mathematics

Article

Coefficient Inequalities of Functions Associated with Petal Type Domains

Sarfraz Nawaz Malik [1,*], Shahid Mahmood [2], Mohsan Raza [3], Sumbal Farman [1] and Saira Zainab [4]

1 Department of Mathematics, COMSATS University Islamabad, Wah Campus 47040, Pakistan; sumbalfarman678@gmail.com
2 Department of Mechanical Engineering, Sarhad University of Science and I.T, Ring Road, Peshawar 25000, Pakistan; shahidmahmood757@gmail.com
3 Department of Mathematics, Government College University Faisalabad, Faisalabad 38000, Pakistan; mohsan976@yahoo.com
4 Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi 46000, Pakistan; sairazainab07@yahoo.com
* Correspondence: snmalik110@yahoo.com

Received: 13 November 2018; Accepted: 1 December 2018; Published: 3 December 2018

Abstract: In the theory of analytic and univalent functions, coefficients of functions' Taylor series representation and their related functional inequalities are of major interest and how they estimate functions' growth in their specified domains. One of the important and useful functional inequalities is the Fekete-Szegö inequality. In this work, we aim to analyze the Fekete-Szegö functional and to find its upper bound for certain analytic functions which give parabolic and petal type regions as image domains. Coefficient inequalities and the Fekete-Szegö inequality of inverse functions to these certain analytic functions are also established in this work.

Keywords: analytic functions; starlike functions; convex functions; Fekete-Szegö inequality

MSC: Primary 30C45, 33C10; Secondary 30C20, 30C75

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ be the class of functions from $\mathcal{A}$ which are univalent in $\mathcal{U}$. One of the classical results regarding univalent functions related to coefficients a_n of a function's Taylor series, named as the Fekete-Szegö problem, introduced by Fekete and Szegö [1], is defined as follows:

If $f \in \mathcal{S}$ and is of the form (1), then

$$\left|a_3 - \lambda a_2^2\right| \leq \begin{cases} 3 - 4\lambda, & \text{if} \quad \lambda \leq 0, \\ 1 + 2\exp\left(\frac{2\lambda}{\lambda - 1}\right), & \text{if} \quad 0 \leq \lambda \leq 1, \\ 4\lambda - 3, & \text{if} \quad \lambda \geq 1. \end{cases}$$

This result is sharp. The Fekete-Szegö problem has a rich history in literature. Several results dealing with maximizing the non-linear functional $\left|a_3 - \lambda a_2^2\right|$ for various classes and subclasses of univalent functions have been proved. The functional has been examined for λ to be both a real and complex number. Several authors used certain classified techniques to maximize the Fekete-Szegö functional $|a_3 - \lambda a_2^2|$ for different types of functions having interesting geometric characteristics of image domains. For more details and results, we refer to [1–11]. The function f is said to be subordinate to the function g, written symbolically as $f \prec g$, if there exists a schwarz function w such that

$$f(z) = g(w(z)), \quad z \in \mathcal{U}, \tag{2}$$

where $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathcal{U}$. Let P denote the class of analytic functions p such that $p(0) = 1$ and $p \prec \frac{1+z}{1-z}$, $z \in \mathcal{U}$. For details, see [12].

In 1991, Goodman [13] initiated the concept of a conic domain by introducing generalized convex functions which generated the first parabolic region as an image domain of analytic functions. He introduced and defined the class UCV of uniformly convex functions as follows:

$$UCV = \left\{ f \in \mathcal{A} : \Re\left(1 + (z - \zeta)\frac{f''(z)}{f'(\zeta)}\right) > 0,\ z, \zeta \in \mathcal{U} \right\}.$$

Later on, Rønning [14], and Ma and Minda [7] independently gave the most suitable one variable characterization of the class UCV and defined it as follows:

$$UCV = \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}.$$

This characterization gave birth to the first conic (parabolic) domain

$$\Omega = \{w : \Re w > |w - 1|\}.$$

This domain was then generalized by Kanas and Wiśniowska [15,16] who introduced the domain

$$\Omega_k = \{w : \Re w > k|w - 1|,\ k \geq 0\}.$$

The conic domain Ω_k represents the right half plane for $k = 0$, hyperbolic regions when $0 < k < 1$, parabolic region for $k = 1$ and elliptic regions when $k > 1$. For more details, we refer [15,16]. This conic domain Ω_k has been extensively studied in [17–19]. The domain Ω was also generalized by Noor and Malik [20] by introducing the domain

$$\begin{aligned}\Omega[A,B] = \Big\{ u + iv : \Big[\left(B^2 - 1\right)\left(u^2 + v^2\right) - 2(AB - 1)u + \left(A^2 - 1\right)\Big]^2 \\ > \left(-2(B+1)\left(u^2+v^2\right) + 2(A+B+2)u - 2(A+1)\right)^2 + 4(A-B)^2 v^2 \Big\}.\end{aligned}$$

The domain $\Omega[A,B]$ represents the petal type region, for more details, we refer to [20]. Now, we consider the following class of functions which take all values from the domain $\Omega[A,B]$, $-1 \leq B < A \leq 1$.

Definition 1. *A function $p(z)$ is said to be in the class $UP[A,B]$, if and only if*

$$p(z) \prec \frac{(A+1)\widetilde{p}(z) - (A-1)}{(B+1)\widetilde{p}(z) - (B-1)}, \quad -1 \leq B < A \leq 1, \tag{3}$$

where $\widetilde{p}(z) = 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$, $z \in \mathcal{U}$.

It can be seen that $\Omega[1,-1] = \Omega_1 = \Omega$. This fact leads us to the following implications of different well-known classes of analytic functions.

1. $UP[A,B] \subset P\left(\frac{3-A}{3-B}\right)$, the well-known class of functions with real part greater than $\frac{3-A}{3-B}$, see [12].
2. $UP[1,-1] = \mathcal{P}(\widetilde{p})$, the well-known class of functions, introduced by Kanas and Wiśniowska [4,21].

Now we consider the following classes $UCV[A,B]$ of uniformly Janowski convex functions and $ST[A,B]$ of corresponding Janowski starlike functions (see [20]) as follows.

Definition 2. *A function $f \in \mathcal{A}$ is said to be in the class $UCV[A,B]$, $-1 \le B < A \le 1$, if and only if*

$$\Re\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}\right) > \left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)}-(A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{(zf'(z))'}{f'(z)} \in UP[A,B]. \tag{4}$$

Definition 3. *A function $f \in \mathcal{A}$ is said to be in the class $ST[A,B]$, $-1 \le B < A \le 1$, if and only if*

$$\Re\left(\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}\right) > \left|\frac{(B-1)\frac{zf'(z)}{f(z)}-(A-1)}{(B+1)\frac{zf'(z)}{f(z)}-(A+1)}-1\right|,$$

or equivalently,

$$\frac{zf'(z)}{f(z)} \in UP[A,B]. \tag{5}$$

It can easily be seen that $f \in UCV[A,B] \iff zf' \in ST[A,B]$. It is clear that $UCV[1,-1] = UCV$ and $ST[1,-1] = ST$, the well-known classes of uniformly convex and corresponding starlike functions respectively, introduced by Goodman [13] and Rønning [22].

In 1994, Ma and Minda [7] found the maximum bound of Fekete-Szegö functional $\left|a_3 - \lambda a_2^2\right|$ for uniformly convex functions of class UCV and then Kanas [21] investigated the same for the functions of class $\mathcal{P}(\widetilde{p})$. Our aim is to solve this classical Fekete-Szegö problem for the functions of classes $UP[A,B]$, $UCV[A,B]$ and $ST[A,B]$. We need the following lemmas (see [7]) to prove our results.

Lemma 1. *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any complex number μ,*

$$\left|p_2 - \mu p_1^2\right| \le 2\max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions

$$p_0(z) = \frac{1+z}{1-z} \quad \text{or} \quad p_*(z) = \frac{1+z^2}{1-z^2}, \qquad (z \in \mathcal{U}).$$

Lemma 2. *If $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ is a function with positive real part in $\mathcal{U}$, then, for any real number v,*

$$\left|p_2 - v p_1^2\right| \le \begin{cases} -4v+2, & v \le 0, \\ 2, & 0 \le v \le 1, \\ 4v-2, & v \ge 1. \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < v < 1$, then, the equality holds if and only if $p(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if,

$$p(z) = \left(\frac{1+\eta}{2}\right)\frac{1+z}{1-z} + \left(\frac{1-\eta}{2}\right)\frac{1-z}{1+z} \quad (0 \le \eta \le 1),$$

or one of its rotations. If $v = 1$, then, the equality holds if and only if $p(z)$ is reciprocal of one of the function such that equality holds in the case of $v = 0$. Although the above upper bound is sharp, when $0 < v < 1$, it can be improved as follows:

$$\left|p_2 - vp_1^2\right| + |p_1|^2 \le 2 \qquad \left(0 < v \le \frac{1}{2}\right)$$

and

$$\left|p_2 - vp_1^2\right| + (1-v)|p_1|^2 \le 2 \quad \left(\frac{1}{2} < v \le 1\right).$$

2. Main Results

Theorem 1. *Let $p \in UP[A,B]$, $-1 \le B < A \le 1$ and of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \le \frac{4}{\pi^2}(A-B).\max\left(1, \left|\frac{4}{\pi^2}(B+1) - \frac{2}{3} + 4\mu\left(\frac{A-B}{\pi^2}\right)\right|\right) \tag{6}$$

and for a real number μ, we have

$$\left|p_2 - \mu p_1^2\right| \le \frac{2(A-B)}{\pi^2} \begin{cases} \frac{4}{3} - \frac{8}{\pi^2}(B+1) - \frac{8}{\pi^2}(A-B)\mu, & \mu \le -\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B}, \\ 2, & -\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B} \le \mu \le \frac{5\pi^2}{12(A-B)} - \frac{B+1}{A-B}, \\ -\frac{4}{3} + \frac{8}{\pi^2}(B+1) + \frac{8}{\pi^2}(A-B)\mu, & \mu \ge \frac{5\pi^2}{12(A-B)} - \frac{B+1}{A-B}. \end{cases} \tag{7}$$

These results are sharp and the equality in (6) holds for the functions

$$p_1(z) = \frac{\frac{2(A+1)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 + 2}{\frac{2(B+1)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 + 2} \tag{8}$$

or

$$p_2(z) = \frac{\frac{2(A+1)}{\pi^2}\left(\log\frac{1+z}{1-z}\right)^2 + 2}{\frac{2(B+1)}{\pi^2}\left(\log\frac{1+z}{1-z}\right)^2 + 2}. \tag{9}$$

When $\mu < -\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B}$ or $\mu > \frac{5\pi^2}{12(A-B)} - \frac{B+1}{A-B}$, the equality in (7) holds for the function $p_1(z)$ or one of its rotations. If $-\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B} < \mu < \frac{5\pi^2}{12(A-B)} - \frac{B+1}{A-B}$, then, the equality in (7) holds for the function $p_2(z)$ or one of its rotations. If $\mu = -\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B}$, the equality in (7) holds for the function

$$p_3(z) = \left(\frac{1+\eta}{2}\right)p_1(z) + \left(\frac{1-\eta}{2}\right)p_1(-z), \quad (0 \le \eta \le 1), \tag{10}$$

or one of its rotations. If $\mu = \frac{5\pi^2}{12(A-B)} - \frac{B+1}{A-B}$, then, the equality in (7) holds for the functions $p(z)$ which is reciprocal of one of the function such that equality holds in the case for $\mu = -\frac{\pi^2}{12(A-B)} - \frac{B+1}{A-B}$.

Proof. For $h \in P$ and of the form $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, we consider

$$h(z) = \frac{1 + w(z)}{1 - w(z)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. It follows easily that

$$\begin{aligned} w(z) &= \frac{h(z) - 1}{h(z) + 1} \\ &= \frac{(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots) - 1}{(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots) + 1} \\ &= \frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots . \end{aligned} \tag{11}$$

Now, if $\widetilde{p}(z) = 1 + R_1 z + R_2 z^2 + \cdots$, then from (11), one may have

$$\begin{aligned} \widetilde{p}(w(z)) &= 1 + R_1 w(z) + R_2 (w(z))^2 + R_3 (w(z))^3 + \cdots \\ &= 1 + R_1 \left(\frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots\right) \\ &\quad + R_2 \left(\frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots\right)^2 \\ &\quad + R_3 \left(\frac{1}{2} c_1 z + \left(\frac{1}{2} c_2 - \frac{1}{4} c_1^2\right) z^2 + \left(\frac{1}{2} c_3 - \frac{1}{2} c_2 c_1 + \frac{1}{8} c_1^3\right) z^3 + \cdots\right)^3 + \cdots , \end{aligned}$$

where $R_1 = \frac{8}{\pi^2}$, $R_2 = \frac{16}{3\pi^2}$ and $R_3 = \frac{184}{45\pi^2}$, see [21]. Using these, the above series reduces to

$$\widetilde{p}(w(z)) = 1 + \frac{4}{\pi^2} c_1 z + \frac{4}{\pi^2} \left(c_2 - \frac{1}{6} c_1^2\right) z^2 + \frac{4}{\pi^2} \left(c_3 - \frac{1}{3} c_2 c_1 + \frac{2}{45} c_1^3\right) z^3 + \cdots . \tag{12}$$

Since $p \in UP[A, B]$, so from relations (2), (3) and (12), one may have

$$\begin{aligned} p(z) &= \frac{(A+1)\widetilde{p}(w(z)) - (A-1)}{(B+1)\widetilde{p}(w(z)) - (B-1)} \\ &= \frac{2 + (A+1)\frac{4}{\pi^2} c_1 z + (A+1)\frac{4}{\pi^2}\left(c_2 - \frac{1}{6} c_1^2\right) z^2 + \cdots}{2 + (B+1)\frac{4}{\pi^2} c_1 z + (B+1)\frac{4}{\pi^2}\left(c_2 - \frac{1}{6} c_1^2\right) z^2 + \cdots}. \end{aligned}$$

This implies that

$$\begin{aligned} p(z) &= 1 + (A-B)\frac{2}{\pi^2} c_1 z + (A-B)\frac{2}{\pi^2}\left(c_2 - \frac{1}{6} c_1^2 - \frac{2}{\pi^2}(B+1) c_1^2\right) z^2 + \\ &\quad (A-B)\frac{8}{\pi^2}\left(\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right) c_1^3 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right) c_2 c_1 + \frac{1}{4} c_3\right) z^3 + \cdots . \end{aligned} \tag{13}$$

If $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then equating coefficients of z and z^2, one may have

$$\begin{aligned} p_1 &= \frac{2}{\pi^2}(A-B) c_1, \\ p_2 &= \frac{2}{\pi^2}(A-B)\left(c_2 - \frac{1}{6} c_1^2 - \frac{2}{\pi^2}(B+1) c_1^2\right). \end{aligned}$$

Now for a complex number μ, consider

$$p_2 - \mu p_1^2 = \frac{2}{\pi^2}(A-B)\left[c_2 - c_1^2\left(\frac{1}{6} + \frac{2}{\pi^2}(B+1) + \mu\frac{2}{\pi^2}(A-B)\right)\right].$$

This implies that

$$\left|p_2 - \mu p_1^2\right| = \frac{2}{\pi^2}(A-B)\left|c_2 - \left(\frac{1}{6} + \frac{2}{\pi^2}(B+1) + \mu\frac{2}{\pi^2}(A-B)\right)c_1^2\right|. \tag{14}$$

Using Lemma 1, one may have

$$\left|p_2 - \mu p_1^2\right| \le \frac{2}{\pi^2}(A-B).2\max(1, |2v-1|),$$

where

$$v = \frac{1}{6} + \frac{2}{\pi^2}(B+1) + \mu\frac{2}{\pi^2}(A-B).$$

This leads us to the required inequality (6) and applying Lemma 2 to the expression (14) for real number μ, we get the required inequality (7). Sharpness follows from the functions $p_i(z)\,;\ i = 1,2,3,$ defined by (8)–(10), and the following series form.

$$\begin{aligned} p_1(z) &= 1 + \frac{4(A-B)}{\pi^2}z + \frac{8(A-B)}{\pi^2}\left(\frac{1}{3} - \frac{2(B+1)}{\pi^2}\right)z^2 + \\ &\quad \frac{16(A-B)}{\pi^2}\left(4\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right) - 2\left(\frac{B+1}{\pi^2} + \frac{1}{12}\right) + \frac{1}{4}\right)z^3 + \cdots, \end{aligned}$$

$$\begin{aligned} p_2(z) &= 1 + \frac{4(A-B)}{\pi^2}z^2 + \frac{8(A-B)}{\pi^2}\left(\frac{1}{3} - \frac{2(B+1)}{\pi^2}\right)z^4 + \\ &\quad \frac{16(A-B)}{\pi^2}\left(4\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right) - 2\left(\frac{B+1}{\pi^2} + \frac{1}{12}\right) + \frac{1}{4}\right)z^6 + \cdots. \end{aligned}$$

□

Corollary 1. *Let $p \in UP[1,-1] = \mathcal{P}(p_1) = \mathcal{P}(\tilde{p})$ and of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$. Then, for a complex number μ, we have*

$$\left|p_2 - \mu p_1^2\right| \le \frac{8}{\pi^2}.\max\left(1, \left|\frac{8\mu}{\pi^2} - \frac{2}{3}\right|\right) \tag{15}$$

and for real number μ, we have

$$\left|p_2 - \mu p_1^2\right| \le \frac{4}{\pi^2}\begin{cases} \frac{4}{3} - \frac{16}{\pi^2}\mu, & \mu \le -\frac{\pi^2}{24}, \\ 2, & -\frac{\pi^2}{24} \le \mu \le \frac{5\pi^2}{24}, \\ -\frac{4}{3} + \frac{16}{\pi^2}\mu, & \mu \ge \frac{5\pi^2}{24}. \end{cases} \tag{16}$$

These inequalities are sharp.

In [4,21], Kanas studied the class $\mathcal{P}(p_k)$ which consists of functions who take all values from the conic domain Ω_k. Kanas [21] found the bound of Fekete-Szegö functional for the class $\mathcal{P}(p_k)$ whose particular case for $k = 1$ is as follows:

Let $p(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \in \mathcal{P}(p_1)$. Then, for real number μ, we have

$$\left| b_2 - \mu b_1^2 \right| \le \frac{8}{\pi^2} \begin{cases} 1 - \frac{8}{\pi^2}\mu, & \mu \le 0, \\ 1, & \mu \in (0,1], \\ 1 + \frac{8}{\pi^2}(\mu - 1), & \mu \ge 1. \end{cases} \tag{17}$$

We observe that Corollary 1 improves the bounds of the Fekete-Szegö functional $\left|p_2 - \mu p_1^2\right|$ for the functions of class $\mathcal{P}(p_1)$.

Theorem 2. *Let $f \in UCV[A,B]$, $-1 \le B < A \le 1$ and of the form (1). Then, for a real number μ, we have*

$$\left| a_3 - \mu a_2^2 \right| \le \frac{A-B}{3\pi^2} \begin{cases} \frac{4}{3} - \frac{8}{\pi^2}(B+1) + \frac{4}{\pi^2}(A-B)(2-3\mu), & \mu \le \frac{2}{3} - \frac{\pi^2}{18(A-B)} - \frac{2(B+1)}{3(A-B)}, \\ 2, & \frac{2}{3} - \frac{\pi^2}{18(A-B)} - \frac{2(B+1)}{3(A-B)} \le \mu \le \frac{2}{3} + \frac{5\pi^2}{18(A-B)} - \frac{2(B+1)}{3(A-B)}, \\ -\frac{4}{3} + \frac{8}{\pi^2}(B+1) - \frac{4}{\pi^2}(A-B)(2-3\mu), & \mu \ge \frac{2}{3} + \frac{5\pi^2}{18(A-B)} - \frac{2(B+1)}{3(A-B)}. \end{cases} \tag{18}$$

This result is sharp.

Proof. If $f \in UCV[A,B]$, $-1 \le B < A \le 1$, then it follows from relations (2)–(4),

$$\frac{(zf'(z))'}{f'(z)} = \frac{(A+1)\widetilde{p}(w(z)) - (A-1)}{(B+1)\widetilde{p}(w(z)) - (B-1)},$$

where $w(z)$ is such that $w(0) = 0$ and $|w(z)| < 1$. The right hand side of above expression gets its series form from (13) and reduces to

$$\begin{aligned} \frac{(zf'(z))'}{f'(z)} &= 1 + (A-B)\frac{2}{\pi^2}c_1 z + (A-B)\frac{2}{\pi^2}\left(c_2 - \frac{1}{6}c_1^2 - \frac{2}{\pi^2}(B+1)c_1^2\right)z^2 + \\ &\quad (A-B)\frac{8}{\pi^2}\left(\left(\frac{(B+1)^2}{\pi^4} + \frac{B+1}{6\pi^2} + \frac{1}{90}\right)c_1^3 - \left(\frac{B+1}{\pi^2} + \frac{1}{12}\right)c_2 c_1 + \frac{1}{4}c_3\right)z^3 + \cdots. \end{aligned} \tag{19}$$

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then one may have

$$\frac{(zf'(z))'}{f'(z)} = 1 + 2a_2 z + \left(6a_3 - 4a_2^2\right)z^2 + \left(12a_4 - 18a_2 a_3 + 8a_2^3\right)z^3 + \cdots. \tag{20}$$

From (19) and (20), comparison of coefficients of z and z^2 gives

$$a_2 = \frac{1}{\pi^2}(A-B)c_1 \tag{21}$$

and

$$6a_3 - 4a_2^2 = (A-B)\frac{2}{\pi^2}\left(c_2 - \frac{1}{6}c_1^2 - \frac{2}{\pi^2}(B+1)c_1^2\right).$$

This implies, by using (21), that

$$a_3 = \frac{1}{3\pi^2}(A-B)\left(c_2 - \frac{1}{6}c_1^2 - \frac{2}{\pi^2}(B+1)c_1^2 + \frac{2}{\pi^2}(A-B)c_1^2\right). \tag{22}$$

Now, for a real number μ, consider

$$\begin{aligned}\left|a_3-\mu a_2^2\right| &= \left|(A-B)\frac{1}{3\pi^2}\left(c_2-\frac{1}{6}c_1^2-\frac{2}{\pi^2}(B+1)c_1^2\right)+\frac{2}{3\pi^4}(A-B)^2c_1^2-\mu\frac{1}{\pi^4}(A-B)^2c_1^2\right| \\ &= \frac{A-B}{3\pi^2}\left|c_2-c_1^2\left(\frac{1}{6}+\frac{2}{\pi^2}(B+1)-\frac{2}{\pi^2}(A-B)+\frac{3\mu}{\pi^2}(A-B)\right)\right| \\ &= \frac{A-B}{3\pi^2}\left|c_2-vc_1^2\right|,\end{aligned}$$

where

$$v=\frac{1}{6}+\frac{2}{\pi^2}(B+1)-\frac{1}{\pi^2}(A-B)(2-3\mu).$$

Applying Lemma 2 leads us to the required result. The inequality (18) is sharp and equality holds for $\mu<\frac{2}{3}-\frac{\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$ or $\mu>\frac{2}{3}+\frac{5\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$ when $f(z)$ is $f_1(z)$ or one of its rotations, where $f_1(z)$ is defined such that $\frac{(zf_1'(z))'}{f_1'(z)}=p_1(z)$. If $\frac{2}{3}-\frac{\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}<\mu<\frac{2}{3}+\frac{5\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$, then, the equality holds for the function $f_2(z)$ or one of its rotations, where $f_2(z)$ is defined such that $\frac{(zf_2'(z))'}{f_2'(z)}=p_2(z)$. If $\mu=\frac{2}{3}-\frac{\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$, the equality holds for the function $f_3(z)$ or one of its rotations, where $f_3(z)$ is defined such that $\frac{(zf_3'(z))'}{f_3'(z)}=p_3(z)$. If $\mu=\frac{2}{3}+\frac{5\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$, then, the equality holds for $f(z)$, which is such that $\frac{(zf'(z))'}{f'(z)}$ is reciprocal of one of the function such that equality holds in the case of $\mu=\frac{2}{3}-\frac{\pi^2}{18(A-B)}-\frac{2(B+1)}{3(A-B)}$. □

For $A=1,\ B=-1$, the above result takes the following form which is proved by Ma and Minda [8].

Corollary 2. *Let $f\in UCV[1,-1]=UCV$ and of the form (1). Then, for a real number μ,*

$$\left|a_3-\mu a_2^2\right|\le\frac{2}{3\pi^2}\begin{cases}\frac{4}{3}+\frac{8}{\pi^2}(2-3\mu), & \mu\le\frac{2}{3}-\frac{\pi^2}{36}, \\ 2, & \frac{2}{3}-\frac{\pi^2}{36}\le\mu\le\frac{2}{3}+\frac{5\pi^2}{36}, \\ -\frac{4}{3}-\frac{8}{\pi^2}(2-3\mu), & \mu\ge\frac{2}{3}+\frac{5\pi^2}{36}.\end{cases}$$

This result is sharp.

Theorem 3. *Let $f\in ST[A,B]$, $-1\le B<A\le 1$ and of the form (1). Then, for a real number μ,*

$$\left|a_3-\mu a_2^2\right|\le\frac{A-B}{\pi^2}\begin{cases}\frac{4}{3}-\frac{8}{\pi^2}(B+1)+\frac{8}{\pi^2}(A-B)(1-2\mu), & \mu\le\frac{1}{2}-\frac{\pi^2}{24(A-B)}-\frac{B+1}{2(A-B)}, \\ 2, & \frac{1}{2}-\frac{\pi^2}{24(A-B)}-\frac{B+1}{2(A-B)}\le\mu\le\frac{1}{2}+\frac{5\pi^2}{24(A-B)}-\frac{B+1}{2(A-B)}, \\ -\frac{4}{3}+\frac{8}{\pi^2}(B+1)-\frac{8}{\pi^2}(A-B)(1-2\mu), & \mu\ge\frac{1}{2}+\frac{5\pi^2}{24(A-B)}-\frac{B+1}{2(A-B)}.\end{cases} \quad (23)$$

This result is sharp.

Proof. The proof follows similarly as in Theorem 2. □

For $A=1,\ B=-1$, the above result reduces to the following form.

Corollary 3. *Let $f \in ST[1,-1]$ and of the form (1). Then, for a real number μ,*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{2}{\pi^2} \begin{cases} \frac{4}{3} + \frac{16}{\pi^2}(1-2\mu), & \mu \leq \frac{1}{2} - \frac{\pi^2}{48}, \\ 2, & \frac{1}{2} - \frac{\pi^2}{48} \leq \mu \leq \frac{1}{2} + \frac{5\pi^2}{48}, \\ -\frac{4}{3} - \frac{16}{\pi^2}(1-2\mu), & \mu \geq \frac{1}{2} + \frac{5\pi^2}{48}. \end{cases}$$

Now we consider the inverse function $\mathcal{F}$ which maps petal type regions to the open unit disk $\mathcal{U}$, defined as $\mathcal{F}(w) = \mathcal{F}(f(z)) = z, z \in \mathcal{U}$ and we find the following coefficient bound for inverse functions. As the classes $UCV[A,B]$ and $ST[A,B]$ are the subclasses of $\mathcal{S}$. Thus the existence of such inverse functions to the functions from $UCV[A,B]$ and $ST[A,B]$ is assured.

Theorem 4. *Let $w = f(z) \in UCV[A,B]$, $-1 \leq B < A \leq 1$ and $\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$. Then,*

$$|d_n| \leq \frac{4(A-B)}{n(n-1)\pi^2} \qquad (n = 2,3,4).$$

Proof. Since $\mathcal{F}(w) = \mathcal{F}(f(z)) = z$, so it is easy to see that

$$d_2 = -a_2, \qquad d_3 = 2a_2^2 - a_3, \qquad d_4 = -a_4 + 5a_2a_3 - 5a_2^3.$$

By using (21) and (22), one can have

$$d_2 = \frac{-1}{\pi^2}(A-B)c_1 \tag{24}$$

and

$$d_3 = \frac{A-B}{3\pi^2}\left[\left(\frac{1}{6} + \frac{2}{\pi^2}(B+1) + \frac{4}{\pi^2}(A-B)\right)c_1^2 - c_2\right]. \tag{25}$$

From (19) and (20), comparison of z^3 gives

$$\begin{aligned} a_4 \quad = \quad & \frac{A-B}{3\pi^2}\left[\left(\frac{1}{45} + \frac{1}{\pi^2}\left(\frac{1}{3}(B+1) - \frac{1}{4}(A-B)\right) + \frac{1}{\pi^4}\left(2(B+1)^2 - 3(A-B)(B+1) + (A-B)^2\right)\right)c_1^3 \right. \\ & \left. -\left(\frac{1}{6} + \frac{1}{\pi^2}\left(2(B+1) - \frac{3}{2}(A-B)\right)\right)c_2c_1 + \frac{1}{2}c_3\right]. \end{aligned}$$

Using the values of $a_n;\ n = 2,3,4$, we get

$$\begin{aligned} d_4 \quad = \quad & -\frac{A-B}{3\pi^2}\left[\left(\frac{1}{45} + \frac{1}{3\pi^2}\left(B+1+\frac{7}{4}(A-B)\right) + \frac{1}{\pi^4}\left(2(B+1)^2 + 7(A-B)(B+1) + 6(A-B)^2\right)\right)c_1^3 \right. \\ & \left. -\left(\frac{1}{6} + \frac{2}{\pi^2}\left(B+1+\frac{7}{4}(A-B)\right)\right)c_2c_1 + \frac{1}{2}c_3\right]. \end{aligned} \tag{26}$$

Now, from (24) and (25), one can have

$$|d_2| \leq \frac{2}{\pi^2}(A-B)$$

and

$$\begin{aligned} |d_3| \quad \leq \quad & \frac{A-B}{3\pi^2}\left|\frac{1}{6} + \frac{2}{\pi^2}(B+1) + \frac{4}{\pi^2}(A-B)\right|\left|c_2 - c_1^2\right| \\ & + \frac{A-B}{3\pi^2}\left|\frac{5}{6} - \frac{2}{\pi^2}(B+1) - \frac{4}{\pi^2}(A-B)\right||c_2|. \end{aligned}$$

Application of the bounds $\left|c_2 - c_1^2\right| \leq 2$ and $|c_2| \leq 2$ (see Lemma 2 for $v = 1$ and $v = 0$) gives $|d_3| \leq \frac{2(A-B)}{3\pi^2}$. Lastly, (26) reduces to

$$|d_4| \leq \frac{A-B}{3\pi^2}\left[|\lambda_1|\left|c_3 - 2c_2c_1 + c_1^3\right| + |\lambda_2|\,|c_3 - c_2c_1| + |\lambda_3|\,|c_3|\right], \tag{27}$$

where

$$\lambda_1 = \frac{1}{45} + \frac{1}{3\pi^2}\left(B + 1 + \frac{7}{4}(A-B)\right) + \frac{1}{\pi^4}\left(2(B+1)^2 + 7(A-B)(B+1) + 6(A-B)^2\right),$$

$$\lambda_2 = \frac{11}{90} + \frac{4}{3\pi^2}\left(B + 1 + \frac{7}{4}(A-B)\right) - \frac{2}{\pi^4}\left(2(B+1)^2 + 7(A-B)(B+1) + 6(A-B)^2\right)$$

and

$$\lambda_3 = \frac{16}{45} - \frac{5}{3\pi^2}\left(B + 1 + \frac{7}{4}(A-B)\right) + \frac{1}{\pi^4}\left(2(B+1)^2 + 7(A-B)(B+1) + 6(A-B)^2\right).$$

Applying the bounds $\left|c_3 - 2c_2c_1 + c_1^3\right| \leq 2$, see [23], $|c_3 - c_2c_1| \leq 2$ and $|c_3| \leq 2$, see [7] to the right hand side of (27) and using the fact that $\lambda_i \geq 0$; $i = 1, 2, 3$, we have $|d_4| \leq \frac{A-B}{3\pi^2}$ and this completes the proof. □

For $A = 1$, $B = -1$, the above result takes the following form which is proved by Ma and Minda [8].

Corollary 4. *Let $w = f(z) \in UCV$ and $\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$. Then,*

$$|d_n| \leq \frac{8}{n(n-1)\pi^2} \qquad (n = 2, 3, 4).$$

Theorem 5. *Let $w = f(z) \in UCV[A, B]$, $-1 \leq B < A \leq 1$ and $\mathcal{F}(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$. Then, for a real number μ, we have*

$$\left|d_3 - \mu d_2^2\right| \leq \frac{A-B}{3\pi^2}\begin{cases} \frac{4}{3} - \frac{8}{\pi^2}(B+1) - \frac{4}{\pi^2}(A-B)(4-3\mu), & \mu \geq \frac{4}{3} + \frac{\pi^2}{18(A-B)} + \frac{2(B+1)}{3(A-B)}, \\ 2, & \frac{4}{3} - \frac{5\pi^2}{18(A-B)} + \frac{2(B+1)}{3(A-B)} \leq \mu \leq \frac{4}{3} + \frac{\pi^2}{18(A-B)} + \frac{2(B+1)}{3(A-B)}, \\ -\frac{4}{3} + \frac{8}{\pi^2}(B+1) + \frac{4}{\pi^2}(A-B)(4-3\mu), & \mu \leq \frac{4}{3} - \frac{5\pi^2}{18(A-B)} + \frac{2(B+1)}{3(A-B)}. \end{cases}$$

This result is sharp.

Proof. The proof follows directly from (24), (25) and Lemma 2. □

Author Contributions: Conceptualization, M.R.; Formal analysis, M.R.; Funding acquisition, S.M.; Investigation, S.F.; Methodology, S.N.M. and S.F.; Supervision, S.N.M.; Validation, S.M.; Visualization, S.Z.; Writing—original draft, S.Z.; Writing—review and editing, S.Z.

Funding: This research is partially supported by Sarhad University of Science and I.T, Ring Road, Peshawar 25000.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Fekete, M.; Szegö, G. Eine bemerkung uber ungerade schlichte funktionen. *J. Lond. Math. Soc.* **1933**, *8*, 85–89. [CrossRef]
2. Agrawal, S. Coefficient estimates for some classes of functions associated with q-function theory. *Bull. Aust. Math. Soc.* **2017**, *95*, 446–456. [CrossRef]
3. Ahuja, O.P.; Jahangiri, M. Fekete–Szegö problem for a unified class of analytic functions. *Panam. Math. J.* **1997**, *7*, 67–78.
4. Kanas, S. An unified approach to the Fekete–Szegö problem. *Appl. Math. Comput.* **2012**, *218*, 8453–8461. [CrossRef]
5. Keogh, F.R.; Merkes, E.P. A coefficient inequality for certain classes of analytic functions. *Proc. Am. Math. Soc.* **1969**, *20*, 8–12. [CrossRef]
6. Koepf, W. On the Fekete-Szegö problem for close to convex functions I and II. *Proc. Am. Math. Soc.* **1987**, *101*, 89–95; reprinted in *Arch. Math. (Basel)* **1987**, *49*, 420–433.
7. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, Tianjin, China, 19–23 June 1992; Conference on Proceedings Lecture Notes for Analysis; International Press: Cambridge, MA, USA, 1994; pp. 157–169.
8. Ma, W.; Minda, D. Uniformly convex functions II. *Ann. Polon. Math.* **1993**, *8*, 275–285. [CrossRef]
9. Raza, M.; Malik, S.M. Upper bound of the third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli. *J. Inequal. Appl.* **2013**, *2013*, 412. [CrossRef]
10. Sokół J.; Darwish, H.E. Fekete–Szegö problem for starlike and convex functions of complex order. *Appl. Math. Lett.* **2010**, *23*, 777–782.
11. Thomas, D.K.; Verma, S. Invaience of the coefficients of strongly convex functions. *Bull. Aust. Math. Soc.* **2017**, *95*, 436–345. [CrossRef]
12. Goodman, A.W. *Univalent Functions*; Mariner Publishing Company: Tempa, FL, USA, 1983; Volumes I–II.
13. Goodman, A.W. On uniformly convex functions. *Ann. Polon. Math.* **1991**, *56*, 87–92. [CrossRef]
14. Rønning, F. On starlike functions associated with parabolic regions. *Ann. Univ. Mariae Curie-Sklodowska Sect. A* **1991**, *45*, 117–122.
15. Kanas, S.; Wiśniowska, A. Conic regions and k-uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [CrossRef]
16. Kanas, S.; Wiśniowska, A. Conic domains and starlike functions. *Rev. Roum. Math. Pures Appl.* **2000**, *45*, 647–657.
17. Malik, S.N. *Some topics in Geometric Function Theory*; LAP LAMBERT Academic Publishing: Saarbrucken, Germany, 2017.
18. Malik, S.N.; Raza, M.; Arif, M.; Hussain, S. Coefficient estimates of some subclasses of analytic functions related with conic domains. *Anal. Univ. Ovidius Const. Ser. Mat.* **2013**, *21*, 181–188. [CrossRef]
19. Noor, K.I.; Malik, S.N. On generalized bounded Mocanu variation associated with conic domain. *Math. Comput. Model.* **2012**, *55*, 844–852. [CrossRef]
20. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Math. Appl.* **2011**, *62*, 2209–2217. [CrossRef]
21. Kanas, S. Coefficient estimates in subclasses of the Caratheodory class related to conical domains. *Acta Math. Univ. Comen.* **2005**, *74*, 149–161.
22. Rønning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [CrossRef]
23. Libera, R.J.; Złotkiewcz, E.J. Early coefficients of the inverse of a regular convex function. *Proc. Amer. Math. Soc.* **1982**, *85*, 225–230. [CrossRef]