

Article

Sub-Sharvin Conductance and Incoherent Shot-Noise in Graphene Disks at Magnetic Field

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Abstract: Highly doped graphene samples show reduced conductance and enhanced shot-noise power compared with standard ballistic systems in two-dimensional electron gas. These features can be understood within a model that assumes incoherent scattering of Dirac electrons between two interfaces separating the sample and the leads. Here we find, by adopting the above model for the edge-free (Corbino) geometry and by computer simulation of quantum transport, that another graphene-specific feature should be observable when the current flow through a doped disk is blocked by a strong magnetic field. When the conductance drops to zero, the Fano factor approaches the value of $F \approx 0.56$, with a very weak dependence on the ratio of the disk radii. The role of finite source-drain voltages and the system behavior when the electrostatic potential barrier is tuned from a rectangular to a parabolic shape are also discussed.

Keywords: graphene; shot noise; Corbino disk; Landauer–Büttiker formalism



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1. Introduction

Although the electronic properties of matter are governed by the rules of quantum mechanics [1], it is very unlikely to find that any measurable characteristic of a macroscopic system is determined solely by the universal constants of nature, such as the elementary charge (e) or the Planck constant (h). In the last century, two notable exceptions arrived with the phenomena of superconductivity [2], namely, the quantization of the magnetic flux piercing the superconducting circuit, which is the multiplicity of the flux quantum $\Phi_0 = h/(2e)$ [3,4], and the *ac* Josephson effect, with the universal frequency-to-voltage ratio given by $2e/h = 1/\Phi_0$ [5]. Later, with the advent of semiconducting heterostructures [6], came the quantum Hall effect [7–12] and the conductance quantization [13], bringing us the conductance quantum $g_0 = se^2/h$ (with the degeneracy $s = 1, 2$, or 4). The further development of nanosystems led to the observation of the Aharonov–Bohm effect, which is manifested by magnetoconductance oscillations with the period $2\Phi_0 = h/e$ [14], as well as the universal conductance fluctuations [15–18], characterized by a variance $\propto \beta^{-1}(se^2/h)^2$, with an additional symmetry-dependent prefactor ($\beta = 1, 2$, or 4). A related but slightly different issue concerns the Wiedemann–Franz (WF) law, which defines the Lorentz number, $\mathcal{L}_0 = \frac{\pi^2}{3}(k_B/e)^2$ (with the Boltzmann constant k_B) [2], as the proportionality coefficient between the electronic part of the thermal conductivity and the electrical conductivity multiplied by the absolute temperature. Although the WF law is obeyed with a few percent accuracy in various condensed matter systems it has never been shown to have metrological accuracy [19–24].

Some new ‘magic numbers’ similar to those mentioned above have arrived with the discovery of graphene, an atomically thin form of carbon [11,12]. For undoped graphene samples, charge transport is dominated by transport via evanescent modes [25], resulting in the universal *dc* conductivity $4e^2/(\pi h)$ accompanied by the sub-Poissonian shot noise,

with a Fano factor $F = 1/3$ [26–31]. For high frequencies, the ac conductivity is given by $\pi e^2/(2h)$, which leads to the quantized visible-light opacity $\pi\alpha$ (where $\alpha \approx 1/137.036$ is the fine-structure constant) [32–34]. A possible new universal value is predicted for the maximum absolute thermopower, which approaches $\approx k_B/e$ near the charge neutrality point, for both monolayer and gapless bilayer graphene [35–39].

Away from the charge-neutrality point, ballistic graphene samples exhibit sub-Sharvin charge transport [40,41], characterized by a conductance reduced by a factor of $\pi/4$ compared with standard Sharvin contacts in two-dimensional electron gas (2DEG) [42,43]. What is more, the shot noise is enhanced (compared with 2DEG) up to $F \approx 1/8$ far from the charge-neutrality [30,31]. The detailed dependence of the above factors on the sample geometry has recently been discussed in analytical terms [41], using the example of an edge-free (Corbino) setup, characterized by the inner radius, R_i , and the outer radius, R_o (see Figure 1). It is further found in Refs. [40,41] that the ballistic values of the conductance and the Fano factor are gradually restored when the potential barrier, which defines a sample area in the effective Dirac–Weyl Hamiltonian, evolves from a rectangular toward a parabolic shape.

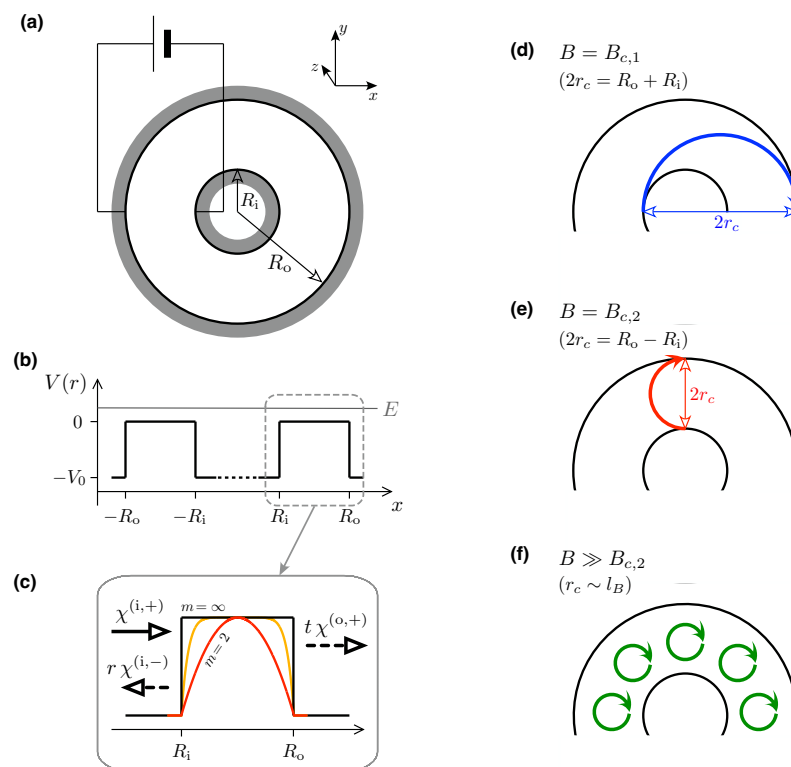


Figure 1. (a) Schematic of Corbino disk in graphene, with the inner radius, R_i , and the outer radius, R_o , contacted by two circular electrodes (dark areas). A voltage source drives a current through the disk. A separate gate electrode (not shown) allows us to tune the carrier concentration around the neutrality point. The coordinate system (x, y, z) is also shown. (b) Cross section of the electrostatic potential profile given by Equation (2) with $m \rightarrow \infty$ (i.e., the rectangular barrier) at $y = z = 0$. (c) Zoom-in of a single barrier, for $x > 0$, showing also the profiles for $m = 2$ and 8 , with symbolic representations of the incident and reflected waves in inner electrode ($x < R_i$) and the transmitted wave in outer electrode ($x > R_o$), with the amplitudes r and t corresponding to the Fermi energy $E > 0$. (d–f) Characteristic values of the magnetic field $\mathbf{B} = (0, 0, B)$ separating different transport regimes. At $B = B_{c,1}$, the cyclotron diameter $2r_c = R_o + R_i$, and the particle leaving the inner lead approaches the outer lead regardless of the initial direction (d). At $B = B_{c,2}$, we have $2r_c = R_o - R_i$, and only the trajectory tangent to the inner lead reaches the outer lead (e). For higher fields, classical trajectories do not contribute to the charge transport, which is possibly only if the resonance with the Landau level occurs for $E \approx E_{nLL}$, with $n = 0, \pm 1, \pm 2, \dots$ (f).

Here we focus on the Corbino geometry, which is often considered when discussing fundamental aspects of graphene [25,44–55]. In this geometry, charge transport at high magnetic fields is unaffected by edge states, allowing bulk transport properties to be studied [48–54]. Recently, we have shown numerically that the thermoelectric properties in such a situation are determined by the energy interval separating consecutive Landau levels rather than by the transport gap (being the energy interval, for which the cyclotron diameter $2r_c < R_o - R_i$) [55]. In this paper, we address the question of how the shot noise behaves when the tunneling conductance regime is entered by increasing the magnetic field at a fixed doping (or decreasing the doping at a fixed field) such that $2r_c \lesssim R_o - R_i$. Going beyond the linear-response regime, we find that the threshold voltage, U_{on} , defined as the source-drain voltage difference that activates the current at minimum doping, is accompanied by the quasi-universal (i.e., weakly dependent on the radii ratio R_o/R_i) value of $F \approx 0.56$. The robustness of the effect is also analyzed when smoothing the electrostatic potential barrier.

The paper is organized as follows. In Section 2 we briefly present the effective Dirac Hamiltonian and the numerical approach used in the remaining parts of the paper. In Section 3, we derive an approximation for the transmission through a doped Corbino disk at non-zero magnetic field and subsequent formulas for the charge-transfer characteristics: the conductance and the Fano factor. Our numerical results, for both the rectangular and the smooth potential barriers, are presented in Section 4. The conclusions are given in Section 5.

2. Model and Methods

2.1. Dirac Equation for the Disk Geometry

Our analysis of the device schematically shown in Figure 1 starts from the effective wave equation for Dirac fermions in graphene, near the K valley,

$$[v_F (\mathbf{p} + e\mathbf{A}) \cdot \boldsymbol{\sigma} + V(r)]\Psi = E\Psi, \quad (1)$$

where the Fermi velocity is given by $v_F = \sqrt{3} t_0 a / (2\hbar)$, with $t_0 = 2.7$ eV the nearest-neighbor hopping integral and $a = 0.246$ nm the lattice parameter, $\mathbf{p} = -i\hbar (\partial_x, \partial_y)$ is the in-plane momentum operator, and we choose the symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x)$ corresponding to the perpendicular, uniform magnetic field $\mathbf{B} = (0, 0, B)$, and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$, where σ_j are the Pauli matrices. For the forthcoming numerical calculations, we set (in the physical units) $\hbar v_F = 0.575214$ eV·nm, and $e/(\pi\hbar) = 2067.83$ T^{−1}·nm^{−2}. The electrostatic potential energy in Equation (1), $V(r)$, is given by

$$V(r) = -V_0 \times \begin{cases} \frac{2^m |r - R_{av}|^m}{|R_o - R_i|^m} & \text{if } |r - R_{av}| \leq \frac{R_o - R_i}{2}, \\ 1 & \text{if } |r - R_{av}| > \frac{R_o - R_i}{2}, \end{cases} \quad (2)$$

where we have defined $R_{av} = (R_i + R_o)/2$. In particular, the limit of $m \rightarrow \infty$ corresponds to the rectangular barrier (with a cylindrical symmetry); any finite $m \geq 2$ defines a smooth potential barrier, interpolating between the parabolic ($m = 2$) and rectangular shapes. In principle, barrier smoothing can be regarded as a feature of a self-consistent solution resulting from carrier diffusion; we expect this feature to be strongly dependent on the experimental details, with graphene-on-hBN devices [49] showing rectangular rather than smooth profiles.

The symmetry of the problem allows one to search for the wave function in the form

$$\Psi_j(r, \varphi) = e^{i(j-1/2)\varphi} \begin{pmatrix} \chi_a \\ \chi_b e^{i\varphi} \end{pmatrix}, \quad (3)$$

where $j = \pm 1/2, \pm 3/2, \dots$ is the total angular-momentum quantum number, the components $\chi_a = \chi_a(r)$, $\chi_b = \chi_b(r)$, and we have introduced the polar coordinates (r, φ) .

Substituting the above into Equation (1) brings us to the system of ordinary differential equations

$$\chi'_a = \left(\frac{j-1/2}{r} + \frac{eBr}{2\hbar} \right) \chi_a + i \frac{E-V(r)}{\hbar v_F} \chi_b, \quad (4)$$

$$\chi'_b = i \frac{E-V(r)}{\hbar v_F} \chi_a - \left(\frac{j+1/2}{r} + \frac{eBr}{2\hbar} \right) \chi_b, \quad (5)$$

where primes denote derivatives with respect to r .

2.2. Analytic Solutions

For the disk area, $R_i < r < R_o$, Equations (4) and (5) must typically be integrated numerically; key details of the procedure are presented in Appendix A. Here we focus on the special case of the rectangular barrier ($m = \infty$), for which some analytical solutions have been reported [44,45,56].

In particular, in the absence of a magnetic field ($B = 0$), the spinors $\chi_j = (\chi_a, \chi_b)^T$ corresponding to different j -s can be written as linear combinations [44]

$$\chi_j^{(\text{disk})} = A_j \begin{pmatrix} H_{j-1/2}^{(2)}(kr) \\ i\eta H_{j+1/2}^{(2)}(kr) \end{pmatrix} + B_j \begin{pmatrix} H_{j-1/2}^{(1)}(kr) \\ i\eta H_{j+1/2}^{(1)}(kr) \end{pmatrix}, \quad (6)$$

where $H_v^{(1)}(\rho)$ [$H_v^{(2)}(\rho)$] is the Hankel function of the first [second] kind, $k = |E|/(\hbar v_F)$, the doping sign $\eta = \text{sgn } E = \pm 1$ (with $\eta = +1$ indicating electron doping and $\eta = -1$ indicating hole doping), and A_j, B_j are arbitrary complex coefficients. For $B > 0$, Equation (6) is replaced by [45,56]

$$\chi_j^{(\text{disk})} = A_j \begin{pmatrix} \zeta_{j\uparrow}^{(1)} \\ i\eta z_{j,1} \zeta_{j\downarrow}^{(1)} \end{pmatrix} + B_j \begin{pmatrix} \zeta_{j\uparrow}^{(2)} \\ i\eta z_{j,2} \zeta_{j\downarrow}^{(2)} \end{pmatrix}, \quad (7)$$

where $z_{j,1} = [2(j + s_j)]^{-2s_j}$, $z_{j,2} = 2(\beta/k^2)^{s_j+1/2}$ (with $s_j \equiv \frac{1}{2}\text{sgn } j$, $\beta = eB/(2\hbar)$), and

$$\zeta_{js}^{(\nu)} = e^{-\beta r^2/2} (kr)^{|l_s|} \begin{cases} M(\alpha_{js}, \gamma_{js}, \beta r^2), & \nu = 1, \\ U(\alpha_{js}, \gamma_{js}, \beta r^2), & \nu = 2, \end{cases} \quad (8)$$

with $l_s = j \mp \frac{1}{2}$ for $s = \uparrow, \downarrow$, $\alpha_{js} = \frac{1}{4}[2(l_s + |l_s| + 1) - k^2/\beta]$, and $\gamma_{js} = |l_s| + 1$. $M(a, b, z)$ and $U(a, b, z)$ are the confluent hypergeometric functions [57].

For the leads, $r < R_i$ or $r > R_o$, the electrostatic potential energy is constant, $V(r) = -V_0$. We further assume $B = 0$ and $E > -V_0$ (electron doping) in the leads, which allows us to adapt the wave function given by Equation (6); i.e., for the inner lead, $r < R_i$,

$$\chi_j^{(\text{inner})} = \begin{pmatrix} H_{j-1/2}^{(2)}(Kr) \\ iH_{j+1/2}^{(2)}(Kr) \end{pmatrix} + r_j \begin{pmatrix} H_{j-1/2}^{(1)}(Kr) \\ iH_{j+1/2}^{(1)}(Kr) \end{pmatrix}, \quad (9)$$

and for the outer lead $r > R_o$,

$$\chi_j^{(\text{outer})} = t_j \begin{pmatrix} H_{j-1/2}^{(2)}(Kr) \\ iH_{j+1/2}^{(2)}(Kr) \end{pmatrix}, \quad (10)$$

where $K = |E + V_0|/(\hbar v_F)$, and we have introduced the reflection and transmission coefficient. The first spinor in each of Equations (9) and (10) represents the incoming (i.e., propagating from $r = 0$) wave, and the second spinor in Equation (9) represents the outgoing (propagating from $r = \infty$) wave.

2.3. Mode-Matching Method

Since the current-density operator following from Equation (1), $\mathbf{j} = ev_F \sigma$, does not involve differentiation, the mode-matching conditions for $r = R_i$ and $r = R_o$ reduce to the equalities for spinor components, namely

$$\chi_j^{(\text{inner})}(R_o) = \chi_j^{(\text{disk})}(R_o) \text{ and } \chi_j^{(\text{disk})}(R_i) = \chi_j^{(\text{outer})}(R_i). \quad (11)$$

The resulting formula for the transmission probability for the j -th mode becomes particularly simple if we consider the limit of heavily doped leads, $U_0 \rightarrow \infty$. In particular, for $B = 0$, substituting Equation (6) into the above gives [58]

$$T_j = |t_j|^2 = \frac{16}{\pi^2 k^2 R_i R_o} \frac{1}{\left[\mathfrak{D}_j^{(+)}\right]^2 + \left[\mathfrak{D}_j^{(-)}\right]^2}, \quad (12)$$

where

$$\begin{aligned} \mathfrak{D}_j^{(\pm)} = \text{Im} \Big[& H_{j-1/2}^{(1)}(kR_i) H_{j\mp 1/2}^{(2)}(kR_o) \\ & \pm H_{j+1/2}^{(1)}(kR_i) H_{j\pm 1/2}^{(2)}(kR_o) \Big]. \end{aligned} \quad (13)$$

Probably, the most surprising feature of the above result is that taking the limit of $U_0 \rightarrow \infty$ does not give $T_j \rightarrow 0$ for all j -s; instead, there is a set of $T_j \sim 1$ for $|j| \lesssim kR_i$. (The corresponding discussion for the Corbino disk in 2DEG can be found in Ref. [44].)

Analogously, for $B > 0$ one finds, using Equations (7) and (8),

$$T_j = |t_j|^2 = \frac{16 (k^2/\beta)^{|2j-1|}}{k^2 R_i R_o (X_j^2 + Y_j^2)} \left[\frac{\Gamma(\gamma_{j\uparrow})}{\Gamma(\alpha_{j\uparrow})} \right]^2, \quad (14)$$

where $\Gamma(z)$ is the Euler Gamma function, and

$$\begin{aligned} X_j &= w_{j\uparrow\uparrow}^- + z_{j,1} z_{j,2} w_{j\downarrow\downarrow}^-, \quad Y_j = z_{j,2} w_{j\uparrow\downarrow}^+ - z_{j,1} w_{j\downarrow\uparrow}^+, \\ w_{jss'}^\pm &= \zeta_{js}^{(1)}(R_i) \zeta_{js'}^{(2)}(R_o) \pm \zeta_{js}^{(2)}(R_o) \zeta_{js'}^{(1)}(R_i). \end{aligned} \quad (15)$$

For $B < 0$, one obtains $T_j(B) = T_{-j}(-B)$.

Details of numerical mode-matching, applicable to smooth potentials, are given in Appendix A.

2.4. Landauer–Büttiker Formalism

If the nanoscopic system is connected to external reservoirs, characterized by the electrochemical potentials μ and $\mu + eU_{\text{eff}}$ (for simplicity, the two reservoirs are considered; for a more general discussion, see Ref. [59]), the conductance of the system is related to the transmission probabilities for normal modes (T_j -s) by

$$G(U_{\text{eff}}) = \frac{\langle I \rangle}{U_{\text{eff}}} = \frac{g_0}{U_{\text{eff}}} \int_{\mu}^{\mu + eU_{\text{eff}}} d\epsilon \sum_j T_j(\epsilon), \quad (16)$$

where $\langle I \rangle$ denotes the average electric current and the zero-temperature limit is taken. The conductance quantum is $g_0 = 4e^2/h$, taking into account spin and valley degeneracies. U_{eff} is the effective voltage difference between the reservoirs (note that the actual voltage applied may differ from U_{eff} due to charge-screening effects). Similarly, the Fano factor, which relates the current variance, $\langle (I - \langle I \rangle)^2 \rangle$, to the value $\langle (I - \langle I \rangle)^2 \rangle_{\text{Poisson}}$, which

one would measure in the absence of correlations between scattering events (occurring, e.g., in the tunneling limit of $T_j \ll 1$ for all j -s), is given by

$$F(U_{\text{eff}}) = \frac{\langle (I - \langle I \rangle)^2 \rangle}{\langle (I - \langle I \rangle)^2 \rangle_{\text{Poisson}}} = \frac{g_0}{GU_{\text{eff}}} \int_{\mu}^{\mu + eU_{\text{eff}}} d\epsilon \sum_j T_j(\epsilon) [1 - T_j(\epsilon)], \quad (17)$$

where $\langle (I - \langle I \rangle)^2 \rangle_{\text{Poisson}} = e\langle I \rangle / \Delta t = eGU_{\text{eff}} / \Delta t$, with Δt denoting the time of the measurement.

For the sake of completeness, we emphasize here that the non-interacting Landauer–Büttiker formalism, as presented above, is valid for weak currents, when the system–reservoir distinction is possible. Generalizations for strong coupling [60,61], as well as for many-body correlated nanosystems [62], have recently been proposed. However, such situations are beyond the scope of the present work.

In the linear-response regime ($U_{\text{eff}} \rightarrow 0$), Equations (16) and (17) reduce to

$$G(U_{\text{eff}} \rightarrow 0) = g_0 \sum_j T_j, \quad (18)$$

and

$$F(U_{\text{eff}} \rightarrow 0) = \frac{\sum_j T_j(1 - T_j)}{\sum_j T_j}, \quad (19)$$

where $T_j = T_j(\mu)$. For the disk geometry, the summation range is limited by the number of propagating modes in the inner lead, $|j| \leq j_{\text{max}} = \lfloor KR_i \rfloor - \frac{1}{2}$ where $\lfloor x \rfloor$ is the floor function of x . (For heavily doped leads, $j_{\text{max}} \rightarrow \infty$.) Experimentally obtained conductance spectra for suspended graphene disks [48] show relatively good agreement with the formula following from Equations (12), (13), and (18), provided that an adjustable parameter, quantifying the contact resistance between the electrodes and the sample, is taken into account. To the best of our knowledge, noise measurements for the disk geometry are still missing.

As a notable example, let us consider the zero-doping limit ($\mu \rightarrow 0$). In such a case, Equation (14) can be simplified as follows [45,63]

$$T_j(\mu \rightarrow 0) = \frac{1}{\cosh^2[(j + \Phi/\Phi_0) \ln(R_o/R_i)]}, \quad (20)$$

where $\Phi = \pi(R_o^2 - R_i^2)B$ is the flux piercing the disk area, and we have defined $\Phi_0 = 2(h/e) \ln(R_o/R_i)$. Assuming the narrow-disk range, $R_o \approx R_i$, we can approximate the sums occurring in Equations (18) and (19) by integrals, obtaining

$$G \approx G_{\text{diff}} = \frac{2\pi\sigma_0}{\ln(R_o/R_i)} \quad \text{and} \quad F \approx F_{\text{diff}} = \frac{1}{3}. \quad (21)$$

The above reproduces the pseudodiffusive conductance and the shot-noise power for a disk geometry [44]. For larger R_o/R_i , both characteristics are predicted to show approximately sinusoidal conductance oscillations with the field B [45,63,64].

The case of the doped disk, for which one may expect to observe some features of the sub-Sharvin charge transport [40,41], is discussed next.

3. Approximate Conductance and Fano Factor at Magnetic Field

Before calculating the conductance, G , and the Fano factor, F , within the mode-matching method described in Section 2, we first present the approximation formulas for the incoherent transport, obtained by adapting the derivation of Ref. [41] for the $B > 0$ case.

In short, the approximation is based on two key assumptions. (i) We neglect the evanescent waves, which are present in the exact solutions from which one obtains $T_j > 0$ for arbitrarily large $|j|$, see Equations (12)–(15). In turn, zero linear-response conductance,

see Equation (18), is expected for $B > B_{c,2}$. In such a range, the approximate Fano factor given by Equation (19) will be undefined (since the actual value of F is governed by evanescent waves); however, the limit of $B \rightarrow B_{c,2}-$ could still be taken (and is not necessarily determined by the evanescent contribution). In addition, (ii) the incoherent scattering is assumed, i.e., the phase gained by the wave propagating in the sample area is assumed to be random, eliminating the resonances with Landau levels present in the exact solution [55].

3.1. Corbino Disk in Graphene as a Double Barrier

A key step in the derivation is the observation that in the multimode regime ($kR_i \gg 1$), for which one can consider well-defined trajectories, the disk symmetry causes the incident angles θ_1 and θ_2 , corresponding to the interfaces at $r = R_i$ and $r = R_o$ (see Figure 2), to remain constant (up to the sign) after multiple scattering. Therefore, the double-contact formula for incoherent transmission [65,66] can be applied, namely

$$\begin{aligned} \{T\}_{\text{incoh}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \frac{T_1 T_2}{2 - T_1 - T_2 + T_1 T_2 - 2\sqrt{(1 - T_1)(1 - T_2)} \cos \phi} \\ &= \frac{T_1 T_2}{T_1 + T_2 - T_1 T_2}, \end{aligned} \quad (22)$$

where the transmission probabilities T_1, T_2 , corresponding to a potential step of infinite height, are given by

$$T_l = \frac{2 \cos \theta_l}{1 + \cos \theta_l}, \quad l = 1, 2, \quad (23)$$

and ϕ is assumed to be a random phase acquired during the propagation between $r = R_i$ and $r = R_o$ (or vice versa). Similarly, we calculate the incoherent squared transmission, which is useful when evaluating the Fano factor,

$$\begin{aligned} \{T^2\}_{\text{incoh}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \left(\frac{T_1 T_2}{2 - T_1 - T_2 + T_1 T_2 - 2\sqrt{(1 - T_1)(1 - T_2)} \cos \phi} \right)^2 \\ &= \frac{(T_1 T_2)^2 (2 - T_1 - T_2 + T_1 T_2)}{(1 + T_1 T_2 - T_1 T_2)^3}. \end{aligned} \quad (24)$$

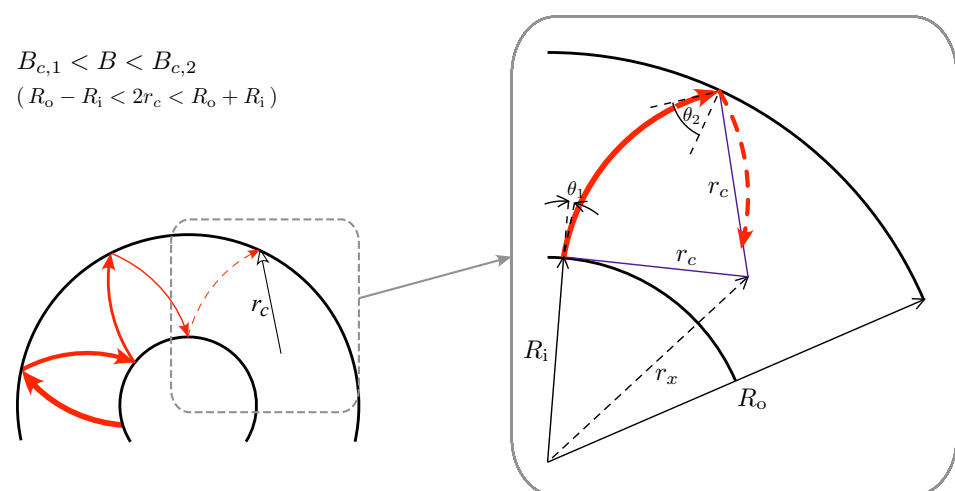


Figure 2. Propagation between consecutive scatterings (red lines with arrows) on interfaces at $r = R_i$ and $r = R_o$ in a uniform magnetic field $B_{c,1} < B < B_{c,2}$. A zoom-in showing an arc of single cyclotron orbit centered at $r = r_x$, with radius r_c , and incident angles θ_1 (for $r = R_i$) and θ_2 (for $r = R_o$).

Next, the incoherent conductance in the linear-response regime is evaluated by inserting $\{T\}_{\text{incoh}}$ (22) into Equation (18),

$$G_{\text{incoh}} = G_{\text{Sharvin}} \langle \{T\}_{\text{incoh}} \rangle_{u=\sin \theta_1}, \quad (25)$$

with

$$G_{\text{Sharvin}} = 2g_0 k R_i. \quad (26)$$

By analogy, for the Fano factor we can derive the following from Equation (19)

$$F_{\text{incoh}} = 1 - \frac{\langle \{T^2\}_{\text{incoh}} \rangle_{u=\sin \theta_1}}{\langle \{T\}_{\text{incoh}} \rangle_{u=\sin \theta_1}}. \quad (27)$$

The summation over $2kR_i$ modes is approximated in Equations (25) and (27) by averaging over the variable $u = \sin \theta_1$, in the range of $-1 \leq u \leq 1$. Explicitly,

$$\langle \{T^n\}_{\text{incoh}} \rangle_{u=\sin \theta_1} = \frac{1}{2} \int_{u_c}^1 du \{T^n\}_{\text{incoh}}, \quad n = 1, 2, \quad (28)$$

where the lower integration limit (u_c) is defined by the value of $\sin \theta_1$, below which the trajectory cannot reach the outer interface ($r = R_o$). (In other words, for $u = \sin \theta_1 < u_c$, the geometric derivation below yields $|\sin \theta_2| > 1$.)

The missing elements, necessary to calculate $\langle \{T^n\}_{\text{incoh}} \rangle_{u=\sin \theta_1}$ in Equation (28), are the dependence of θ_2 on θ_1 and B (see Equations (22)–(24)), as well as the dependence of u_c on B .

Since we have assumed constant electrostatic potential energy in the disk area, the trajectory between successive scatterings (see Figure 2) forms an arc, with the constant radius

$$r_c = \hbar k / (eB) = |E| / (v_F e B), \quad (29)$$

(i.e., the cyclotron radius for a massless Dirac particle at $B > 0$), centered at the distance r_x from the origin. Now, solving the two triangles with a common edge, r_x (dashed line) and the opposite vertices in two scattering points, we find

$$r_x^2 = R_i^2 + r_c^2 + 2R_i r_c \sin \theta_1 \quad (30)$$

(for the triangle containing a scattering point at $r = R_i$), and

$$r_x^2 = R_o^2 + r_c^2 - 2R_o r_c \sin \theta_2 \quad (31)$$

(for the triangle containing the scattering point at $r = R_o$). Together, Equations (30) and (31) lead to

$$\sin \theta_2 = \frac{R_o^2 - R_i^2 - 2R_i r_c u}{2R_o r_c}. \quad (32)$$

Subsequently, the value of u_c in Equation (28) is given by

$$u_c = \begin{cases} -1, & \text{if } B \leq B_{c,1} \\ \frac{R_o^2 - R_i^2}{2R_i r_c} - \frac{R_o}{R_i}, & \text{if } B_{c,1} < B \leq B_{c,2}, \\ 1, & \text{if } B > B_{c,2} \end{cases} \quad (33)$$

where we have additionally defined

$$B_{c,m} = \frac{2\hbar k}{e[R_o - (-1)^m R_i]}, \quad m = 1, 2. \quad (34)$$

3.2. The Zero-Field Limit

Typically, averages occurring in Equations (25) and (27) must be evaluated numerically. Analytical expressions are available, e.g., for zero magnetic field [41]

$$G_{\text{incoh}}(B \rightarrow 0) = G_{\text{Sharvin}} \frac{(2a + \frac{1}{a}) \arcsin a + 3\sqrt{1-a^2} - \frac{\pi}{2}(a^2 + 2)}{1 - a^2}, \quad (35)$$

$$F_{\text{incoh}}(B \rightarrow 0) = \left\{ 2a\sqrt{1-a^2}(53 + 279a^2 + 88a^4) - 3\pi a(12 + 82a^2 + 45a^4 + a^6) \right. \\ \left. + 6(1 + 45a^2 + 82a^4 + 12a^6) \arcsin a \right\} \\ / \left\{ 6(1-a^2)^2 \left[\pi a(a^2 + 2) - 6a\sqrt{1-a^2} - 2(2a^2 + 1) \arcsin a \right] \right\}, \quad (36)$$

where we have defined the inverse radii ratio $a = R_i/R_o$.

3.3. The Zero-Conductance Limit

In this paper, we focus on the limit of $B \rightarrow B_{c,2}-$ (i.e., B approaching $B_{c,2}$ from below), for which $G_{\text{incoh}} \rightarrow 0$. By introducing the dimensionless $0 < \varepsilon \ll 1$, one can express the cyclotron diameter, see Equation (29), as

$$2r_c = R_o - R_i + \varepsilon(R_o - R_i). \quad (37)$$

In turn, the value u_c , see Equation (33), can be approximated (up to the leading order in ε) as follows

$$u_c \approx 1 - \varepsilon \left(1 + \frac{R_o}{R_i} \right). \quad (38)$$

It is now convenient to define the variable

$$\alpha = \frac{1 - u}{\varepsilon(1 + R_o/R_i)}, \quad (39)$$

so that the integration over $u_c \leq u \leq 1$, which occurs when evaluating $\langle \{T^n\}_{\text{incoh}} \rangle_{u=\sin \theta_1}$ from Equation (28), can be replaced by an integration over $1 \geq \alpha \geq 0$. The transmission probabilities (T_1 , T_2) for the interfaces at $r = R_i$ and $r = R_o$, see Equations (22)–(24) and (32), can now be approximated as

$$T_1 \approx 2\sqrt{2\alpha(1 + \frac{1}{a})} \varepsilon^{1/2}, \quad (40)$$

$$T_2 \approx 2\sqrt{2(1 - \alpha)(1 + a)} \varepsilon^{1/2}, \quad (41)$$

where we have used $a = R_i/R_o$ again.

Using the above expressions, we can now rewrite the averages in Equation (28), up to the leading order in ε , as follows

$$\langle \{T\}_{\text{incoh}} \rangle_u = \frac{G_{\text{incoh}}}{G_{\text{Sharvin}}} \approx \frac{2(1 + \frac{1}{a})}{\sqrt{2}} \varepsilon^{3/2} \int_0^1 d\alpha \frac{\sqrt{\alpha(1 + \frac{1}{a})} \sqrt{(1 - \alpha)(1 + a)}}{\sqrt{\alpha(1 + \frac{1}{a})} + \sqrt{(1 - \alpha)(1 + a)}}, \quad (42)$$

$$\langle \{T^2\}_{\text{incoh}} \rangle_u \approx \frac{2(1 + \frac{1}{a})}{\sqrt{2}} \varepsilon^{3/2} \int_0^1 d\alpha \frac{\alpha(1 - \alpha)(2 + \frac{1}{a} + a)}{\left[\sqrt{\alpha(1 + \frac{1}{a})} + \sqrt{(1 - \alpha)(1 + a)} \right]^3}. \quad (43)$$

Remarkably, both quantities decay as $\propto \varepsilon^{3/2}$, but their ratio, which appears in Equation (27) for the Fano factor, remains constant (for a given a). The integrals in Equations (42) and (43) can be calculated analytically, resulting in

$$F_{\text{incoh}}(B \rightarrow B_{c,2-}) = 1 - \left\{ 4\sqrt{a} + 39a - 58a^{3/2} - 23a^2 - 23a^{5/2} - 58a^3 + 39a^{7/2} + 4a^4 \right. \\ \left. + (69a^2 - 18a^3 - 18a)\sqrt{1+a} \operatorname{artanh}\left(\frac{\sqrt{a}}{\sqrt{1+a}}\right) \right. \\ \left. + (69a^2 - 18a - 18a^3)\sqrt{1+a} \operatorname{artanh}\left(\frac{1}{\sqrt{1+a}}\right) \right\} \\ \left/ \left\{ (1+a)^3(1+\sqrt{a})(1-3\sqrt{a}+a) + 3a(1+a)^{5/2} \operatorname{artanh}\left[\frac{(1+\sqrt{a})\sqrt{1+a}}{1+\sqrt{a}+a}\right] \right\} \right\}. \quad (44)$$

In particular, for $a \rightarrow 1$, which represents the narrow-disk limit of $R_o \approx R_i$, the above reduces to

$$F_{\text{incoh}}(B \rightarrow B_{c,2-})_{a \rightarrow 1} = 1 + \frac{38 - 33\sqrt{2} \operatorname{artanh}\left(\frac{1}{2}\sqrt{2}\right)}{6\sqrt{2} \operatorname{artanh}\left(\frac{2}{3}\sqrt{2}\right) - 8} \approx 0.549708. \quad (45)$$

Numerical values of $F_{\text{incoh}}(B \rightarrow 0)$ and $F_{\text{incoh}}(B \rightarrow B_{c,2-})$ for selected $a = R_i/R_o$ are given in Table 1. For $F_{\text{incoh}}(B \rightarrow B_{c,2-})$, we see that the Poissonian value of $F_{\text{incoh}} = 1$, which could be expected due to the vanishing conductance, is reconstructed only for $a \rightarrow 0$ (i.e., for $R_o \gg R_i$). For finite radii ratios, non-trivial values of $0 < F_{\text{incoh}} < 1$ occur. Remarkably, for moderate disk proportions ($a \geq 0.5$), $F_{\text{incoh}}(B \rightarrow B_{c,2-})$ shows very weak dependence on a , decaying by less than 2% (from $F_{\text{incoh}} \approx 0.56$ at $a = 0.5$ to $F_{\text{incoh}} \approx 0.55$ for $a \rightarrow 1$).

For this reason, in the following numerical analysis, we fixed the disk radii ratio at $a = 0.5$ (i.e., $R_o = 2R_i$). We also emphasize that the above derivation holds true for the parameter $\varepsilon \rightarrow 0+$, which quantifies the ratio of the cyclotron diameter, $2r_c$, to the radius difference, $R_o - R_i$,

see Equation (37). Therefore, it is irrelevant whether one increases the magnetic field at a fixed chemical potential, or decreases the chemical potential at a fixed $B > 0$ (as long as the system stays in a multimode range, $kR_i \gg 1$).

Table 1. Selected numerical values of $F_{\text{incoh}}(B \rightarrow 0)$, see Equation (36), and $F_{\text{incoh}}(B \rightarrow B_{c,2-})$, see Equation (44). Box marks the values for $a = 0.5$ (i.e., $R_o = 2R_i$) to be compared with the results following from numerical simulations of quantum transport presented Section 4.

$a = R_i/R_o$	$F_{\text{incoh}}(B \rightarrow 0)$	$F_{\text{incoh}}(B \rightarrow B_{c,2-})$
0	0.106528	1
0.1	0.106705	0.630994
0.2	0.107239	0.591829
0.3	0.108136	0.573885
0.4	0.109409	0.563905
0.5	0.111074	0.557898
0.6	0.113151	0.554178
0.7	0.115663	0.551894
0.8	0.118619	0.550565
0.9	0.121963	0.549899
1.0	0.125000	0.549708

4. Results and Discussion

The main doubt that arises when we consider the applicability of Equation (44) to real quantum systems concerns the possible role of evanescent waves, which are completely neglected in our derivation. Obviously, they should not play an important role when the

system is highly conducting (such as in the zero-field case [41]); however, since the Fano factor is determined by the ratio of two cumulants, both of which vanish for sufficiently high fields, it is not entirely clear which contribution (from propagating or from evanescent modes) would determine the value of F for $B \rightarrow B_{c,2}$. On the other hand, resonances with Landau levels are not expected to play a significant role, as they form very narrow transmission peaks whose contributions are immediately smeared out beyond the linear-response regime.

In the remaining parts of the paper, we compare the results of the computer simulation of quantum transport through the disk in graphene with the predictions for incoherent scattering presented in Section 3, in an attempt to propose an experimental procedure that allows one to extract the nontrivial value of $F \approx 0.56$ from the data plagued by other contributions.

4.1. The Rectangular Barrier of an Infinite Height

As a first numerical example, we took the limit of $V_0 \rightarrow \infty$ and $m \rightarrow \infty$ in Equation (2), for which close-form expressions for the transmission probabilities were presented in Section 2.

In Figure 3, we compare the linear-response conductance, $G(U_{\text{eff}} \rightarrow 0)$, see Equation (18), with $G(U_{\text{eff}})$ calculated from Equation (16) for a small but non-zero value of $U_{\text{eff}} = 0.01$ V, both shown as functions of the chemical potential. Also in Figure 3 the same comparison is presented for the Fano factor, $F(U_{\text{eff}})$ (see Equations (19) and (17)). It is easy to see that the prominent aperiodic oscillations visible for both charge-transfer cumulants in the $U_{\text{eff}} \rightarrow 0$ limit are significantly reduced, even for small $U_{\text{eff}} > 0$. In fact, for $U_{\text{eff}} = 0.01$ V and $B > 0$, the values of F_{incoh} calculated from Equation (27) (black lines) are closely followed by $F(U_{\text{eff}})$ obtained from the numerical mode matching, as long as the former can be defined, i.e., for $B < B_{c,2}$ at a given μ . We also note that the value of μ for which $B \approx B_{c,2}$ and $F(U_{\text{eff}}) \approx 0.56$ is accompanied by $G(U_{\text{eff}}) \sim g_0$ (up to the order of magnitude). For smaller μ , such that $B > B_{c,2}$ and F_{incoh} is undefined, $F(U_{\text{eff}})$ saturates near the value ≈ 0.75 , apparently below the Poissonian limit of $F = 1$.

To better understand the nature of the results, we now (see Figure 4) go further beyond the linear-response regime, and calculate $G(U_{\text{eff}})$ and $F(U_{\text{eff}})$ for $\mu = -eU_{\text{eff}}/2$ (note that for an infinite rectangular barrier we have the particle-hole symmetry, and both cumulants are even upon $\mu \leftrightarrow -\mu + eU_{\text{eff}}$), and display them as functions of U_{eff} .

Next, we introduce the activation voltage $U_{\text{on}} = U_{\text{on}}(B)$, the meaning of which can be understood as follows. The cyclotron diameter, see Equation (29), naturally defines the range of energies for which $2r_c(E) < R_o - R_i$ and the system shows $G \approx 0$ (up to the evanescent modes). On the other hand, since we have set $\mu = -eU_{\text{eff}}/2$, the effective voltage defines the energy range of $|E| \leq eU_{\text{eff}}/2$, which is the integration interval in Equations (16) and (17). As a consequence, $G(U_{\text{eff}}) > 0$ is expected for $U_{\text{eff}} \geq U_{\text{on}}$, which can be approximated as

$$U_{\text{on,incoh}} = v_F B (R_o - R_i), \quad (46)$$

where we have simply rewritten the equality $2r_c(eU_{\text{on}}) = R_o - R_i$, neglecting the evanescent modes.

Looking at the conductance spectra shown in Figure 4a, we see for $B > 0$ that a wide range of lower U_{eff} , for which $G \approx 0$, is attached (via a cusp region) to the range of (approximately linearly) increasing G . To determine the value of $U_{\text{on}}(B)$ directly from the conductance spectra, $G(U_{\text{eff}})$, we numerically find the value of $U_{\text{on}}^{(1)}$ such that $G(U_{\text{on}}^{(1)}) = g_0$,

and $U_{\text{on}}^{(2)}$ such that $G(U_{\text{on}}^{(2)}) = 2g_0$, see the datapoints in Figure 4a. Then the linear extrapolation is performed to obtain

$$\begin{aligned} U_{\text{on}}^{(0)} &= U_{\text{on}}^{(1)} - (U_{\text{on}}^{(2)} - U_{\text{on}}^{(1)}) \frac{G(U_{\text{on}}^{(1)})}{G(U_{\text{on}}^{(2)}) - G(U_{\text{on}}^{(1)})} \\ &= 2U_{\text{on}}^{(1)} - U_{\text{on}}^{(2)}, \end{aligned} \quad (47)$$

so that $G(U_{\text{on}}^{(0)}) \approx 0$. The resulting values of $U_{\text{on}}^{(i)}$, depicted in Figure 4c (datapoints), stay close to $U_{\text{on, incoh}}$ given by Equation (46) (dashed line).

Remarkably, the values of the Fano factor corresponding to $U_{\text{eff}} = U_{\text{on}}^{(i)}$, $i = 1, 2$, see Figure 4b, are close to $F_{\text{incoh}}(B \rightarrow B_{c,2}-) \approx 0.56$. Similar observation holds for all studied values of $B \leq 0.5$ T, see Figure 4d; a typical deviation does not exceed 5%.

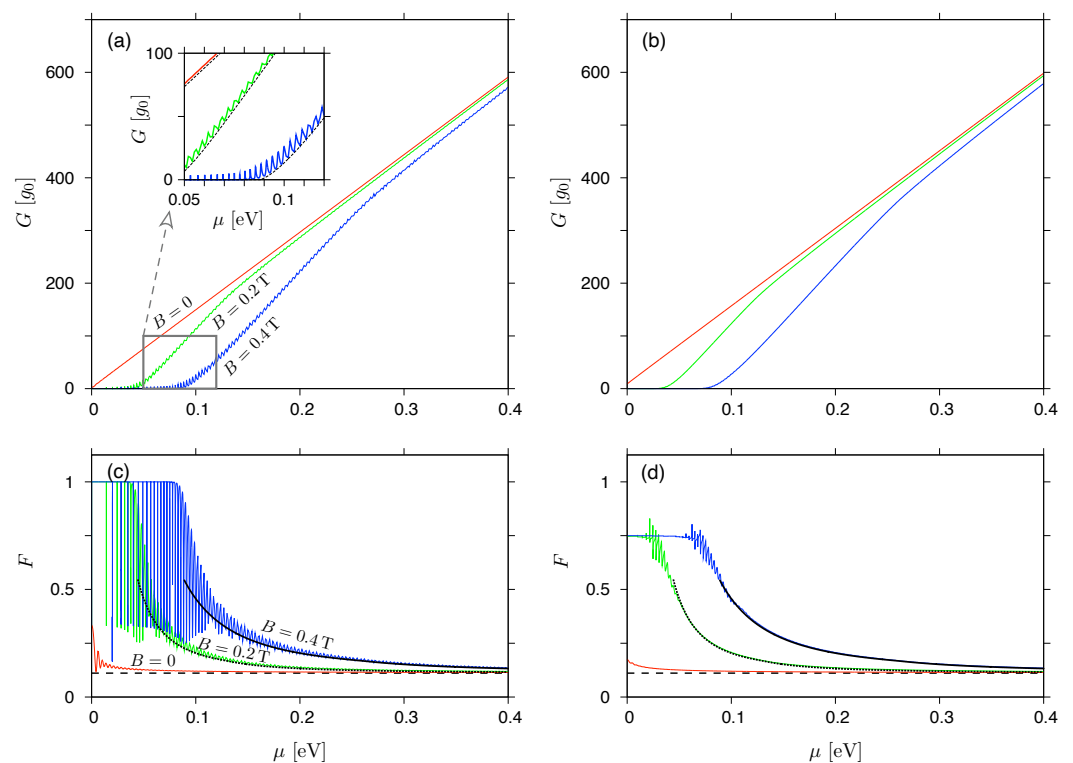


Figure 3. (a,b) Conductance and (c,d) the Fano factor for the Corbino disk in graphene with the radii $R_o = 2R_i = 1000$ nm, and the rectangular potential barrier (i.e., $V_0 \rightarrow \infty$ and $m \rightarrow \infty$ in Equation (2)) displayed as functions of the chemical potential. The values of magnetic field are $B = 0$ (red solid lines in all plots), $B = 0.2$ T (green solid lines), and $B = 0.4$ T (blue solid lines). Inset in (a) is a zoom-in, with black dashed lines depicting the incoherent conductance, see Equation (25). (a,c) show the linear-response results, see Equations (18) and (19); the datasets in (b,d) are obtained from Equations (16) and (17) with $U_{\text{eff}} = 0.01$ V. Remaining lines in (c,d) (black solid, black dotted, and black dashed) mark the incoherent Fano factor, see Equation (27); the values of magnetic field are specified for lines in (c), and are the same in (d). (For $B = 0$, horizontal lines mark $F_{\text{incoh}}(B \rightarrow 0) = 0.111074$, corresponding to $R_o = 2R_i$, see Table 1).

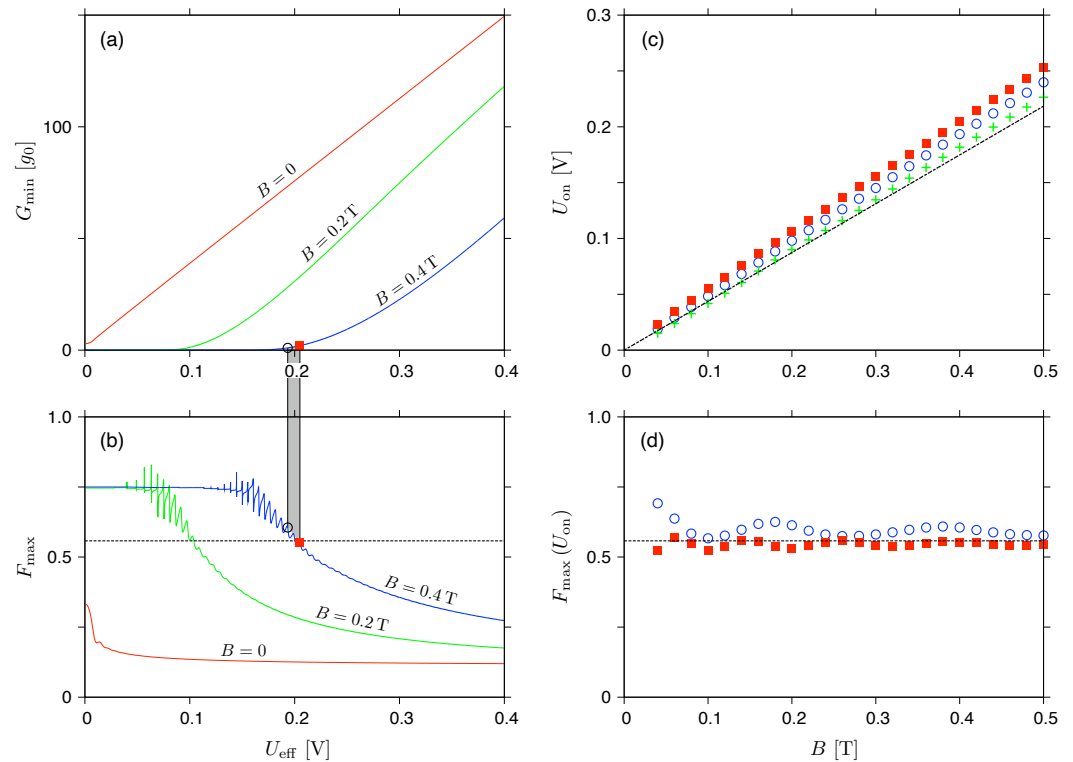


Figure 4. (a,b) Minimal conductivity ($G_{\min}$) and maximal Fano factor ($F_{\max}$), see Equations (16) and (17), corresponding to the chemical potential fixed at $\mu = -eU_{\text{eff}}/2$, versus the effective voltage. The magnetic field (B) is specified for each line. (c) The activation voltage, defined via $G_{\min}(U_{\text{on}}) = g_0$ (blue open circles), $G_{\min}(U_{\text{on}}) = 2g_0$ (red solid squares), or obtained from scaling according to Equation (47) (green crosses), displayed versus the magnetic field. (d) The Fano factor corresponding to $U_{\text{eff}} = U_{\text{on}}$ shown in (c). Horizontal dashed lines in (b,d) mark the value of $F_{\text{incoh}}(B \rightarrow B_{c,2-}) = 0.557898$ for $R_o = 2R_i$ (see Table 1). Dashed line in (c) depicts the approximation given in Equation (46). The remaining system parameters are same as in Figure 3.

4.2. Smooth Potential Barriers

In this subsection, we extend our numerical analysis to smooth potential barriers, defined by choosing $2 \leq m < \infty$ in Equation (2). Moreover, the barrier height is now finite, i.e., $V_0 = t_0/2 = 1.35$ eV, being not far from the results of some first-principles calculations for graphene–metal structures [67,68]. To the best of our knowledge, such a model, first proposed in Ref. [40], seems to be the simplest, providing a qualitatively correct description of the conductance spectrum asymmetry observed in existing experiments [46,48,51,54], in which the conductance for $\mu < 0$ is noticeably suppressed compared with the $\mu > 0$ range, due to the presence of two circular p-n junctions in the former case. (Such a feature is also correctly reproduced by a simpler model assuming the trapezoidal potential barrier [69], which allows a fully analytical treatment, but this approach produces an artificial conductance maximum near $\mu = 0$.)

The conductance spectra for five selected values of m are displayed in Figure 5, both for the linear-response regime (see Figure 5a,c) and beyond (Figure 5b,d). This time, we have limited our presentation to a single value of the magnetic field, i.e., $B = 0.2$ T. It should be noted that a finite value of V_0 leads to a small but visible asymmetry of the spectrum, even for $m = \infty$.

The finite-voltage results, $G(U_{\text{eff}})$ at $\mu = -eU_{\text{eff}}/2$, allow us to determine the activation voltage, $U_{\text{on}}(B)$, in a similar manner as for the infinite barrier case (see previous

subsection). When attempting to apply the incoherent-scattering approximation to smooth potentials, some modification is required for Equation (46), which now can be rewritten as

$$U_{\text{on, incoh}} = v_F B L_{\text{diff}}(m). \quad (48)$$

In the above, we have introduced the m -dependent effective sample length given by [40,41]

$$L_{\text{diff}}(m) = |R_o - R_i| \left(\frac{\hbar v_F}{|R_o - R_i| V_0} \right)^{1/m}, \quad (49)$$

which reduces to $L_{\text{diff}}(\infty) = R_o - R_i$ for a rectangular barrier, and gives $L_{\text{diff}}(m = 2) \ll R_o - R_i$ for the parabolic case. In brief, Equation (49) can be derived from $V(\pm L_{\text{diff}}/2) = -E_{\text{diff}}$, where E_{diff} denotes the value of the Fermi energy above which the Sharvin conductance overrules the pseudodiffusive conductance, namely,

$$E_{\text{diff}} = \frac{\hbar v_F}{R_o - R_i} \approx 1 \text{ meV} \quad \text{for } R_o - R_i = 500 \text{ nm}. \quad (50)$$

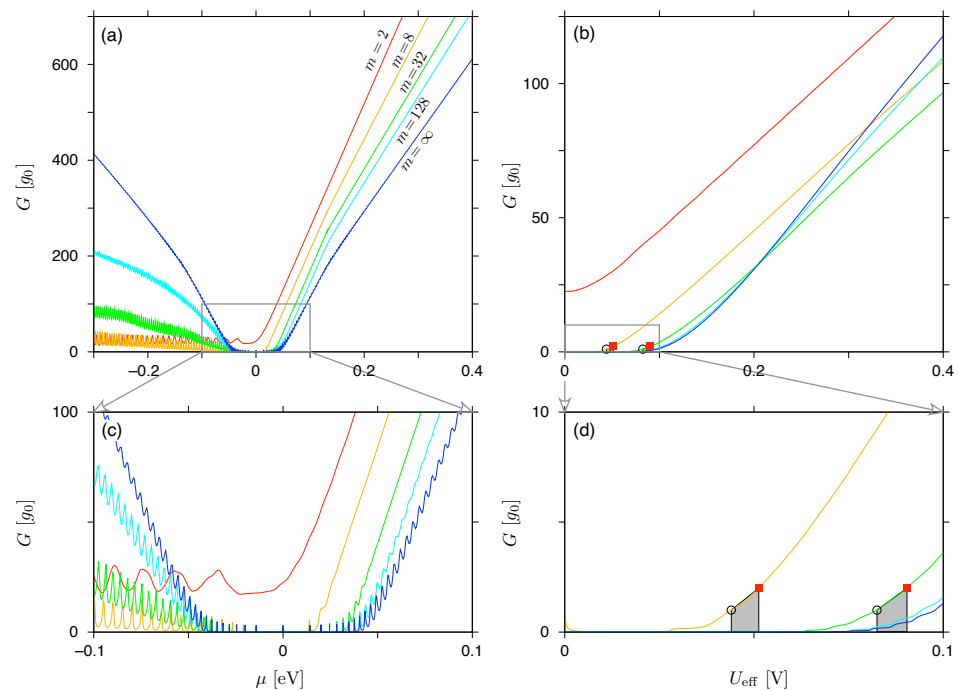


Figure 5. (a) Linear-response conductance (G) as a function of the chemical potential (μ) and (b) finite-voltage conductance, for $\mu = -eU_{\text{eff}}/2$, as a function of the voltage. The magnetic field is $B = 0.2 \text{ T}$ for all plots. The disk radii are the same as in Figure 3, but the barrier height, see Equation (2), is now fixed at $V_0 = t_0/2 = 1.35 \text{ eV}$; the parameter m is specified for each line. (c,d) Zoom-in, for low energies, with same datasets as in (a,b). Datapoints in (b,d) mark the values of $G(U_{\text{on}}^{(i)}) = i g_0$, $i = 1, 2$, defining the activation voltages $U_{\text{eff}} = U_{\text{on}}^{(i)}$.

In Figure 6 we show the Fano factor, for the same five values of m as previously used for the conductance (see Figure 5) and $B = 0.2 \text{ T}$, as a function μ in the linear-response limit ($U_{\text{eff}} \rightarrow 0$), as well as a function U_{eff} for $\mu = -U_{\text{eff}}/2$ (see left or right side of Figure 6, respectively). Again, the aperiodic oscillations almost vanish when entering the nonlinear response regime; in fact, the shape of $F_{\text{max}}(U_{\text{eff}})$ appears to be much less sensitive to the value of m than the linear-response, $F(\mu)$. The datapoints on the right side of Figure 6, identifying the values of $F(U_{\text{on}}^{(i)})$, $i = 1, 2$, such that $G(U_{\text{on}}^{(i)}) = i g_0$ (see Figure 5), are available starting from $m = 8$ (although the deviation from $F_{\text{incoh}}(B \rightarrow B_{c,2}-) \approx 0.56$ is significant in such a case), whereas strong asymmetry of $F(\mu)$ is visible up to $m = 32$.

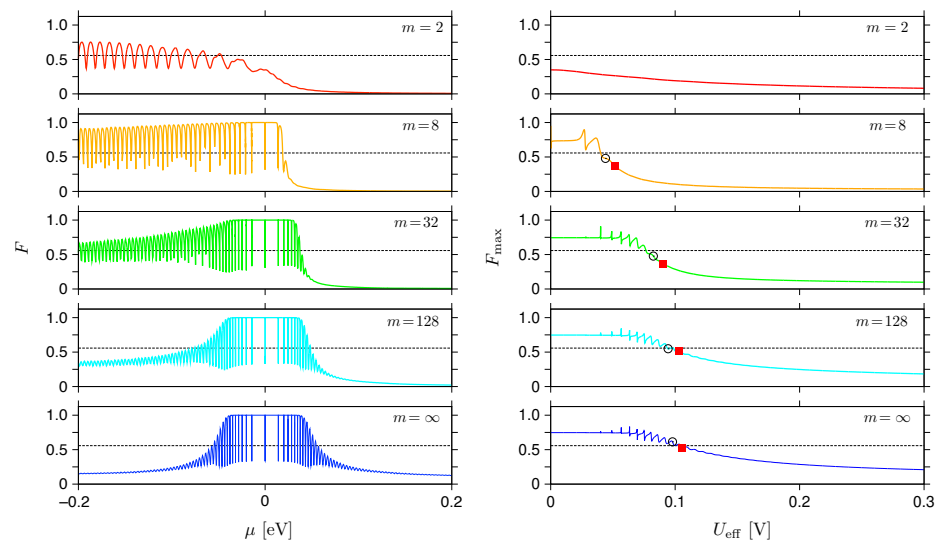


Figure 6. (Left): linear-response Fano factor as a function of the chemical potential. (Right): finite-voltage Fano factor, for $\mu = -eU_{\text{eff}}/2$, as a function of the voltage. The magnetic field is $B = 0.2$ T for all plots, the value of exponent m is specified at each plot, and remaining parameters are same as in Figure 5. Horizontal line at each plot marks the value of $F_{\text{incoh}}(B \rightarrow B_{c,2}-) = 0.557898$, see Table 1. Datapoints (right) mark the values of $F(U_{\text{on}}^{(i)})$, $i = 1, 2$, corresponding to activation voltages $U_{\text{eff}} = U_{\text{on}}^{(i)}$, for which $G(U_{\text{on}}^{(i)}) = ig_0$ (see also Figure 5).

The values of $U_{\text{on}}^{(i)}$ and the corresponding $F(U_{\text{on}}^{(i)})$, for the magnetic fields up to $B \leq 0.5$ T, are displayed in Figure 7. It can be noticed that the voltages, $U_{\text{on}}^{(i)}$, see datapoints in Figure 7a–d, show relatively good agreement with the approximation given by Equation (48) (purple solid lines); in fact, significant deviation from Equation (46) relevant for the rectangular barrier (black dashed lines) can be observed only for $m = 8$. On the contrary, the corresponding Fano factors, $F(U_{\text{on}}^{(i)})$, see datapoints in Figure 7e–h, remain close to the value of $F_{\text{incoh}}(B \rightarrow B_{c,2}-) \approx 0.56$ only for $m = \infty$ and $m = 128$, showing that the incoherent treatment of the shot-noise power that we propose in Section 3 is applicable only when the potential profiles are close to (but not necessarily perfectly matching) the rectangular shape.

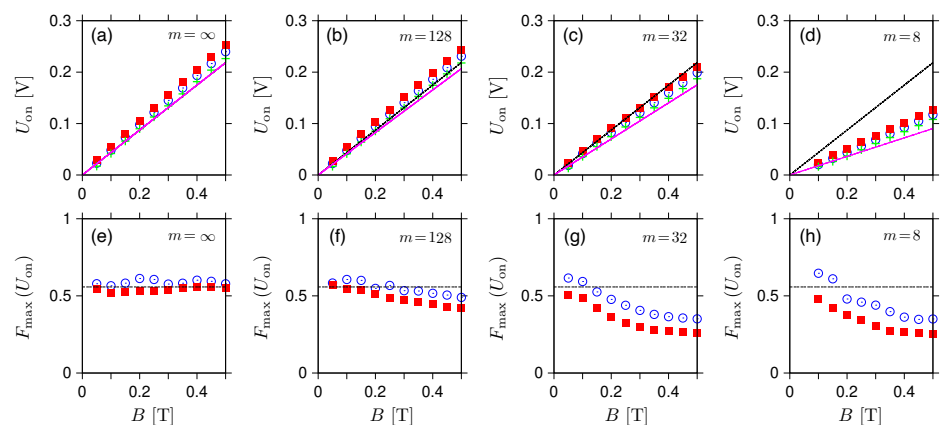


Figure 7. (a–d) The activation voltage (for the definition, see Figure 4) and (e–h) the corresponding Fano factor for $\mu = -eU_{\text{eff}}/2$, displayed as functions of the magnetic field (datapoints). The value of exponent m is specified at each plot; remaining parameters are same as in Figure 5. Lines in (a–d) depict the approximation given by Equation (48) (purple solid) and Equation (46) (black dashed) coinciding in the $m \rightarrow \infty$ limit. Horizontal lines in (e–h) mark the value of $F_{\text{incoh}}(B \rightarrow B_{c,2}-) = 0.557898$, see Table 1.

5. Conclusions

We have proposed the analytical description of the shot-noise power in graphene-based disks in high magnetic field and doping. Assuming the incoherent scattering of Dirac fermions between two potential steps of an infinite height, both characterized by *a priori* non-zero transmission probability due to the Klein tunneling, we find that the vanishing conductance should be accompanied by the Fano factor $F \approx 0.56$, which is weakly dependent on the disk proportions.

Next, the results of analytical considerations are confronted with the outcome of computer simulations, including both rectangular and smooth shapes of the electrostatic potential barrier in the disk area. Calculating both linear-response and finite-voltage transport cumulants, within the zero-temperature Landauer–Büttiker formalism, we point out that the role of evanescent waves (earlier ignored in the analytic approach) is significant in the linear-response regime, but one should be able to detect the quasi-universal $F \approx 0.56$ noise in a properly designed experiment going beyond the linear-response regime. To achieve this goal, the following procedure is suggested. First, the activation voltage (for a fixed magnetic field) must be determined, by finding a cusp position on the conductance-versus-voltage plot, above which the conductance grows rapidly with voltage (the average chemical potential is controlled by the gate so that the conductance is minimal for a given voltage). Having determined the activation voltage, one measures the noise for such a voltage, expecting the Fano factor to be close to $F \approx 0.56$.

We expect that the effect we describe should be observable in ultraclean samples and at sub-kelvin temperatures (such as in Ref. [49]); at higher temperatures, hydrodynamic effects may noticeably alter the measurable quantities [54]. Since the noise-related characteristics seem to be generally more sensitive to the potential shape than the conductance (or the thermoelectric properties discussed earlier in Ref. [55]), the experimental study following the scenario presented here may be a suitable way to check whether the flat-potential area of a mesoscopic size is present or not in a given graphene-based structure.

Although the work focuses primarily on graphene, recent progress in the fabrication of semiconductor artificial graphenes [70,71] suggests that our results may also be relevant to such systems.

Author Contributions: A.R. designed the algorithm, A.R. and P.W. developed the code and performed preliminary computations, K.R. organized the computations on the PL-Grid supercomputing infrastructure; all authors were involved in data analysis and manuscript preparation. All authors have read and agreed to the published version of the manuscript.

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Appendix A. Numerical Mode Matching for Smooth Potentials

Here we summarize the numerical approach presented earlier in Ref. [55].

In a typical situation, the system of ordinary differential equations for the spinor components (χ_a, χ_b) , see Equations (4) and (5), needs to be integrated numerically for all j -s. In order to reduce round-off errors that can occur in finite-precision arithmetic

due to exponentially growing (or decaying) solutions, one can divide the entire interval $R_i < r < R_o$ into M parts bounded by

$$R_c^{(l)} = R_i + l \frac{R_o - R_i}{M} < r < R_c^{(l+1)},$$

with $l = 0, 1, \dots, M - 1$.

(A1)

(In particular, $R_c^{(0)} = R_i$ and $R_c^{(M)} = R_o$).

The wave function in the disk area $\chi_j^{(\text{disk})}$ is now given by the set of functions $\{\chi_j^{(l)}\}$ for M consecutive intervals given by Equation (A1). For the l -th interval,

$$\chi_j^{(l)} = A_j^{(l)} \chi_j^{(l),I} + B_j^{(l)} \chi_j^{(l),II},$$
(A2)

where $\chi_j^{(l),I}$, $\chi_j^{(l),II}$ are two linearly independent solutions obtained by integration of Equations (4) and (5) with two different initial conditions, $\chi_j^{(l),I}|_{r=R_i^{(l)}} = (1, 0)^T$ and $\chi_j^{(l),II}|_{r=R_i^{(l)}} = (0, 1)^T$. $A_j^{(l)}$ and $B_j^{(l)}$ are complex coefficients (to be determined later).

In particular, for $R_o = 2R_i = 1000 \text{ nm}$ and $B < 0.5 \text{ T}$ considered in this paper, it is sufficient to set $M = 20$ and employ a standard fourth-order Runge–Kutta (RK4) algorithm with a spatial step of 0.5 pm . (For such a choice, the output numerical uncertainties of the transmission probabilities T_j are smaller than 10^{-7}).

The matching conditions for the $M + 1$ interfaces at $r = R_i$, $r = R_c^{(1)}, \dots, r = R_c^{(M-1)}$, and $r = R_o$, can now be written as

$$\chi_j^{(\text{inner})}(R_i) = \chi_j^{(0)}(R_i),$$
(A3)

$$\chi_j^{(l)}(R_c^{(l+1)}) = \chi_j^{(l+1)}(R_c^{(l+1)}), \quad l = 0, \dots, M - 2,$$
(A4)

$$\chi_j^{(M-1)}(R_o) = \chi_j^{(\text{outer})}(R_o),$$
(A5)

and are equivalent to the Cramer's system of $2(M + 1)$ linear equations for the unknowns $A_j^{(0)}$, $B_j^{(0)}$, $\dots$, $A_j^{(M-1)}$, $B_j^{(M-1)}$, r_j , and t_j .

Writing down explicitly the spinor components appearing in Equations (A3)–(A5), we arrive to

$$\begin{bmatrix} -\chi_{j,a}^{\text{out}}(R_i) & \chi_{j,a}^{(0),I}(R_c^{(0)}) & \chi_{j,a}^{(0),II}(R_c^{(0)}) & & & & & & \\ -\chi_{j,b}^{\text{out}}(R_i) & \chi_{j,b}^{(0),I}(R_c^{(0)}) & \chi_{j,b}^{(0),II}(R_c^{(0)}) & & & & & & \\ 0 & \chi_a^{(0),I}(R_c^{(1)}) & \chi_{j,a}^{(0),II}(R_c^{(1)}) & & & & & & \\ 0 & \chi_b^{(0),I}(R_c^{(1)}) & \chi_{j,b}^{(0),II}(R_c^{(1)}) & \ddots & & & & & \\ & & & \ddots & \chi_{j,a}^{(\overline{M}),I}(R_c^{(\overline{M})}) & \chi_{j,a}^{(\overline{M}),II}(R_c^{(\overline{M})}) & & 0 & \\ & & & & \chi_{j,b}^{(\overline{M}),I}(R_c^{(\overline{M})}) & \chi_{j,b}^{(\overline{M}),II}(R_c^{(\overline{M})}) & & 0 & \\ & & & & \chi_a^{(\overline{M}),I}(R_c^{(M)}) & \chi_{j,a}^{(\overline{M}),II}(R_c^{(M)}) & -\chi_{j,a}^{\text{in}}(R_o) & & \\ & & & & \chi_b^{(\overline{M}),I}(R_c^{(M)}) & \chi_{j,b}^{(\overline{M}),II}(R_c^{(M)}) & -\chi_{j,b}^{\text{in}}(R_o) & & \end{bmatrix}$$

$$\times \begin{bmatrix} r_j \\ A_j^{(0)} \\ B_j^{(0)} \\ \vdots \\ A_j^{(M-1)} \\ B_j^{(M-1)} \\ t_j \end{bmatrix} = \begin{bmatrix} \chi_{j,a}^{\text{in}}(R_i) \\ \chi_{j,b}^{\text{in}}(R_i) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (\text{A6})$$

where we have defined $\overline{M} = M - 1$,

$$\chi_j^{\text{in}} = \begin{pmatrix} H_{j-1/2}^{(2)}(Kr) \\ iH_{j+1/2}^{(2)}(Kr) \end{pmatrix}, \quad \chi_j^{\text{out}} = \begin{pmatrix} H_{j-1/2}^{(1)}(Kr) \\ iH_{j+1/2}^{(1)}(Kr) \end{pmatrix}. \quad (\text{A7})$$

For heavily doped leads ($V_0 \rightarrow \infty$), the wave functions given by Equation (A7) simplify to

$$\chi_j^{(\text{in})} = \frac{e^{iKr}}{\sqrt{r}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \chi_j^{(\text{out})} = \frac{e^{-iKr}}{\sqrt{r}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (\text{A8})$$

with $K = |E + V_0|/(\hbar v_F) \rightarrow \infty$.

As the linear systems for different values of j -s are decoupled, standard software packages can be used to find their solutions. We have chosen the double precision LAPACK routine `zgesv`, see Ref. [72].

References

1. Datta, S. *Quantum Transport: Atom to Transistor*; Cambridge University Press: Cambridge, UK, 2005.
2. Kittel, C. *Introduction to Solid State Physics*, 8th ed.; John Wiley and Sons: New York, NY, USA, 2005.
3. Deaver, B.S., Jr.; Fairbank, W.M. Experimental Evidence for Quantized Flux in Superconducting Cylinders. *Phys. Rev. Lett.* **1961**, *7*, 43. [CrossRef]
4. Doll, R.; Näbauer, M. Experimental Proof of Magnetic Flux Quantization in a Superconducting Ring. *Phys. Rev. Lett.* **1961**, *7*, 51. [CrossRef]
5. Josephson, B.D. The discovery of tunneling supercurrents. *Rev. Mod. Phys.* **1974**, *46*, 251. [CrossRef]
6. Imry, Y. *Introduction to Mesoscopic Physics*, 2nd ed.; Oxford University Press: Oxford, UK, 2002; Chapter 5.
7. Ando, T.; Matsumoto, Y.; Uemura, Y. Theory of Hall Effect in a Two-Dimensional Electron System. *J. Phys. Soc. Jpn.* **1975**, *39*, 279. [CrossRef]
8. Klitzing, K.; Dorda, G.; Pepper, M. New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance. *Phys. Rev. Lett.* **1980**, *45*, 494. [CrossRef]
9. Laughlin, R.B. Quantized Hall conductivity in two dimensions. *Phys. Rev. B* **1981**, *23*, 5632. [CrossRef]
10. Tsui, D.C.; Stormer, H.L.; Gossard, A.C. Two-Dimensional Magnetotransport in the Extreme Quantum Limit. *Phys. Rev. Lett.* **1982**, *48*, 1559. [CrossRef]
11. Novoselov, K.S.; Geim, A.K.; Morozov, S.V.; Jiang, D.; Katsnelson, M.I.; Grigorieva, I.V.; Dubonos, S.V.; Firsov, A.A. Two-dimensional gas of massless Dirac fermions in graphene. *Nature* **2005**, *438*, 197. [CrossRef] [PubMed]
12. Zhang, Y.; Tan, Y.-W.; Stormer, H.L.; Kim, P. Experimental observation of the quantum Hall effect and Berry's phase in graphene. *Nature* **2005**, *438*, 201. [CrossRef]
13. van Wees, B.J.; van Houten, H.; Beenakker, C.W.J.; Williamson, J.G.; Kouwenhoven, L.P.; van der Marel, D.; Foxon, C.T. Quantized conductance of point contacts in a two-dimensional electron gas. *Phys. Rev. Lett.* **1988**, *60*, 848. [CrossRef]
14. Webb, R.A.; Washburn, S.; Umbach, C.P.; Laibowitz, R.B. Observation of h/e Aharonov-Bohm Oscillations in Normal-Metal Rings. *Phys. Rev. Lett.* **1985**, *54*, 2696. [CrossRef] [PubMed]
15. Lee, P.A.; Stone, A.D. Universal Conductance Fluctuations in Metals. *Phys. Rev. Lett.* **1985**, *55*, 1622. [CrossRef] [PubMed]
16. Altshuler, B.L.; Shklovskii, B.I. Repulsion of energy levels and conductivity of small metal samples. *Zh. Eksp. Teor. Fiz.* **91**, 220 **1986** *Sov. Phys. JETP* **1986**, *64*, 127. Available online: <http://jetp.ras.ru/cgi-bin/e/index/e/64/1/p127?a=list> (accessed on 20 June 2024).
17. Pal, A.N.; Kochat, V.; Ghosh, A. Direct Observation of Valley Hybridization and Universal Symmetry of Graphene with Mesoscopic Conductance Fluctuations. *Phys. Rev. Lett.* **2012**, *109*, 196601. [CrossRef] [PubMed]
18. Hu, Y.; Liu, H.; Jiang, H.; Xie, X.C. Numerical study of universal conductance fluctuations in three-dimensional topological semimetals. *Phys. Rev. B* **2017**, *96*, 134201. [CrossRef]

19. Sharapov, S.G.; Gusynin, V.P.; Beck, H. Transport properties in the d-density-wave state in an external magnetic field: The Wiedemann-Franz law. *Phys. Rev. B* **2003**, *67*, 144509. [\[CrossRef\]](#)
20. Mahajan, R.; Barkeshli, M.; Hartnoll, S.A. Non-Fermi liquids and the Wiedemann-Franz law. *Phys. Rev. B* **2013**, *88*, 125107. [\[CrossRef\]](#)
21. Yoshino, H.; Murata, K. Significant Enhancement of Electronic Thermal Conductivity of Two-Dimensional Zero-Gap Systems by Bipolar-Diffusion Effect. *J. Phys. Soc. Jpn.* **2015**, *84*, 024601. [\[CrossRef\]](#)
22. Crossno, J.; Shi, J.K.; Wang, K.; Liu, X.; Harzheim, A.; Lucas, A.; Sachdev, S.; Kim, P.; Taniguchi, T.; Watanabe, K.; et al. Observation of the Dirac fluid and the breakdown of the Wiedemann-Franz law in graphene. *Science* **2016**, *351*, 1058. [\[CrossRef\]](#)
23. Rycerz, A. Wiedemann-Franz Law for Massless Dirac Fermions with Implications for Graphene. *Materials* **2021**, *14*, 2704. [\[CrossRef\]](#)
24. Tu, Y.-T.; Das Sarma, S. Wiedemann-Franz law in graphene. *Phys. Rev. B* **2023**, *107*, 085401. [\[CrossRef\]](#)
25. Katsnelson, M.I. *The Physics of Graphene*, 2nd ed.; Cambridge University Press: Cambridge, UK, 2020; Chapter 3. [\[CrossRef\]](#)
26. Katsnelson, M.I.; Novoselov, K.S.; Geim, A.K. Chiral tunnelling and the Klein paradox in graphene. *Nature Phys.* **2006**, *2*, 620. [\[CrossRef\]](#)
27. Tworzydło, J.; Trauzettel, B.; Titov, M.; Rycerz, A.; Beenakker, C.W.J. Sub-Poissonian shot noise in graphene. *Phys. Rev. Lett.* **2006**, *96*, 246802. [\[CrossRef\]](#)
28. Miao, F.; Wijeratne, S.; Zhang, Y.; Coskun, U.; Bao, W.; Lau, C.N. Phase-Coherent Transport in Graphene Quantum Billiards. *Science* **2007**, *317*, 5844. [\[CrossRef\]](#) [\[PubMed\]](#)
29. Sonin, E.B. Charge transport and shot noise in ballistic graphene sheet. *Phys. Rev. B* **2008**, *77*, 233408. [\[CrossRef\]](#)
30. Danneau, R.; Wu, F.; Craciun, M.F.; Russo, S.; Tomi, M.Y.; Salmilehto, J.; Morpurgo, A.F.; Hakonen, P.J. Shot Noise in Ballistic Graphene. *Phys. Rev. Lett.* **2008**, *100*, 196802. [\[CrossRef\]](#) [\[PubMed\]](#)
31. Laitinen, A.; Paraoanu, G.S.; Oksanen, M.; Craciun, M.F.; Russo, S.; Sonin, E.; Hakonen, P. Contact doping, Klein tunneling, and asymmetry of shot noise in suspended graphene. *Phys. Rev. B* **2016**, *93*, 115413. [\[CrossRef\]](#)
32. Nair, R.R.; Blake, P.; Grigorenko, A.N.; Novoselov, K.S.; Booth, T.J.; Stauber, T.; Peres, N.M.R.; Geim, A.K. Fine Structure Constant Defines Visual Transparency of Graphene. *Science* **2008**, *320*, 1308. [\[CrossRef\]](#)
33. Skulason, H.S.; Gaskell, P.E.; Szkopek, T. Optical reflection and transmission properties of exfoliated graphite from a graphene monolayer to several hundred graphene layers. *Nanotechnology* **2010**, *21*, 295709. [\[CrossRef\]](#) [\[PubMed\]](#)
34. Merthe, D.J.; Kresin, V.V. Transparency of graphene and other direct-gap two-dimensional materials. *Phys. Rev. B* **2016**, *94*, 205439. [\[CrossRef\]](#)
35. Wang, C.R.; Lu, W.-S.; Hao, L.; Lee, W.-L.; Lee, T.-K.; Lin, F.; Cheng, I.-C.; Chen, J.-Z. Enhanced thermoelectric power in dual-gated bilayer graphene. *Phys. Rev. Lett.* **2011**, *107*, 186602. [\[CrossRef\]](#)
36. Suszalski, D.; Rut, G.; Rycerz, A. Lifshitz transition and thermoelectric properties of bilayer graphene. *Phys. Rev. B* **2018**, *97*, 125403. [\[CrossRef\]](#)
37. Suszalski, D.; Rut, G.; Rycerz, A. Thermoelectric properties of gapped bilayer graphene. *J. Phys. Condens. Matter* **2019**, *31*, 415501. [\[CrossRef\]](#) [\[PubMed\]](#)
38. Zong, P.; Liang, J.; Zhang, P.; Wan, C.; Wang, Y.; Koumoto, K. Graphene-Based Thermoelectrics. *ACS Appl. Energy Mater.* **2020**, *3*, 2224. [\[CrossRef\]](#)
39. Jayaraman, A.; Hsieh, K.; Ghawri, B.; Mahapatra, P.S.; Watanabe, K.; Taniguchi, T.; Ghosh, A. Evidence of Lifshitz Transition in the Thermoelectric Power of Ultrahigh-Mobility Bilayer Graphene. *Nano Lett.* **2021**, *21*, 1221. [\[CrossRef\]](#) [\[PubMed\]](#)
40. Rycerz, A.; Witkowski, P. Sub-Sharvin conductance and enhanced shot noise in doped graphene. *Phys. Rev. B* **2021**, *104*, 165413. [\[CrossRef\]](#)
41. Rycerz, A.; Witkowski, P. Theory of sub-Sharvin charge transport in graphene disks. *Phys. Rev. B* **2022**, *106*, 155428. [\[CrossRef\]](#)
42. Sharvin, Y.V. A possible method for studying Fermi surfaces. *Zh. Eksp. Teor. Fiz.* **1965**, *48*, 984. *Sov. Phys. JETP* **1965**, *21*, 655. Available online: <http://www.jetp.ras.ru/cgi-bin/e/index/e/21/3/p655?a=list> (accessed on 20 June 2024).
43. Beenakker, C.W.J.; van Houten, H. Quantum transport in semiconductor nanostructures. *Solid State Phys.* **1991**, *44*, 1. [\[CrossRef\]](#)
44. Rycerz, A.; Recher, P.; Wimmer, M. Conformal mapping and shot noise in graphene. *Phys. Rev. B* **2009**, *80*, 125417. [\[CrossRef\]](#)
45. Rycerz, A. Magnetoconductance of the Corbino disk in graphene. *Phys. Rev. B* **2010**, *81*, 121404. [\[CrossRef\]](#)
46. Peters, E.C.; Giesbers, A.J.M.; Burghard, M.; Kern, K. Scaling in the quantum Hall regime of graphene Corbino devices. *Appl. Phys. Lett.* **2014**, *104*, 203109. [\[CrossRef\]](#)
47. Abdollahipour, B.; Moomivand, E. Magnetopumping current in graphene Corbino pump. *Physica E* **2017**, *86*, 204. [\[CrossRef\]](#)
48. Kumar, M.; Laitinen, A.; Hakonen, P.J. Unconventional fractional quantum Hall states and Wigner crystallization in suspended Corbino graphene. *Nat. Commun.* **2018**, *9*, 2776. [\[CrossRef\]](#) [\[PubMed\]](#)
49. Zeng, Y.; Li, J.I.A.; Dietrich, S.A.; Ghosh, O.M.; Watanabe, K.; Taniguchi, T.; Hone, J.; Dean, C.R. High-Quality Magnetotransport in Graphene Using the Edge-Free Corbino Geometry. *Phys. Rev. Lett.* **2019**, *122*, 137701. [\[CrossRef\]](#) [\[PubMed\]](#)
50. Suszalski, D.; Rut, G.; Rycerz, A. Mesoscopic valley filter in graphene Corbino disk containing a p-n junction. *J. Phys. Mater.* **2020**, *3*, 015006. [\[CrossRef\]](#)
51. Kamada, M.; Gall, V.; Sarkar, J.; Kumar, M.; Laitinen, A.; Gornyi, I.; Hakonen, P. Strong magnetoresistance in a graphene Corbino disk at low magnetic fields. *Phys. Rev. B* **2021**, *104*, 115432. [\[CrossRef\]](#)

52. Rycerz, A.; Suszalski, D. Graphene disk in a solenoid magnetic potential: Aharonov-Bohm effect without a two-slit-like setup. *Phys. Rev. B* **2020**, *101*, 245429. [\[CrossRef\]](#)
53. Bouhlal, A.; Jellal, A.; Mansouri, M. Quantum tunneling in graphene Corbino disk in a solenoid magnetic potential with wedge disclination. *Physica B* **2022**, *639*, 413904. [\[CrossRef\]](#)
54. Kumar, C.; Birkbeck, J.; Sulpizio, J.A.; Perello, D.; Taniguchi, T.; Watanabe, K.; Reuven, O.; Scaffidi, T.; Stern, A.; Geim, A.K.; et al. Imaging Hydrodynamic Electrons Flowing Without Landauer-Sharvin Resistance. *Nature* **2022**, *609*, 276. [\[CrossRef\]](#)
55. Rycerz, A.; Rycerz, K.; Witkowski, P. Thermoelectric Properties of the Corbino Disk in Graphene. *Materials* **2023**, *16*, 4250. [\[CrossRef\]](#) [\[PubMed\]](#)
56. Recher, P.; Nilsson, J.; Burkard, G.; Trauzettel, B. Bound states and magnetic field induced valley splitting in gate-tunable graphene quantum dots. *Phys. Rev. B* **2009**, *79*, 085407. [\[CrossRef\]](#)
57. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions*; Dover Publications, Inc.: New York, NY, USA, 1965; Chapter 13.
58. Abramowitz, M.; Stegun, I.A. (Eds.) An important step in the derivation of Eq. (tj)hank is the recognition of the Wronskian of Hankel functions; In *Handbook of Mathematical Functions*; Dover Publications, Inc.: New York, NY, USA, 1965; Chapter 9, Eq. 9.1.17.
59. Nazarov, Y.V.; Blanter, Y.M. *Quantum Transport: Introduction to Nanoscience*; Cambridge University Press: Cambridge, UK, 2009; Chapter 1.
60. Pekola, J. Towards quantum thermodynamics in electronic circuits. *Nat. Phys.* **2015**, *11*, 118. [\[CrossRef\]](#)
61. Bruch, A.; Lewenkopf, C.; von Oppen, F. Landauer-Büttiker Approach to Strongly Coupled Quantum Thermodynamics: Inside-Outside Duality of Entropy Evolution. *Phys. Rev. Lett.* **2018**, *120*, 107701. [\[CrossRef\]](#) [\[PubMed\]](#)
62. Rams, M.M.; Zwolak, M. Breaking the Entanglement Barrier: Tensor Network Simulation of Quantum Transport. *Phys. Rev. Lett.* **2020**, *124*, 137701. [\[CrossRef\]](#) [\[PubMed\]](#)
63. Katsnelson, M.I. Aharonov-Bohm effect in undoped graphene: Magnetotransport via evanescent waves. *Europhys. Lett.* **2010**, *89*, 17001. [\[CrossRef\]](#)
64. Rycerz, A. Aharonov-Bohm and Relativistic Corbino Effects in Graphene: A Comparative Study of Two Quantum Interference Phenomena. *Acta Phys. Polon. A* **2012**, *121*, 1242. [\[CrossRef\]](#)
65. Datta, S. *Electronic Transport in Mesoscopic Systems*; Cambridge University Press: Cambridge, UK, 1997; Chapter 3, p. 129. [\[CrossRef\]](#)
66. We use the identity $I(a, b) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{du}{a+b\cos u} = \frac{1}{\sqrt{a^2-b^2}}$, for $a > |b|$, and the First Derivative of the Above Over a ; See Also Gradshteyn, I.S.; Ryzhik, I.M. In *Table of Integrals, Series, and Products*, 7th ed.; Academic Press: New York, NY, USA, 2007; p. 172, Eq. 2.553.3. Available online: <https://archive.org/details/GradshteynI.S.RyzhikI.M.TablesOfIntegralsSeriesAndProducts/page/n1/mode/2up> (accessed on 20 June 2024).
67. Giovannetti, G.; Khomyakov, P.A.; Brocks, G.; Karpan, V.M.; van den Brink, J.; Kelly, P.J. Doping Graphene with Metal Contacts. *Phys. Rev. Lett.* **2008**, *101*, 026803. [\[CrossRef\]](#) [\[PubMed\]](#)
68. Cusati, T.; Fiori, G.; Gahoi, A.; Passi, V.; Lemme, M.C.; Fortunelli, A.; Iannaccone, G. Electrical properties of graphene-metal contacts. *Sci. Rep.* **2017**, *7*, 5109. [\[CrossRef\]](#)
69. Paraoanu, G.S. Klein tunneling through the trapezoidal potential barrier in graphene: Conductance and shot noise. *New J. Phys.* **2021**, *23*, 043027. [\[CrossRef\]](#)
70. Du, L.; Liu, Z.; Wind, S.J.; Pellegrini, V.; West, K.W.; Fallahi, S.; Pfeiffer, L.N.; Manfra, M.J.; Pinczuk, A. Observation of Flat Bands in Gated Semiconductor Artificial Graphene. *Phys. Rev. Lett.* **2021**, *126*, 106402. [\[CrossRef\]](#) [\[PubMed\]](#)
71. Biborski, A.; Zegrodnik, M.; Wójcik, P.; Nowak, M.P. Honeycomb antidot artificial lattice as a prototypical correlated Dirac fermion system. *Phys. Rev. B* **2023**, *108*, 235119. [\[CrossRef\]](#)
72. Anderson, E.; Bai, Z.; Bischof, C.; Blackford, S.; Demmel, J.; Dongarra, J.; Croz, J.D.; Greenbaum, A.; Hammarling, S.; McKenney, A.; et al. *LAPACK Users' Guide*, 3rd ed.; Society for Industrial and Applied Mathematics: Philadelphia, PA, USA, 1999.

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