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Abstract: In this paper, inspired by the concept of $b$-metric space, we introduce the concept of extended $b$-metric space. We also establish some fixed point theorems for self-mappings defined on such spaces. Our results extend/generalize many pre-existing results in literature.

Keywords: fixed point; $b$-metric

1. Introduction

The idea of $b$-metric was initiated from the works of Bourbaki [1] and Bakhtin [2]. Czerwik [3] gave an axiom which was weaker than the triangular inequality and formally defined a $b$-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [4] discussed some kind of relaxation in triangular inequality and called this new distance measure as non-linear elastic mathing (NEM). Similar type of relaxed triangle inequality was also used for trade measure [5] and to measure ice floes [6]. All these applications intrigued and pushed us to introduce the concept of extended $b$-metric space. So that the results obtained for such rich spaces become more viable in different directions of applications.

Definition 1. Let $X$ be a non empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called $b$-metric (Bakhtin [2], Czerwik [3]) if it satisfies the following properties for each $x, y, z \in X$.

(b1): $d(x, y) = 0 \iff x = y$;
(b2): $d(x, y) = d(y, x)$;
(b3): $d(x, z) \leq s [d(x, y) + d(y, z)]$.

The pair $(X, d)$ is called a $b$-metric space.

Example 1. 1. Let $X := l_p(\mathbb{R})$ with $0 < p < 1$ where $l_p(\mathbb{R}) := \{ \{ x_n \} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \}$. Define $d : X \times X \to \mathbb{R}^+$ as:

$$d(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

where $x = \{ x_n \}, y = \{ y_n \}$. Then $d$ is a $b$-metric space [7–9] with coefficient $s = 2^{1/p}$.

2. Let $X := L_p[0, 1]$ be the space of all real functions $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ with $0 < p < 1$. Define $d : X \times X \to \mathbb{R}^+$ as:

$$d(x, y) = \left( \int_0^1 |x(t) - y(t)|^p dt \right)^{1/p}$$
Then $d$ is $b$-metric [7–9] with coefficient $s = 2^{1/p}$.

The above examples show that the class of $b$-metric spaces is larger than the class of metric spaces. When $s = 1$, the concept of $b$-metric space coincides with the concept of metric space. For some details on subject see [7–12].

**Definition 2.** Let $(X, d)$ be a $b$-metric space. A sequence $\{x_n\}$ in $X$ is said to be:

(I) Cauchy [12] if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;

(II) Convergent [12] if and only if there exist $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write

$$\lim_{n \to \infty} x_n = x;$$

(III) The $b$-metric space $(X, d)$ is complete [12] if every Cauchy sequence is convergent.

In the following we recollect the extension of Banach contraction principle in case of $b$-metric spaces.

**Theorem 1.** Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$, such that $b$-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in [0, 1)$ such that $ks < 1$. Then $T$ has a unique fixed point [13].

2. Results

In this section, we introduce a new type of generalized metric space, which we call as an extended $b$-metric space. We also establish some fixed point theorems arising from this metric space.

**Definition 3.** Let $X$ be a non empty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended $b$-metric if for all $x, y, z \in X$ it satisfies:

1. $(d_\theta 1)$ $d_\theta(x, y) = 0$ iff $x = y$;
2. $(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;
3. $(d_\theta 3)$ $d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair $(X, d_\theta)$ is called an extended $b$-metric space.

**Remark 1.** If $\theta(x, y) = s$ for $s \geq 1$ then we obtain the definition of a $b$-metric space.

**Example 2.** Let $X = \{1, 2, 3\}$. Define $\theta : X \times X \to \mathbb{R}^+$ and $d_\theta : X \times X \to \mathbb{R}^+$ as:

$$\theta(x, y) = 1 + x + y$$

$$d_\theta(1, 1) = d_\theta(2, 2) = d_\theta(3, 3) = 0$$

$$d_\theta(1, 2) = d_\theta(2, 1) = 80, d_\theta(1, 3) = d_\theta(3, 1) = 1000, d_\theta(2, 3) = d_\theta(3, 2) = 600$$

**Proof.** $(d_\theta 1)$ and $(d_\theta 2)$ trivially hold. For $(d_\theta 3)$ we have:

$$d_\theta(1, 2) = 80, \theta(1, 2)[d_\theta(1, 3) + d_\theta(3, 2)] = 4(1000 + 600) = 6400$$

$$d_\theta(1, 3) = 1000, \theta(1, 3)[d_\theta(1, 2) + d_\theta(2, 3)] = 5(80 + 600) = 3400$$

Similar calculations hold for $d_\theta(2, 3)$. Hence for all $x, y, z \in X$

$$d_\theta(x, z) \leq \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$$

Hence $(X, d_\theta)$ is an extended $b$-metric space. □
Example 3. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions define on $[a,b]$. Note that $X$ is complete extended $b$-metric space by considering $d_\theta(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$, with $\theta(x,y) = |x(t)| + |y(t)| + 2$, where $\theta : X \times X \to [1,\infty)$.

The concepts of convergence, Cauchy sequence and completeness can easily be extended to the case of an extended $b$-metric space.

Definition 4. Let $(X,d_\theta)$ be an extended $b$-metric space.

(i) A sequence $\{x_n\}$ in $X$ is said to converge to $x \in X$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n,x) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in $X$ is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m,x_n) < \epsilon$, for all $m,n \geq N$.

Definition 5. An extended $b$-metric space $(X,d_\theta)$ is complete if every Cauchy sequence in $X$ is convergent.

Note that, in general a $b$-metric is not a continuous functional and thus so is an extended $b$-metric.

Example 4. Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by [14]:

$$d(x,y) = \begin{cases} 0 & \text{if } m = n \\ \frac{1}{m} - \frac{1}{n} & \text{if } m, n \text{ are even or } mn = \infty \\ 5 & \text{if } m, n \text{ are odd and } m \neq n \\ 2 & \text{otherwise} \end{cases}$$

Then $(X,d)$ is a $b$-metric with $s = 3$ but it is not continuous.

Lemma 1. Let $(X,d_\theta)$ be an extended $b$-metric space. If $d_\theta$ is continuous, then every convergent sequence has a unique limit.

Our first theorem is an analogue of Banach contraction principle in the setting of extended $b$-metric space. Throughout this section, for the mapping $T : X \to X$ and $x_0 \in X$, $O(x_0) = \{x_0, T^2x_0, T^3x_0, \cdots \}$ represents the orbit of $x_0$.

Theorem 2. Let $(X,d_\theta)$ be a complete extended $b$-metric space such that $d_\theta$ is a continuous functional. Let $T : X \to X$ satisfy:

$$d_\theta(Tx,Ty) \leq kd_\theta(x,y) \quad \text{for all } x,y \in X$$

where $k \in [0,1)$ be such that for each $x_0 \in X$, $\lim_{n,m\to\infty} \theta(x_n,x_m) < \frac{1}{k}$, here $x_n = T^n x_0$, $n = 1,2,\ldots$. Then $T$ has precisely one fixed point $\xi$. Moreover for each $y \in X$, $T^n y \to \xi$.

Proof. We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0, \ T x_0 = x_1, \ x_2 = T x_1 = T(T x_0) = T^2(x_0), \ldots, x_n = T^n x_0, \ldots$$

Then by successively applying inequality (1) we obtain:

$$d_\theta(x_n,x_{n+1}) \leq k^n d_\theta(x_0,x_1)$$

By triangular inequality and (2), for $m > n$ we have:
Then Theorem 3.

\[
d_\theta(x_n, x_m) \leq \theta(x_n, x_m) k^n d_\theta(x_0, x_1) + \theta(x_n, x_m) \theta(x_{n+1}, x_m) k^{n+1} d_\theta(x_0, x_1) + \ldots + \\
\theta(x_n, x_m) \theta(x_{n+1}, x_m) \theta(x_{n+2}, x_m) \ldots \theta(x_{m-2}, x_m) \theta(x_{m-1}, x_m) k^{m-1} d_\theta(x_0, x_1)
\]

\[
\leq d_\theta(x_0, x_1) \left[ \theta(x_1, x_m) \theta(x_2, x_m) \ldots \theta(x_{n}, x_m) \theta(x_{n+1}, x_m) k^n + \right. \\
\left. \theta(x_1, x_m) \theta(x_2, x_m) \ldots \theta(x_{n}, x_m) \theta(x_{n+1}, x_m) k^{n+1} + \ldots + \right. \\
\left. \theta(x_1, x_m) \theta(x_2, x_m) \ldots \theta(x_{n}, x_m) \theta(x_{n+1}, x_m) \theta(x_{m-1}, x_m) k^{m-1} \right]
\]

Since, \( \lim_{n,m \to \infty} \theta(x_{n+1}, x_m) k < 1 \) so that the series \( \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x_m) \) converges by ratio test for each \( m \in \mathbb{N} \). Let:

\[
S = \sum_{n=1}^{\infty} k^n \prod_{i=1}^{n} \theta(x_i, x_m), \quad S_n = \sum_{j=1}^{n} k^j \prod_{i=1}^{j} \theta(x_i, x_m)
\]

Thus for \( m > n \) above inequality implies:

\[
d_\theta(x_n, x_m) \leq d_\theta(x_0, x_1) \left[ S_{m-1} - S_n \right]
\]

Letting \( n \to \infty \) we conclude that \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete let \( x_n \to \xi \in X \):

\[
d_\theta(T_\xi, \xi) \leq \theta(T_\xi, \xi) [d_\theta(T_\xi, x_n) + d_\theta(x_n, \xi)] \\
\leq \theta(T_\xi, \xi) [kd_\theta(\xi, x_{n-1}) + d_\theta(x_n, \xi)] \\
d_\theta(T_\xi, \xi) \leq 0 \quad \text{as} \quad n \to \infty \\
d_\theta(T_\xi, \xi) = 0
\]

Hence \( \xi \) is a fixed point of \( T \). Moreover uniqueness can easily be invoked by using inequality (1), since \( k < 1 \). \( \Box \)

In the following we include another variant which is analogue to fixed point theorem by Hicks and Rhoades [15]. We need the following definition.

**Definition 6.** Let \( T : X \to X \) and for some \( x_0 \in X \), \( O(x_0) = \{x_0, f x_0, f^2 x_0, \ldots \} \) be the orbit of \( x_0 \). A function \( G \) from \( X \) into the set of real numbers is said to be \( T \)-orbitally lower semi-continuous at \( t \in X \) if \( \{x_n\} \subset O(x_0) \) and \( x_n \to t \) implies \( G(t) \leq \lim_{n \to \infty} \inf G(x_n) \).

**Theorem 3.** Let \((X, d_\theta)\) be a complete extended \( b \)-metric space such that \( d_\theta \) is a continuous functional. Let \( T : X \to X \) and there exists \( x_0 \in X \) such that:

\[
d_\theta(Ty, T^2 y) \leq kd_\theta(y, Ty) \quad \text{for each} \quad y \in O(x_0)
\]

where \( k \in [0, 1) \) be such that for \( x_0 \in X \), \( \lim_{n,m \to \infty} \theta(x_n, x_m) < \frac{1}{k} \), here \( x_n = T^n x_0 \), \( n = 1, 2, \ldots \). Then \( T^n x_0 \to \xi \in X \) (as \( n \to \infty \)). Furthermore \( \xi \) is a fixed point of \( T \) if and only if \( G(x) = d(x, Tx) \) is \( T \)-orbitally lower semi continuous at \( \xi \).

**Proof.** For \( x_0 \in X \) we define the iterative sequence \( \{x_n\} \) by:

\[
x_0, \quad T x_0 = x_1, \quad x_2 = T x_1 = T(T x_0) = T^2(x_0), \ldots, x_n = T^n x_0, \ldots
\]

Now for \( y = T x_0 \) by successively applying inequality (3) we obtain:

\[
d_\theta(T^n x_0, T^{n+1} x_0) = d_\theta(x_n, x_{n+1}) \leq k^n d_\theta(x_0, x_1)
\]
Following the same procedure as in the proof of Theorem 2 we conclude that \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete then \( x_n = T^n x_0 \to \xi \in X \). Assume that \( G \) is orbitally lower semi continuous at \( \xi \in X \), then:

\[
d_{\theta}(\xi, T^2 \xi) \leq \liminf_{n\to\infty} d_{\theta}(T^n x_0, T^{n+1} x_0) \leq \liminf_{n\to\infty} k^n d_{\theta}(x_0, x_1) = 0
\]  

(5) (6)

Conversely, let \( \xi = T^2 \xi \) and \( x_n \in O(x) \) with \( x_n \to \xi \). Then:

\[
G(\xi) = d(\xi, T^2 \xi) = 0 \leq \liminf_{n \to \infty} G(x_n) = d(T^n x_0, T^{n+1} x_0)
\]  

(7)

\[\square\]

Remark 2. When \( \theta(x, y) = 1 \) a constant function then Theorem 3 reduces to main result of Hicks and Rhoades ([15] (Theorem 1)). Hence Theorem 3 extends/generalizes ([15] (Theorem 1)).

Example 5. Let \( X = [0, \infty) \). Define \( d_{\theta}(x, y) : X \times X \to \mathbb{R}^+ \) and \( \theta : X \times X \to [1, \infty) \) as:

\[
d_{\theta}(x, y) = (x - y)^2, \quad \theta(x, y) = x + y + 2
\]

Then \( d_{\theta} \) is a complete extended \( b \)-metric on \( X \). Define \( T : X \to X \) by \( Tx = \frac{x}{2} \). We have:

\[
d_{\theta}(Tx, Ty) = \left( \frac{x}{2} - \frac{y}{2} \right)^2 \leq \frac{1}{3} (x - y)^2 = k d_{\theta}(x, y)
\]

Note that for each \( x \in X \), \( T^n x = \frac{x}{2^n} \). Thus we obtain:

\[
\lim_{m, n \to \infty} \theta(T^m x, T^n x) = \lim_{m, n \to \infty} \left( \frac{x}{2^n} + \frac{x}{2^m} + 2 \right) < 3
\]

Therefore, all conditions of Theorem 3 are satisfied hence \( T \) has a unique fixed point.

Example 6. Let \( X = [0, \frac{1}{2}] \). Define \( d_{\theta}(x, y) : X \times X \to \mathbb{R}^+ \) and \( \theta : X \times X \to [1, \infty) \) as:

\[
d_{\theta}(x, y) = (x - y)^2, \quad \theta(x, y) = x + y + 2
\]

Then \( d_{\theta} \) is a complete extended \( b \)-metric on \( X \). Define \( T : X \to X \) by \( Tx = x^2 \). We have:

\[
d_{\theta}(Tx, Ty) \leq \frac{1}{4} d_{\theta}(x, y)
\]

Note that for each \( x \in X \), \( T^n x = x^{2^n} \). Thus we obtain:

\[
\lim_{m, n \to \infty} \theta(T^m x, T^n x) < 4
\]

Therefore, all conditions of Theorem 3 are satisfied hence \( T \) has a unique fixed point.

3. Application

In this section, we give existence theorem for Fredholm integral equation. Let \( X = C([a, b], \mathbb{R}) \) be the space of all continuous real valued functions define on \( [a, b] \). Note that \( X \) is complete extended \( b \)-metric space by considering \( d_{\theta}(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|^2 \), with \( \theta(x, y) = |x(t)| + |y(t)| + 2 \), where \( \theta : X \times X \to [1, \infty) \). Consider the Fredholm integral equation as:
\[ x(t) = \int_a^b M(t, s, x(s)) ds + g(t), \quad t, s \in [a, b] \]  \hspace{1cm} (8)

where \( g : [a, b] \to \mathbb{R} \) and \( M : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R} \) are continuous functions. Let \( T : X \to X \) the operator given by:

\[ Tx(t) = \int_a^b M(t, s, x(s)) ds + g(t) \quad \text{for} \quad t, s \in [a, b] \]

where, the function \( g : [a, b] \to \mathbb{R} \) and \( M : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R} \) are continuous. Further, assume that the following condition hold:

\[ |M(t, s, x(s)) - M(t, s, Tx(s))| \leq \frac{1}{2} |x(s) - Tx(s)| \quad \text{for each} \quad t, s \in [a, b] \quad \text{and} \quad x \in X \]

Then the integral Equation (8) has a solution.

We have to show that the operator \( T \) satisfies all the conditions of Theorem 3. For any \( x \in X \) we have:

\[
|Tx(t) - T(Tx(t))|^2 \leq \left( \int_a^b |M(t, s, x(s)) - M(t, s, Tx(s))| ds \right)^2 \\
\leq \frac{1}{4} d_\theta(x, Tx)
\]

All conditions of Theorem 3 follows by the hypothesis. Therefore, the operator \( T \) has a fixed point, that is, the Fredholm integral Equation (8) has a solution.

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