) demonstrate significantly improved spatial homogeneity across the operational workspace. The 2D contour maps of the manipulability index w_{RFB} across the flexion/extension ($\alpha \in [-70^\circ, 80^\circ]$) and radial/ulnar deviation ($\beta \in [-20^\circ, 30^\circ]$) axes are evaluated at five discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$), as depicted in Figure 15. At the neutral pronation pose ($\gamma = 0^\circ$), the mechanism exhibits near-perfect kinematic isotropy, maintaining an almost constant $w_{\text{RFB}} \approx 1.0$ across the entire operational domain. As the forearm

pronation/supination angle varies ($|\gamma| \rightarrow 40^\circ, 80^\circ$), spatial cross-axis coupling remains exceptionally low, with only minor contour variations appearing near extreme extension ($\alpha \rightarrow 80^\circ$ at $\gamma = \pm 40^\circ$) where w_{RFB} reaches a modest peak of approximately 1.4. Throughout the entire clinical rehabilitation zone, the manipulability index displays near-flat, highly uniform contours, preventing sharp gradient spikes and ensuring exceptionally smooth velocity transmission during therapeutic exercises.

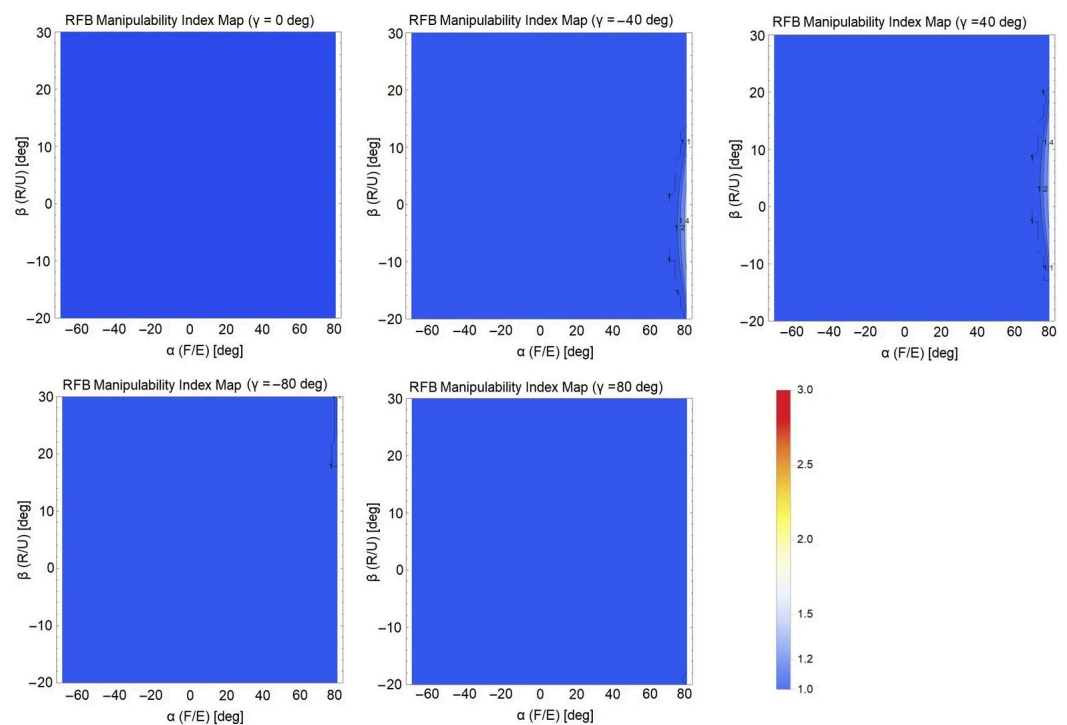


Figure 15. Manipulability index (w_{RFB}) contour maps across the anatomical flexion/extension (α) and radial/ulnar deviation (β) axes evaluated at discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$).

To evaluate workspace reliability against the same established binary safety threshold of $w_{\text{th}} = 2.0$, Figure 16 presents the resulting Binary Safe Workspace Map for the RFB model across all evaluated pronation/supination orientations ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$). The green region represents the Pass Zone ($w \leq 2.0$), indicating safe kinematic states suitable for active/passive therapy. Remarkably, across all five discrete γ angles, 100% of the targeted anatomical ROM falls squarely within the safe zone, with zero occurrence of Fail Zone regions ($w > 2.0$). The RFB model guarantees that the manipulator operates well clear of degradation thresholds throughout the functional workspace.

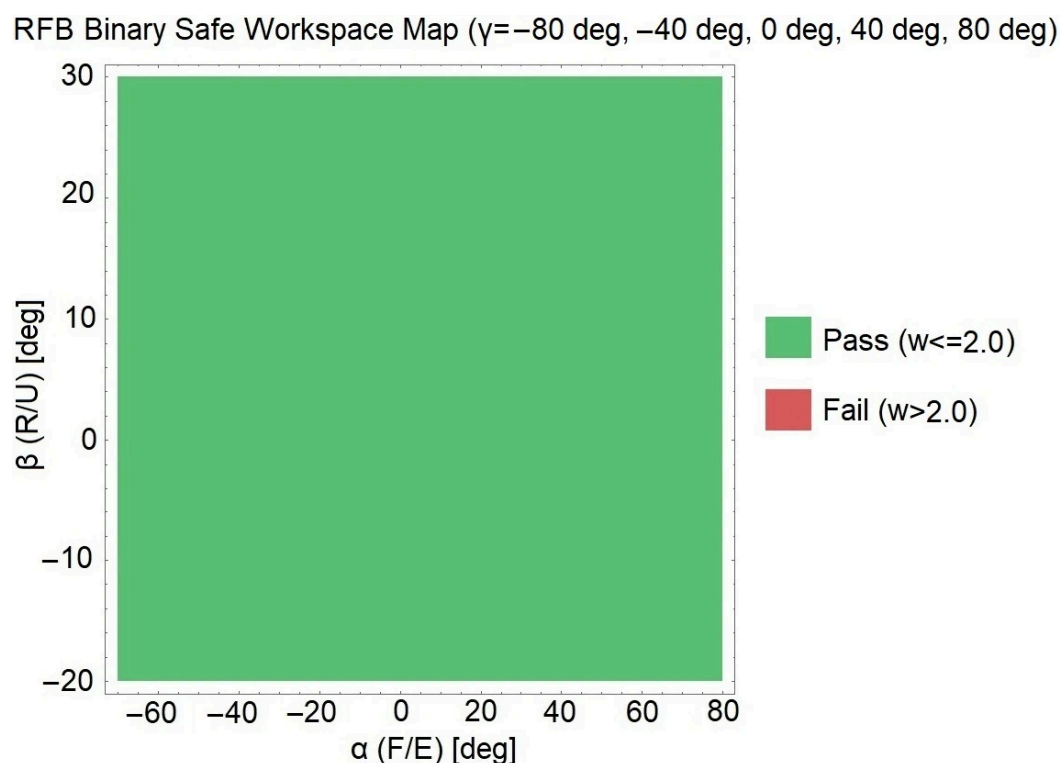


Figure 16. RFB binary safe workspace maps evaluating operational safety against the threshold criterion $w \leq 2.0$ (green = Pass, red = Fail) across discrete pronation/supination orientations.

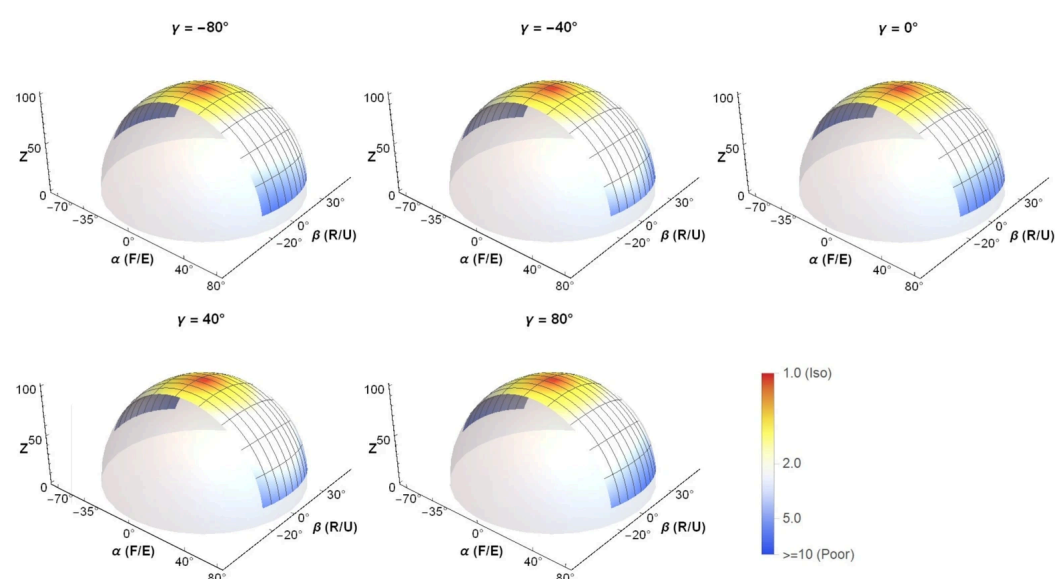


Figure 17. 3D hemispherical workspace heatmaps of the condition number (κ) distribution under RFB formulation evaluated at discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$).

The directional uniformity of motion and force transmission under the RFB formulation was evaluated using the condition number κ . Figure 16 depicts the 3D hemispherical spatial distribution of κ mapped onto the physical spherical workspace shell at five

discrete pronation/supination angles ($\gamma \in \{-80^\circ, -40^\circ, 0^\circ, 40^\circ, 80^\circ\}$). At around the neutral pose for α, β , the condition number approaches unity ($\kappa \approx 1.0$, deep red/orange zone at the top of the hemisphere), indicating the potential to provide isotropic force feedback and haptic transparency during patient interaction. Moving outward toward workspace boundaries, κ increases smoothly through yellow ($\kappa \approx 2.0$) to blue ($\kappa \geq 5.0$). Across all five pronation/supination slices ($\gamma = -80^\circ$ to $+80^\circ$), the 3D isotropy maps exhibit similar spatial topology.

7.3. Quantitative Workspace Coverage and Performance Comparison

To evaluate the overall clinical utility and kinematic efficiency of the parallel exoskeleton across the full wrist ROM, the task-space workspace coverage was quantified using Equation (88) (in Section 5.3). While the parallel mechanism achieves 100% theoretical geometric reachability across the target anatomical ROM, evaluating performance quality requires assessing the condition number (κ) to identify performance degradation toward boundary orientations. To differentiate between high-performance operational zones and acceptable workspace limits, two distinct threshold criteria are defined. High-Dexterity Workspace ($\kappa < 5.0$) indicates better haptic transparency and near-isotropic force/velocity transmission, highly suitable for active rehabilitation exercises. Usable Workspace ($\kappa < 10.0$) defines the boundary beyond which directional distortion becomes significant, serving as the limit for safe passive motion assistance [40,46,47].

The numerical grid evaluations conducted across the 3D wrist ROM ($\alpha \in [-70^\circ, 80^\circ]$, $\beta \in [-20^\circ, 30^\circ]$, $\gamma \in [-80^\circ, 80^\circ]$) yield the comparative results summarized in Table 2. Both formulations achieve a perfect minimum condition number ($\kappa_{\min} = 1.00$) at the neutral postural origin ($\alpha = 0^\circ$, $\beta = 0^\circ$, $\gamma = -68^\circ$), confirming ideal haptic transparency near the home configuration. Under the IFB formulation, extrinsic cross-axis coupling causes κ to spike up to a maximum of 54.39 at extreme combined extension/radial deviation/pronation ($\alpha = 80^\circ$, $\beta = 30^\circ$, $\gamma = 80^\circ$). In contrast, the RFB model suppresses this peak to $\kappa_{\max} = 13.24$, representing a 75.7% reduction in maximum condition number spike. For the High-Dexterity Zone ($\kappa < 5.0$), the RFB model covers 94.13% of the ROM compared to 78.80% for IFB, demonstrating that intrinsic coordinate parameterization expands the effective active therapy zone by over 15%. For the Usable Workspace ($\kappa < 10.0$), the RFB model achieves virtually complete coverage (99.78%), whereas the IFB model drops to 92.18% due to boundary representation singularities. This numerical verification demonstrates that while both models physically

reach 100 % of the anatomical workspace, parameterizing controller kinematics using the Rotated Frame Based (RFB) formulation preserves kinematic isotropy and provides a dramatically wider high-dexterity operating window for clinical rehabilitation.

Table 2. Workspace coverage and condition number metrics for IFB and RFB formulations.

Metric/Parameter	Initial Frame Based (IFB)	Rotated Frame Based (RFB)
Minimum Condition Number ($\kappa_{\min}$)	1.00 ($\alpha=0^\circ$, $\beta=0^\circ$, $\gamma=-68^\circ$)	1.00 ($\alpha=0^\circ$, $\beta=0^\circ$, $\gamma=-68^\circ$)
Maximum Condition Number ($\kappa_{\max}$)	54.39 ($\alpha=80^\circ$, $\beta=30^\circ$, $\gamma=80^\circ$)	13.24 ($\alpha=80^\circ$, $\beta=30^\circ$, $\gamma=60^\circ$)
High-Dexterity Coverage ($\kappa<5.0$)	78.80 %	94.13 %
Usable Workspace Coverage ($\kappa<10.0$)	92.18 %	99.78 %

8. Conclusions

In this study, a novel 3-DOF spherical parallel exoskeleton with base-fixed actuators was designed and its kinematic validation was performed for wrist rehabilitation. The proposed architecture successfully addresses two major challenges in wearable robotics: minimizing moving mass through a low-inertia design and ensuring kinematic compatibility with the human anatomy via the virtual center concept. By mounting all actuators on a fixed base, the system allows the patient to interact with the device without the burden of heavy motor weights. The comparative kinematic and singularity analyses demonstrate the distinct operational characteristics of the wrist rehabilitation manipulator under Rotated Frame Based (RFB) and Initial Frame Based (IFB) formulations. IFB and RFB models differ in the way the wrist anatomical rotations are defined, either with respect to the world frame or with respect to the body-fixed frames in the kinematic chain, respectively. Both formulations confirm ideal kinematic isotropy ($\kappa=1.0$) at around the neutral pose, having the potential to provide stable motion transmission in the core region where primary therapeutic exercises are conducted.

Quantitative performance metrics derived from the condition number (κ) and manipulability indices (w) reveal complementary workspace characteristics. The RFB formulation provides a highly uniform performance profile with a low peak condition number of $\kappa=13.23$ and a 99.78 % usable workspace coverage ($\kappa<10.0$). In contrast, the IFB formulation expands the high-dexterity envelope ($\kappa<5.0$) to 78.80 % of the workspace (compared to 54.13 % in RFB), while maintaining 92.18 % usable operational coverage despite a higher peak condition number ($\kappa=54.39$) at extreme physiological limits. Furthermore, the model-specific analytical manipulability metrics verify that the system remains inherently singularity-free ($w\leq 2.0$ across primary operational zones) throughout the entire physiological range of motion (ROM) of the human wrist.

It has been reported [48] that there was some coupling between radial/ulnar (R/U) deviation and flexion/extension (F/E) leading to an egg-shaped plot rather than a rectan-

gular plot when the R/U ROM was plotted against F/E ROM in healthy young male volunteers. Hence, wrist rotation in one direction affected the ROM in the other direction. Therefore, when the proposed parallel exoskeleton is to be used in clinical trials, this coupling must be taken into account in producing a rehabilitation trajectory.

The proposed parallel exoskeleton is basically a spherical mechanism which has two slotted arcs forming two concentric spheres with different radii, and hence these arcs never collide. Since the exoskeleton is a spherical mechanism, all joint axes intersect at one center which is the virtual center (VC). This is a strong feature of the design because its kinematic characteristics and performance are based on the joint angles, not on link lengths. Thus, the dimensions of the exoskeleton depend only on the size of the hand. As shown in Figure 2, the main dimensions of the exoskeleton are specified to accommodate the hand of a 95th percentile male so that the device can be comfortably used by any patient [49].

In a practical application of this device as shown in Figure 18, the physician will manually adjust the VC by manipulating the patient's hand with the aid of a laser pointer. Large errors in the VC position are strictly avoided; only minor positional errors may occur. Since the skin tends to slide over bone, any minor positional errors that arise will be compensated by the flexibility (i.e., compliance) of the skin and flesh. Moreover, there will be the flexibility of the cuff that rotates the forearm in the proposed device. The forearm is rotated at around its proximal end as there is more soft tissue in this region to provide sufficient flexibility (compliance) in all three directions. As shown in Figures 2 and 18, the handle involves a plate to which the hand is attached. This plate can be wearable and to provide firm grip, bandages can be used to secure the hand to the plate and the handle.

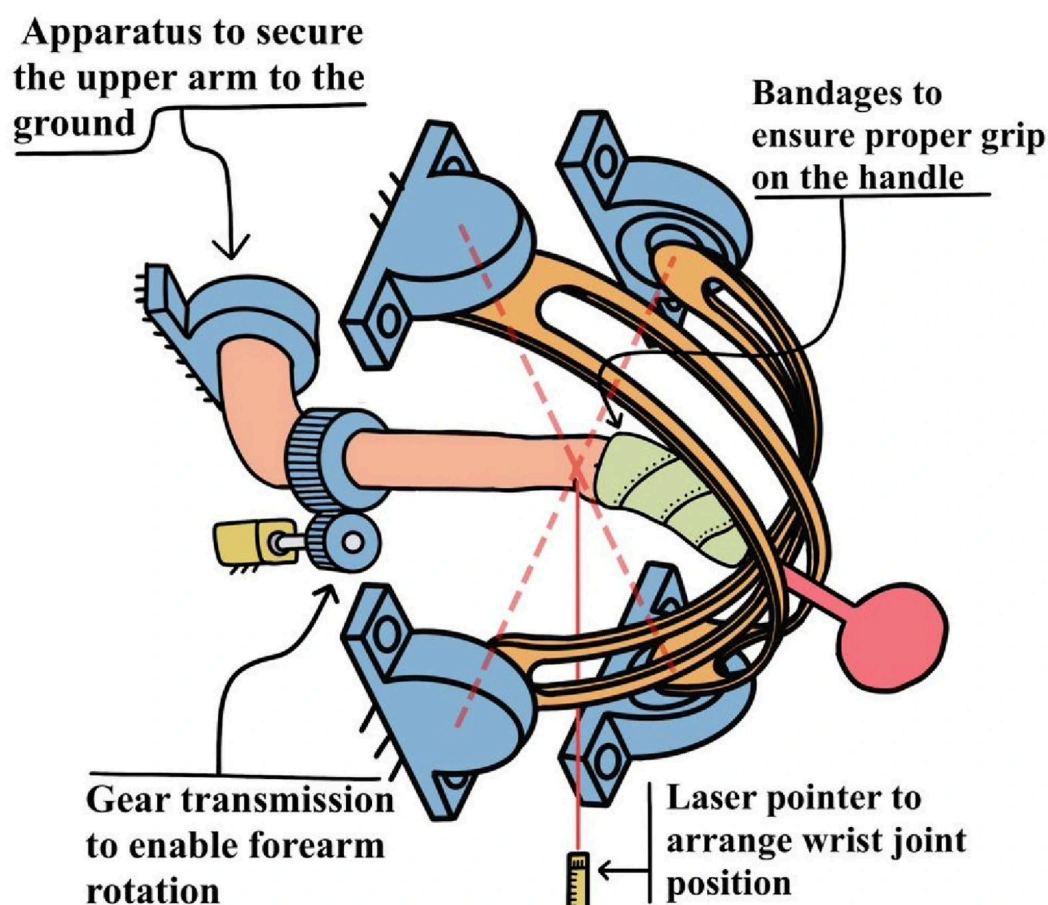


Figure 18. Conceptual prototype design for future work.

Future work will focus on the development of a physical prototype to conduct experimental trials (Figure 18). Integrating force/torque sensors for advanced impedance control and performing clinical pilot studies with post-stroke patients will be the next steps in evaluating the real-world therapeutic efficacy of this virtual center exoskeleton.

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Abbreviations

The following abbreviations are used in this manuscript:

DOF	Degree of freedom
VC	Virtual center
ROM	Range of motion
ADLs	Activities of Daily Living
AHA	American Heart Association
DALYs	Disability-adjusted life years
P/S	Pronation/supination
F/E	Flexion/extension
R/U	Radial/ulnar (deviation)
RFB	Rotated Frame Based
IFB	Initial Frame Based

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