Article

Reproducing Kernel Hilbert Space *vs.* Frame Estimates

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Abstract: We consider conditions on a given system $\mathcal{F}$ of vectors in Hilbert space $\mathcal{H}$, forming a frame, which turn $\mathcal{H}$ into a reproducing kernel Hilbert space. It is assumed that the vectors in $\mathcal{F}$ are functions on some set Ω. We then identify conditions on these functions which automatically give $\mathcal{H}$ the structure of a reproducing kernel Hilbert space of functions on Ω. We further give an explicit formula for the kernel, and for the corresponding isometric isomorphism. Applications are given to Hilbert spaces associated to families of Gaussian processes.

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1. Introduction

A reproducing kernel Hilbert space (RKHS) is a Hilbert space $\mathcal{H}$ of functions on a set, say Ω, with the property that $f(t)$ is continuous in f with respect to the norm in $\mathcal{H}$. There is then an associated kernel. It is called reproducing because it reproduces the function values for f in $\mathcal{H}$. Reproducing kernels and their RKHSs arise as inverses of elliptic PDOs, as covariance kernels of stochastic processes, in the study of integral equations, in statistical learning theory, empirical risk minimization, as potential

kernels, and as kernels reproducing classes of analytic functions, and in the study of fractals, to mention only some of the current applications. They were first introduced in the beginning of the 20ties century by Stanisaw Zaremba and James Mercer, Gábor Szegö, Stefan Bergman, and Salomon Bochner. The subject was given a global and systematic presentation by Nachman Aronszajn in the early 1950s. The literature is by now vast, and we refer to the following items from the literature, and the papers cited there [1,4,7,12,15,16]. Our aim in the present paper is to point out an intriguing use of reproducing kernels in the study of frames in Hilbert space.

2. An Explicit Isomorphism

Let $\mathcal{H}$ be a separable Hilbert space, and let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a system of vectors in $\mathcal{H}$. Then we shall study relations of $\mathcal{H}$ as a reproducing kernel Hilbert space (RKHS) subject to properties imposed on the system $\{\varphi_n\}_{n\in\mathbb{N}}$. An RKHS is a Hilbert space $\mathcal{H}$ of functions on some set Ω such that for all $t \in \Omega$, there is a (unique) $K_t \in \mathcal{H}$ with $f(t) = \langle K_t, f\rangle_{\mathcal{H}}$, for all $t \in \Omega$, for all $f \in \mathcal{H}$. In the theorem below we study what systems of functions

$$\varphi_n \in \mathcal{H} \cap \{\text{functions on some set } \Omega\} \tag{1}$$

yield RKHSs; *i.e.*, if $\{\varphi_n\}_{n\in\mathbb{N}}$ satisfies (1), what additional conditions are required to guarantee that $\mathcal{H}$ is an RKHS?

Given $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$, we shall introduce the Gramian $G = (\langle\varphi_i, \varphi_j\rangle_{\mathcal{H}})$ considered as an $\infty \times \infty-$matrix.

Under mild restrictions on $\{\varphi_n\}_{n\in\mathbb{N}}$, it turns out that G defines an unbounded (generally) selfadjoint linear operator

$$l^2 \xrightarrow{G} l^2$$

$$(G(c_j))_k = \sum_{j\in\mathbb{N}} \langle\varphi_k, \varphi_j\rangle_{\mathcal{H}} c_j \tag{2}$$

Let $\mathcal{F}$ denote finitely supported sequence with (2) defined on all finitely supported sequence (c_j) $\mathcal{F}$ in l^2, *i.e.*, $(c_j) \in \mathcal{F}$ if and only if there exists $n \in \mathbb{Z}_+$ such that $c_j = 0$, for all $j \geq n$; but note that n depends the sequences. Denoting δ_j the canonical basis in l^2, $\delta_j(j) = \delta_{i,j}$, note $\mathcal{F} = span\{\delta_j | j \in \mathbb{N}\}$.

Further, note that the RHS in (2) is well defined when $\sum_j |\langle\varphi_k, \varphi_j\rangle_{\mathcal{H}}|^2 < \infty$, for all $k \in \mathbb{N}$.

Remark 1. *1. Since the statement of our result entails G^{-1} (inverse Gramian, it is actually a pseudo-inverse, see below), we will add a few comments about this point. It is known to specialists in frame theory, but to make the presentation self-contained we add a lemma giving equivalent properties and conditions for G^{-1}.*

2. The Gramian of a frame is invertible if and only if the frame is a Riesz basis.

3. By definition a frame refers to a fixed Hilbert space, say $\mathcal{H}$, and then a system of vectors in $\mathcal{H}$ satisfying an axiomatic a priori estimate, see (11). But because of applications to stochastic processes (see Section 3) it is natural to consider frames in some specified Hilbert space which in fact consists of function on some set, say Ω (given at the outset).

4. Our concern here is to make the connection between the initial Hilbert space $\mathcal{H}$ (with frame vectors) and an associated reproducing kernel Hilbert space of functions on Ω. An application to the study of Gaussian processes is motivating our approach.

Theorem 1. *Suppose $\mathcal{H}$, $\{\varphi_n\}$ are given. Assume that*

(a) each φ_n is a function on Ω where Ω is a given set

(b) $\{\varphi_n\}$ is a frame in $\mathcal{H}$, see (10) and (11), and that

(c) $\{\varphi_n(t)\} \in l^2$, for all $t \in \Omega$

then $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) with kernel

$$K^G(s,t) = \langle l(s), G^{-1}l(t) \rangle_2 = l(s)^* G^{-1} l(t) \tag{3}$$

where $l(t) = \{\varphi_n(t)\} \in l^2$, and where G is the Gramian of $G = (\langle \varphi_n, \varphi_m \rangle_{\mathcal{H}})$. Moreover, G defines selfadjoint operator in l^2 with dense domain, and we get an isometric isomorphism

$$\mathcal{H}_{RK} \overset{T_G}{\to} \mathcal{H} \tag{4}$$

$T_G(\sum_n c_n \varphi_n) = T^* c$ *where T is the frame operator.*

Proof of Theorem 1. Overview: Since $\{\varphi_n\} \subset \mathcal{H}$ is a frame, the Gramian $G_{mn} := \langle \varphi_m, \varphi_n \rangle_{\mathcal{H}}$, is an $\infty \times \infty$ matrix defining a bounded operator $l^2 \to l^2$, invertible with $(G^{-1})_{mn}$ such that

$$\sum_{k=1}^{\infty} (G^{-1})_{mk} \langle \varphi_k, \varphi_n \rangle_{\mathcal{H}} = \delta_{m,n}$$

and the reproducing kernel of $\mathcal{H}$ is $\sum_m \sum_n \overline{\varphi_m(s)} G_{mn}^{-1} \varphi_n(t) = \langle l(s), G^{-1}l(t) \rangle_2$ $\quad \square$

Proof of Theorem 1. (details) By (4) and Lemma 10, the frame operators T and T^* are as follows: Given $\mathcal{H}$, $\{\varphi_n\}$, set

$$\begin{cases} T : \mathcal{H} \to l^2 \\ T^* : l^2 \to \mathcal{H} \end{cases} \tag{5}$$

to be the two linear operators

$$Tf = (\langle \varphi_n, f \rangle_{\mathcal{H}})$$

and adjoint T^* as follows:

$$T^* c = \sum_n c_n \varphi_n$$

Lemma 2. *We have*

$$\langle Tf, c \rangle_{l^2} = \langle f, T^*c \rangle_{\mathcal{H}}, \quad and \quad T^*Tf = \sum \langle \varphi_n, f \rangle \varphi_n, \quad \forall f \in \mathcal{H}, \quad \forall c \in l^2 \tag{6}$$

$$(TT^*c)_n = (Gc)_n = \sum_m G_{nm} c_m, \quad \forall c \in l^2 \tag{7}$$

Do the real case first, then it is easy to extend to complex valued functions.

Note that TT^* is an operator in l^2, *i.e.*,

$$l^2 \overset{TT^*}{\to} l^2$$

It has a matrix-representation as follows

$$(TT^*)_{i,j} = \langle \delta_i, TT^* \delta_j \rangle_{l^2} \qquad (8)$$

Lemma 3. *We have*

$$(TT^*)_{i,j} = G_{i,j} = \langle \varphi_i, \varphi_j \rangle_{\mathcal{H}}, \quad \forall (i,j) \in \mathbb{N} \times \mathbb{N} \qquad (9)$$

Proof of Lemma 3. By (8), we have

$$
\begin{aligned}
(TT^*)_{i,j} &= \langle \delta_i, TT^* \delta_j \rangle_{l^2} \\
&= \langle T^* \delta_i, T^* \delta_j \rangle_{\mathcal{H}} \\
&= \langle \varphi_i, \varphi_j \rangle_{\mathcal{H}} = G_{i,j}
\end{aligned}
$$

which is the desired conclusion (9). $\square$

Both T^*T and TT^* are self-adjoint: If B_i, $i = 1, 2$ are the constants from the frame estimates, then:

$$B_1 \|c\|_2^2 \le \|T^* c\|_{\mathcal{H}}^2 \le B_2 \|c\|_2^2 \quad \forall c \in l^2, \quad \text{and} \qquad (10)$$

$$B_1 \|f\|_{\mathcal{H}}^2 \le \|Tf\|_{l^2}^2 \le B_2 \|f\|_{\mathcal{H}}^2 \quad \forall f \in \mathcal{H} \qquad (11)$$

equivalently

$$B_1 \|f\|_{\mathcal{H}}^2 \le \sum_n |\langle \varphi_n, f \rangle_{\mathcal{H}}|^2 \le B_2 \|f\|_{\mathcal{H}}^2$$

Set

$$K(s,t) = \sum_{n=1}^{\infty} \varphi_n(s)^* \varphi_n(t) = l(s)^* l(t) = \langle l(s), l(t) \rangle_2 \qquad (12)$$

We have

$$B_1 I_{l^2} \le TT^* \le B_2 I_{l^2}, \quad \text{and}$$

$$B_1 I_{\mathcal{H}} \le T^*T \le B_2 I_{\mathcal{H}}$$

If $B_1 = B_2 = 1$, then we say that $\{\varphi_n\}_{n \in \mathbb{N}}$ is a Parseval frame.

For the theory of frames and some of their applications, see e.g., [5,6] and the papers cited there.

By the polar-decomposition theorems, see e.g., [11] we conclude that there is a unitary isomorphism $u : \mathcal{H} \to l^2$ such that $T = u(T^*T)^{1/2} = (TT^*)^{1/2}u$; and so in particular, the two s.a. operators T^*T and TT^* are unitarily equivalent.

Definition 4.

$$l(t) = (\varphi_n(t)) \in l^2 \qquad (13)$$

Therefore $(T^*T)^{-1/2}$ is well defined $\mathcal{H} \to \mathcal{H}$. Now (6) holds if and only if

$$f = \sum \langle (T^*T)^{-1/2}\varphi, f \rangle (T^*T)^{-1/2}\varphi_n$$

or equivalently:

$$f = \sum \langle \psi_n, f \rangle_{\mathcal{H}} \psi_n \tag{14}$$

where

$$\psi_n := (T^*T)^{-1/2}\varphi_n \tag{15}$$

Here we used that T^*T is a selfadjoint operator in $\mathcal{H}$, and it has a positive spectral lower bound; where $\{\varphi_j\}_{j\in\mathbb{N}}$ is assumed to be a frame. $\square$

Lemma 5. *There is an operator $L : \mathcal{H} \to \mathcal{H}$ (the Lax-Milgram operator) such that*

$$\sum_{n=1}^{\infty} \langle f, \varphi_n \rangle_{\mathcal{H}} \langle \varphi_n Lg \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}} \tag{16}$$

holds for all $f \in \mathcal{H}$.

Proof of Lemma 5. We shall apply the Lax-Milgram lemma [11], p. 57 to the sesquilinear form

$$\mathcal{B}(f,g) = \sum_{n=1}^{\infty} \langle f, \varphi_n \rangle_{\mathcal{H}} \langle \varphi_n, g \rangle_{\mathcal{H}}, \quad \forall f, g\mathcal{H} \tag{17}$$

Since $\{\varphi_n\}_{n=1}^{\infty}$ is given to be a frame in $\mathcal{H}$, then our frame-bounds $B_1 > 0$ and $B_2 < \infty$ such that (11) holds. Introducing $\mathcal{B}$ from (17) this into

$$B_1 \|f\|_{\mathcal{H}}^2 \leq \mathcal{B}(f,f) \leq B_2 \|f\|_{\mathcal{H}}^2, \quad \forall f\mathcal{H} \tag{18}$$

The existence of the operator L as stated in (16) now follows from the Lax-Milgram lemma. $\square$

Corollary 6. *Let $\mathcal{H}$, $\{\varphi_n\}$, T, T^* be as in Lemma 8; and let L be the Lax-Milgram operator; then $L = (T^*T)^{-1}$.*

Lemma 7. *The kernel $K^G(\cdot, \cdot)$ on $\Omega \times \Omega$ from (3) is well-defined and positive definite.*

Proof of Lemma 7. We must show that all the finite double summations

$$\sum_i \sum_j \overline{c_i} c_j K^G(t_i, t_j)$$

are ≥ 0, whenever (c_i) is a finite system of coefficients, and (t_i) is a finite sample of points in Ω. Now fix (c_i) and (t_i) as specified, and, for $n \in \mathbb{N}$, set

$$F_n := \sum_i c_i \varphi_n(t_i)$$

then we have the following:

$$\sum_i \sum_j \overline{c_i} c_j K^G(t_i, t_j) = \sum_i \sum_j \overline{c_i} c_j \langle l(t_i), G^{-1} l(t_j) \rangle_{l^2}$$

$$= \sum_i \sum_j \overline{c_i} c_j \sum_m \sum_n \overline{\varphi_m(t_i)} G_{m,n}^{-1} \varphi_n(t_j)$$

$$= \sum_m \sum_n \overline{F_m} G_{m,n}^{-1} F_n \geq 0.$$

□

Lemma 8. *We have the following:*

$$\psi_n(t) = (G^{-1/2}\varphi)_n(t) = \sum_{m=1}^{\infty} (G_{nm}^{-1/2}\varphi_m)(t) = G^{-1/2} l(t)_n \tag{19}$$

and these functions are in the RKHS of the kernel K^G from (3).

Proof of Lemma 8. Begin with (the frame identity):

$$(T^*T)\varphi_n \underset{by(6)}{=} \sum_{m=1}^{\infty} \langle \varphi_m, \varphi_n \rangle \varphi_m = (Gl)_n, \quad \forall n \in \mathbb{Z}_+, \text{ where } l = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \end{bmatrix} \tag{20}$$

if and only if

$$(T^*T)l(t) = G(l(t))$$

Now approximate $\sqrt{x}$ with polynomials (Weierstrass), and we get

$$(T^*T)^{-1/2} l(t) = G^{-1/2} l(t) \tag{21}$$

Recall, $\psi_n = (T^*T)^{-1/2}\varphi_n$. $\psi_n(t) = (G^{-1/2} l(t))_n$. Now rewrite (14) as

$$f(t) = \sum_{n=1}^{\infty} \langle \psi_n, f \rangle_{\mathcal{H}} \psi_n(t) = \sum_{n=1}^{\infty} \langle G^{-1/2}\varphi_n, f \rangle_{\mathcal{H}} (G^{-1/2}\varphi)_n(t) = \langle K_t^G, f \rangle \tag{22}$$

where

$$K_t^G = \sum_{n=1}^{\infty} G^{-1/2}\varphi_n(\cdot)(G^{-1/2}\varphi_n)(t)$$

$$\underset{by(21)}{=} K^G(s, t) = \sum_{n=1}^{\infty} (G^{-1/2}\varphi_n)(s)(G^{-1/2}\varphi_n)(t)$$

$$= \langle G^{-1/2} l(s), G^{-1/2} l(t) \rangle_2$$

$$= \langle l(s), G^{-1} l(t) \rangle_2$$

For the complex case, the result still holds, mutatis mutandis; one only needs to add the complex conjugations. □

Note that (22) is the reproducing property.

Corollary 9. *The function* $(\psi_n(t))$ *from (19) in Lemma 8 satisfy*

$$\sum_{n\in\mathbb{N}} \overline{\psi_n(s)}\psi_n(t) = K^G(s,t), \quad \forall (s,t) \in \Omega \times \Omega. \tag{23}$$

Proof of Corollary 9.

$$
\begin{aligned}
LHS_{(23)} &= \langle G^{-1/2}l(s), G^{-1/2}l(t)\rangle_2 \\
&= \langle l(s), (G^{-1/2})^2 l(t)\rangle_2 \\
&= \langle l(s), G^{-1}(l(t)), \rangle_2 \\
&= K^G(s,t).
\end{aligned}
$$

$\square$

Lemma 10. *The following isometric property holds:*

$$
\begin{aligned}
\left\| \sum_{n=1} c_n\varphi_n(\cdot) \right\|_{\mathcal{H}}^2 &= \sum\sum c_m c_n \langle\varphi_n, \varphi_m\rangle_{\mathcal{H}} \\
&= c^T G c = \langle c, Gc\rangle_2 \\
&= \langle c, TT^*c\rangle_2 = \|T^*c\|_{\mathcal{H}}^2, \quad T^*c \in \mathcal{H}, \quad c \in l^2
\end{aligned}
$$

where T and T^* are the frame operators $\mathcal{H} \xleftrightarrow[T^*]{T} l^2$, i.e., $Tf = (\langle\varphi_n, f\rangle_{\mathcal{H}})_n \in l^2$

Corollary 11. *The Lax operator L satisfies $Lf := \sum_n (T^{*-1}f)_n\varphi_n(\cdot)$, for all $f \in \mathcal{H}$ and it is isometric $\mathcal{H} \to \mathcal{H}$.*

The following partial converse holds to Corollary 9.

Let $\mathcal{H}_{RK}(\Omega)$ be a reproducing kernel Hilbert space associated with a fixed kernel

$$K : \Omega \times \Omega \to \mathbb{C}$$

then a system

$$\{\psi_n(\cdot)\}_{n\in\mathbb{N}} \subset \mathcal{H}_{RK}(\Omega) \tag{24}$$

is a Parseval-frame in $\mathcal{H}_{RK}(\Omega)$ if and only if

$$\sum_{n\in\mathbb{N}} \psi_n(s)\overline{\psi_n(t)} = K(s,t) \tag{25}$$

holds for all $(s,t) \in \Omega \times \Omega$; with absolute convergence in (25).

Proof of Lemma 10. The non-trivial part. Suppose a system of functions $(\varphi_n(\cdot)) \subset \mathcal{H}_{RK}(\Omega)$ is given, and that it is a Parseval frame in $\mathcal{H}_{RK}(\Omega)$. Then for all $f \in \mathcal{H}_{RK}(\Omega)$, we have

$$\|f\|_{RK}^2 = \sum_n |\langle\psi_n, f\rangle_{RK}|^2 . \tag{26}$$

Apply this to $f_t = K(t, \cdot) \in \mathcal{H}_{RK}(\Omega)$; then by polarization of (26), we get for all $(s, t) \in \Omega \times \Omega$,

$$
\begin{aligned}
K(s, t) &= \langle K_s, K_t \rangle_{RK} \\
&= \sum_n \langle K_s, \psi_n \rangle_{RK} \langle \psi_n, K_t \rangle_{RK} \\
&\underset{by(26)}{=} \sum_n \psi_n(s) \overline{\psi_n(t)} \quad \text{by the reproducing property in RK}
\end{aligned}
$$

which is the desired conclusion in (25). $\quad\square$

Example 1. *In the theorem, we assume that the given Hilbert space $\mathcal{H}$ has a frame $\{\varphi_n\} \subset \mathcal{H}$ consisting of functions on a set Ω. So this entails a lower, and an upper frame bound, i.e., $0 < B_1 \leq B_2 < \infty$.*

The following example shows that the conclusion in the theorem is false if there is not a positive lower frame-bound.

Set $\mathcal{H} = L^2(0, 1)$, $\Omega = (0, 1)$ the open unit-inbound, and $\varphi_n(t) = t^n$, $n \in \{0\} \cup \mathbb{N} = \mathbb{N}_0$. In this case, the Gramian

$$
G_{nm} = \int_0^1 x^{n+m} dx = \frac{1}{n + m + 1}
$$

is the $\infty \times \infty$ Hilbert matrix, see ([10,13,14]). In this case it is known that there is an upper frame bound $B_2 = \pi$, i.e.,

$$
\sum_{n=0}^{\infty} \left| \int_0^1 f(x) x^n dx \right|^2 \leq \pi \int_0^1 |f(x)|^2 dx
$$

in fact, for the operator-norm, we have

$$
\|G\|_{l^2 \to l^2} = \pi
$$

but there is not a lower frame bound. Moreover, G define a selfadjoint operator in $l^2(\mathbb{N}_0)$ with spectrum $[0, \pi] =$ the closed interval. This implies that there cannot be a positive lower frame-bound.

Moreover, it is immediate by inspection that $\mathcal{H} = L^2(0, 1)$ is not a RKHS.

3. Frames and Gaussian Processes

In [2] and [3], it was shown that for every positive Borel measure σ on $\mathbb{R}$ such that

$$
\int_{\mathbb{R}} \frac{d\sigma(u)}{1 + u^2} < \infty \tag{27}
$$

there is a unique (up to measure isomorphism) Gaussian proess X as follows:

(i) $X = X_\varphi$ is indexed by the Schwartz-space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ of function φ on $\mathbb{R}$, i.e., $\varphi \in \mathcal{S} \iff \varphi \in \mathbb{C}^\infty$, and for all $N, M \in \mathbb{N}$ we have

$$
\underset{\substack{m \leq M \\ x \in \mathbb{R}}}{maxsup} \left| x^N \left(\frac{\partial}{\partial x} \right)^m \varphi(x) \right| < \infty \tag{28}
$$

with $\mathbb{E}(X_\varphi) = 0$, and $\mathbb{E}(X_\varphi^2) = \int_{\mathbb{R}} |\hat{\varphi}|^2 d\sigma$ for all $\varphi \in \mathcal{S}$.

(ii) Let $\Omega := \mathcal{S}' =$ the dual $=$ the Schwartz space of all tempered distribution, then X_φ is defined on $\mathcal{S}'$, by

$$X_\varphi(w) = w(\varphi), \quad \varphi \in \mathcal{S}, \quad w \in \mathcal{S}'$$

It is real valued Gaussian random variable.

(iii) [2,3] there is a unique measure $\mathbb{P} = \mathbb{P}_\sigma$ on $\mathcal{S}'$ such that

(a) X_φ is Gaussian for all $\varphi \in \mathcal{S}$,

(b)

$$\mathbb{E}(X_\varphi) = 0, \quad \text{and} \tag{29}$$

$$\mathbb{E}_\sigma(e^{iX_\varphi}) = \int_{\mathcal{S}'} e^{iX_\varphi} d\mathbb{P}_\sigma = e^{-\frac{1}{2}\|\widehat{\varphi}\|_\sigma^2} = e^{-\frac{1}{2}\int_{\mathbb{R}} |\widehat{\varphi}(u)|^2 d\sigma(u)} \tag{30}$$

where $\widehat{\varphi} =$ the Fourier transform,

$$\widehat{\varphi}(u) = \int_{\mathbb{R}} e^{ixu} \varphi(x) dx \tag{31}$$

Theorem 12. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a real-valued frame in $L^2(\mathbb{R}, \sigma)$ $(= \{f$ on $\mathbb{R}$, such that $\|f\|_\sigma^2 := \int_{\mathbb{R}} |f(u)|^2 d\sigma(u) < \infty\})$ with frame bounds a, b such that $0 < a \le b < \infty$, so*

$$a\|f\|_\sigma^2 \le \sum_{n\in\mathbb{N}} \left| \int_{\mathbb{R}} f(u) f_n(u) d\sigma(u) \right|^2 \le b\|f\|_\sigma^2 \tag{32}$$

Let $\{B_n\}_{n\in\mathbb{N}}$ be a system of i.i.d. (independent identically distributed) $N(0,1)$ Gaussian random variables on $(\Omega, \mathcal{F}, \mathbb{P}_\sigma)$, and set

$$Y_\varphi(\cdot) = \sum_{n\in\mathbb{N}} \langle f_n, \widehat{\varphi} \rangle_{L^2(\sigma)} B_n(\cdot), \quad \textit{(Karhunen-Loève)} \tag{33}$$

then

$$a \, \mathbb{E}_\sigma \left(|X_\varphi|^2\right) \le \mathbb{E}_\sigma \left(|Y_\varphi|^2\right) \le b \, \mathbb{E}_\sigma \left(|X_\varphi|^2\right) \tag{34}$$

Proof of Theorem 12. Using the i.i.d. $N(0,1)$ property of $\{B_n\}_{n\in\mathbb{N}}$, we get

$$\mathbb{E}_\sigma \left(|Y_\varphi|^2\right) = \sum_{n\in\mathbb{N}} |\langle f_n, \widehat{\varphi} \rangle_{L^2(\sigma)}|^2 \tag{35}$$

The desired conclusion (34) now follows from (32) combined with

$$\mathbb{E}_\sigma \left(|X_\varphi|^2\right) = \|\widehat{\varphi}\|_{L^2(\sigma)}^2, \quad \text{see [2,3]} \tag{36}$$

while (36) is immediate from (30). $\square$

Corollary 13. *The property for $\{Y_\varphi\}_{\varphi\in\mathcal{S}}$ in (33) agrees with $\{X_\varphi\}_{\varphi\in\mathcal{S}}$ if and only if $\{f_n\}_{n\in\mathbb{N}}$ is a Parseval frame in $L^2(\sigma)$.*

Proof of Corollary 13. This follows from the Karhunen-Loève theorem; see [2,3,8,9]. $\square$

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Author Contributions

Palle E. T. Jorgensen and Myung-Sin Song discussed together and wrote the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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