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Some Identities Involving Fibonacci Polynomials and Fibonacci Numbers

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Abstract: The aim of this paper is to research the structural properties of the Fibonacci polynomials and Fibonacci numbers and obtain some identities. To achieve this purpose, we first introduce a new second-order nonlinear recursive sequence. Then, we obtain our main results by using this new sequence, the properties of the power series, and the combinatorial methods.

Keywords: Fibonacci polynomials; Fibonacci numbers; recursive sequence; combinatorial method; power series; identity

MSC: 11B39; 11B50

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1. Introduction

For any real number x, the Fibonacci polynomials $F_n(x)$ are defined by $F_0(x) = 1$, $F_1(x) = x$, $F_2(x) = x^2 + 1$, $F_3(x) = x^3 + 2x$, $F_4(x) = x^4 + 3x^2 + 1$, and the second order linear recursive formula:

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x),\ n \geq 1.$$

If $x = 1$, then $F_n(1) = F_{n+1}$ is the famous Fibonacci sequence. Its initial values are $F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13, \cdots$.

The general expression of $F_n(x)$ is:

$$\begin{aligned} F_n(x) &= \frac{1}{\sqrt{x^2+4}}\left[\left(\frac{x+\sqrt{x^2+4}}{2}\right)^{n+1} - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^{n+1}\right] \\ &= \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(n-k)!}{k!\cdot(n-2k)!}\cdot x^{n-2k},\ n = 0,\ 1,\ 2, \cdots. \end{aligned} \tag{1}$$

The generating function of the Fibonacci polynomials $F_n(x)$ is:

$$\frac{1}{1-xt-t^2} = \sum_{n=0}^{\infty} F_n(x)\cdot t^n.$$

These polynomials play a very important role in the theory and application of mathematics. Because of this, many number theory experts study the properties of $F_n(x)$ and F_n and obtain a series of important results. For example, Yuan Yi and Wenpeng Zhang [1] studied the computational problem of the summation:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)\cdot F_{a_2}(x)\cdots F_{a_h}(x)\cdot F_{a_{h+1}}(x), \tag{2}$$

where the summation is taken over all $(h+1)$-dimension nonnegative integer coordinates $(a_1, a_2, \cdots, a_{h+1})$ such that $a_1 + a_2 + \cdots + a_{h+1} = n$.

They used the elementary and combination methods to give a meaningful identity involving $F_n(x)$ and its derivative $F'_n(x)$.

Zhengang Wu and Wenpeng Zhang [2,3] studied the calculation problem of the reciprocal sum of the Fibonacci numbers and Fibonacci polynomials and obtained many interesting identities and calculation formulae.

Other achievements related to Fibonacci numbers and Fibonacci polynomials can also be found in [4–18]; here, we will not repeat them one by one.

Very recently, Yixue Zhang and Zhuoyu Chen [19] used the elementary method and combination skill to research the calculating problem of one kind sums of the second kind Chebyshev polynomials and then obtained a very interesting identity.

This paper is inspired by [19]. As a note of [1,4], we are going to introduce a new second order non-linear recursive sequence, then use this sequence to acquire a meaningful expression for (2). That is, we shall prove the following:

Theorem 1. *Let h be a positive integer. Then, for any integer $n \geq 0$, we have the identity:*

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)F_{a_2}(x)\cdots F_{a_{h+1}}(x) = \frac{1}{h!}\cdot\sum_{j=1}^{h}\frac{(-1)^{h-j}\cdot S(h,j)}{x^{2h-j}}$$
$$\times\left(\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\binom{2h+i-j-1}{i}\cdot\frac{(-1)^i\cdot 2^i\cdot F_{n-i+j}(x)}{x^i}\right),$$

where $S(h,i)$ is a second order non-linear recurrence sequence defined by $S(h,0) = 0$, $S(h,h) = 1$, and $S(h+1,i+1) = 2\cdot(2h-1-i)\cdot S(h,i+1) + S(h,i)$ for all positive integers $1 \leq i \leq h-1$.

In particular, taking $n = 0$ and $x = 1$, we can also deduce the following corollaries from Theorem 1.

Corollary 1. *For any positive integer $h \geq 1$, we have the identity:*

$$\sum_{j=1}^{h}(-1)^{h-j}\cdot S(h,j)\cdot j!\cdot x^j\cdot F_j(x) = h!\cdot x^{2h}.$$

Corollary 2. *For any positive integer $h \geq 1$, we have the identity:*

$$\sum_{j=1}^{h}(-1)^{h-j}\cdot S(h,j)\cdot j!\cdot F_{j+1} = h!.$$

The formula in Corollary 1 profoundly reveals the close relationship among the Fibonacci polynomials. Taking $h = 2$, then from Theorem 1, we can also infer the following:

Corollary 3. *For any integer $n \geq 0$, we obtain:*

$$\sum_{a+b+c=n} F_{a+1}\cdot F_{b+1}\cdot F_{c+1} = \frac{1}{2}\sum_{i=0}^{n}(-1)^i 2^i(n+1-i)(n+2-i)(i+1)\cdot F_{n+3-i}$$
$$-\sum_{i=0}^{n}(-1)^i 2^i(n+1-i)(i+1)(i+2)\cdot F_{n+2-i}.$$

To further comprehend the properties of the sequence $S(h,i)$, here we give first several terms of $S(h,i)$ in the following Table 1:

Table 1. Values of $S(k,i)$.

$S(h,i)$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
$h=1$	1							
$h=2$	2	1						
$h=3$	12	6	1					
$h=4$	120	60	12	1				
$h=5$	1680	840	180	20	1			
$h=6$	30,240	15,120	3360	420	30	1		
$h=7$	665,280	332,640	75,600	10,080	840	42	1	
$h=8$	17,297,280	8,648,640	1,995,840	277,200	25,200	1512	36	1

By observing this table, we can easily find that for primes $p = 2, 3, 5, 7$, we have the congruences $S(p,i) \equiv 0 \bmod p(p-1)$ for $0 \le i \le p-1$. In fact, this conclusion is also correct for all primes p. That is, we obtain the following:

Theorem 2. *Let p be a prime. Then, for any integer i with $1 \le i \le p-1$, we acquire the congruence:*

$$S(p,i) \equiv 0 \bmod p(p-1).$$

2. Several Simple Lemmas

In this part, we will give several necessary lemmas in the proof process of our theorems. First of all, we have the following:

Lemma 1. *Let function $f(t) = \frac{1}{1-xt-t^2}$. Then, for any positive integer n and real numbers x and t with $|2t| < |x|$, we have the identity:*

$$n! \cdot f^{n+1}(t) = \sum_{i=1}^{n} \frac{(-1)^{n-i} \cdot S(n,i) \cdot f^{(i)}(t)}{(x+2t)^{2n-i}},$$

where $S(n,i)$ is defined in the theorem and $f^{(h)}(t)$ indicates the h-order derivative of $f(t)$ with respect to variable t.

Proof. In fact, we can prove this Lemma 1 by mathematical induction. By the properties of the derivative, we have:

$$f'(t) = (x+2t) \cdot \left(1 - xt - t^2\right)^{-2} = (x+2t) \cdot f^2(t)$$

or:

$$f^2(t) = \frac{f'(t)}{x+2t} = S(1,1) \cdot \frac{f'(t)}{x+2t}.$$

That is, Lemma 1 is true for $n = 1$. Assuming Lemma 1 holds for $1 \le n = h$, we have,

$$h! \cdot f^{h+1}(t) = \sum_{i=1}^{h} (-1)^{h-i} \cdot S(h,i) \cdot \frac{f^{(i)}(t)}{(x+2t)^{2h-i}}. \quad (3)$$

Next, from (3) and the definitions of $S(k,i)$ and the derivative, we obtain:

$$\begin{aligned} & (h+1)! \cdot f^h(t) \cdot f'(t) = (h+1)! \cdot (x+2t) \cdot f^{h+2}(t) \\ = & \sum_{i=1}^{h} (-1)^{h-i+1} \frac{2(2h-i) \cdot S(h,i)}{(x+2t)^{2h+1-i}} \cdot f^{(i)}(t) + \sum_{i=1}^{h} \frac{(-1)^{h-i} \cdot S(h,i)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^h 2(2h-1)S(h,1)}{(x+2t)^{2h}} \cdot f'(t) + \sum_{i=1}^{h-1} (-1)^{h-i} \frac{2(2h-1-i)S(h,i+1)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \\
&\quad + \frac{S(h,h)}{(x+2t)^h} \cdot f^{(h+1)}(t) + \sum_{i=1}^{h-1} \frac{(-1)^{h-i} \cdot S(h,i)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \\
&= \frac{(-1)^h 2(2h-1)S(h,1)}{(x+2t)^{2h}} \cdot f'(t) + \frac{S(h,h)}{(x+2t)^h} \cdot f^{(h+1)}(t) \\
&\quad + \sum_{i=1}^{h-1} (-1)^{h-i} \frac{2(2h-1-i)S(h,i+1)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \\
&\quad + \sum_{i=1}^{h-1} \frac{(-1)^{h-i} \cdot S(h,i)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \\
&= \frac{(-1)^h \cdot S(h+1,1)}{(x+2t)^{2h}} \cdot f'(t) + \frac{S(h+1,h+1)}{(x+2t)^h} \cdot f^{(h+1)}(t) \\
&\quad + \sum_{i=1}^{h-1} \frac{(-1)^{h-i} \cdot S(h+1,i+1)}{(x+2t)^{2h-i}} \cdot f^{(i+1)}(t) \\
&= (-1)^h \cdot \frac{S(h+1,1)}{(x+2t)^{2h}} \cdot f'(t) + \frac{S(h+1,h+1)}{(x+2t)^h} \cdot f^{(h+1)}(t) \\
&\quad + \sum_{i=2}^{h} \frac{(-1)^{h+1-i} S(h+1,i)}{(x+2t)^{2h+1-i}} \cdot f^{(i)}(t) \\
&= \sum_{i=1}^{h+1} (-1)^{h+1-i} \cdot S(h+1,i) \cdot \frac{f^{(i)}(t)}{(x+2t)^{2h+1-i}}. \qquad (4)
\end{aligned}
$$

Thus, from (4), we deduce that:

$$(h+1)! \cdot (x+2t) \cdot f^{h+2}(t) = \sum_{i=1}^{h+1} (-1)^{h+1-i} \cdot S(h+1,i) \cdot \frac{f^{(i)}(t)}{(x+2t)^{2h+1-i}}$$

or:

$$(h+1)! \cdot f^{h+2}(t) = \sum_{i=1}^{h+1} (-1)^{h+1-i} \cdot S(h+1,i) \cdot \frac{f^{(i)}(t)}{(x+2t)^{2h+2-i}}.$$

That is to say, Lemma 1 also applies for $n = h+1$.

Now, Lemma 1 follows from the mathematical induction. □

Lemma 2. *For any positive integers h and k, we have the power series expansion:*

$$\frac{f^{(h)}(t)}{(x+2t)^k} = \frac{1}{x^k} \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{(n-i+h)!}{(n-i)!} \cdot \frac{(-1)^i \cdot 2^i \cdot F_{n-i+h}(x)}{x^i} \cdot \binom{i+k-1}{i} \right) t^n,$$

where t and x are any real numbers with $|t| < |x|$.

Proof. In fact, from the definition of the Fibonacci polynomials $F_n(x)$, we have:

$$f(t) = \frac{1}{1-xt-t^2} = \sum_{n=0}^{\infty} F_n(x) \cdot t^n.$$

For any positive integer h, from the properties of the power series, we can obtain:

$$\begin{aligned} f^{(h)}(t) &= \sum_{n=0}^{\infty}(n+h)(n+h-1)\cdots(n+1)\cdot F_{n+h}(x)\cdot t^n \\ &= \sum_{n=0}^{\infty}\frac{(n+h)!}{n!}\cdot F_{n+h}(x)\cdot t^n. \end{aligned} \quad (5)$$

Let k be a positive integer. Then, for all real t and x with $|2t| < |x|$, noting that the power series expansion:

$$\frac{1}{x+2t} = \frac{1}{x}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{(2t)^n}{x^n},$$

we have:

$$\frac{1}{(x+2t)^k} = \frac{1}{x^k}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\binom{n+k-1}{n}\cdot\frac{(2t)^n}{x^n}. \quad (6)$$

Applying (5), (6), and the multiplicative properties of the power series, we have:

$$\begin{aligned} &\frac{f^{(h)}(t)}{(x+2t)^k} \\ =\ & \frac{1}{x^k}\cdot\left(\sum_{n=0}^{\infty}\frac{(n+h)!}{n!}\cdot F_{n+h}(x)\cdot t^n\right)\left(\sum_{n=0}^{\infty}(-1)^n\binom{n+k-1}{n}\cdot\frac{(2t)^n}{x^n}\right) \\ =\ & \frac{1}{x^k}\sum_{n=0}^{\infty}\left(\sum_{i+j=n}\frac{(j+h)!}{j!}\cdot F_{j+h}(x)\cdot(-1)^i\cdot\binom{i+k-1}{i}\cdot\frac{2^i}{x^i}\right)t^n \\ =\ & \frac{1}{x^k}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(n-i+h)!}{(n-i)!}\cdot F_{n-i+h}(x)\cdot(-1)^i\cdot\binom{i+k-1}{i}\cdot\frac{2^i}{x^i}\right)t^n. \end{aligned}$$

This proves Lemma 2. □

3. Proofs of the Theorems

In this section, the proofs of our theorems will be completed. Firstly, we prove Theorem 1. For any positive integer h, from (1), Lemma 1, the definition of $f(t)$, and the properties of the power series, we have:

$$\begin{aligned} h!\cdot f^{h+1}(t) &= h!\cdot\left(\sum_{n=0}^{\infty}F_n(x)\cdot t^n\right)^{h+1} \\ &= h!\cdot\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_{h+1}=n}F_{a_1}(x)F_{a_2}(x)\cdots F_{a_{h+1}}(x)\right)\cdot t^n. \end{aligned} \quad (7)$$

On the other hand, applying Lemma 2, we also have:

$$\begin{aligned} &h!\cdot f^{h+1}(t) = \sum_{j=1}^{h}(-1)^{h-j}\cdot S(h,j)\cdot\frac{f^{(j)}(t)}{(x+2t)^{2h-j}} \\ =\ & \sum_{j=1}^{h}\frac{(-1)^{h-j}S(h,j)}{x^{2h-j}} \\ &\times\left(\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot F_{n-i+j}(x)\cdot\binom{2h+i-j-1}{i}\cdot\frac{(-1)^i\cdot 2^i}{x^i}\right)t^n\right) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{j=1}^{h} \frac{(-1)^{h-j} \cdot S(h,j)}{x^{2h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{(-1)^i \cdot 2^i \cdot F_{n-i+j}(x)}{x^i} \times \binom{2h+i-j-1}{i} \cdot t^n. \quad (8)$$

Combined with (7), (8), and Lemma 1 and compared with the coefficients of the power series, we may immediately deduce the identity:

$$h! \sum_{a_1+a_2+\cdots+a_{h+1}=n} F_{a_1}(x)F_{a_2}(x)F_{a_3}(x)\cdots F_{a_h}(x) \cdot F_{a_{h+1}}(x)$$
$$= \sum_{j=1}^{h} \frac{(-1)^{h-j} \cdot S(h,j)}{x^{2h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \binom{2h+i-j-1}{i} \frac{(-1)^i \cdot 2^i \cdot F_{n-i+j}(x)}{x^i}.$$

This proves Theorem 1.

Now, we use the mathematical induction to prove Theorem 2. Taking $h = p$ in Corollary 1, we have:

$$\sum_{j=1}^{p} (-1)^{p-j} \cdot S(p,j) \cdot j! \cdot x^j \cdot F_j(x) = p! \cdot x^{2p}$$

or from (1), we have:

$$\sum_{j=1}^{p-1} (-1)^{p-j} \cdot S(p,j) \cdot j! \cdot x^j \cdot F_j(x) = -p! \cdot \sum_{k=1}^{\left[\frac{p-1}{2}\right]} \frac{(p-k)!}{k! \cdot (p-2k)!} \cdot x^{2p-2k}. \quad (9)$$

For convenience, we let:

$$W(x) = \sum_{j=1}^{p-1} (-1)^{p-j} \cdot S(p,j) \cdot j! \cdot x^j \cdot F_j(x)$$

and:

$$H(x) = -p! \cdot \sum_{k=1}^{\left[\frac{p-1}{2}\right]} \frac{(p-k)!}{k! \cdot (p-2k)!} \cdot x^{2p-2k}.$$

Then, from the definition of the derivative, we have:

$$W^{(2p-2)}(0) = -S(p,p-1) \cdot (p-1)! \cdot (2p-2)! \quad (10)$$

and:

$$H^{(2p-2)}(0) = -p! \cdot (p-1) \cdot (2p-2)!. \quad (11)$$

Combining (9), (10), and (11), we have:

$$S(p,p-1) = p(p-1) \equiv 0 \bmod p(p-1). \quad (12)$$

That is, Theorem 2 is true for $i = p-1$.

Suppose that Theorem 2 is true for all integers $1 < h \leq i \leq p-1$. That is,

$$S(p,i) \equiv 0 \bmod p(p-1), \ h \leq i \leq p-1. \quad (13)$$

Now, we prove that Theorem 2 is also correct for integer $i = h - 1$. Let $H = \min\{p-1, 2h-2\}$. Then, note that:

$$W^{(2h-2)}(0) = \sum_{j=h-1}^{H} (-1)^{p-j} \cdot S(p,j) \cdot j! \cdot \frac{(h-1)! \cdot (2h-2)!}{(j+1-h)! \cdot (2h-2-j)!}, \tag{14}$$

$$H^{(2h-2)}(0) = \begin{cases} 0 & \text{if } h < \frac{p+3}{2}, \\ -p! \cdot \dfrac{(h-1)! \cdot (2h-2)!}{(p+1-h)! \cdot (2h-2-p)!} & \text{if } h \geq \frac{p+3}{2} \end{cases} \tag{15}$$

and:

$$\begin{aligned} & W^{(2h-2)}(0) = \sum_{j=h-1}^{H} (-1)^{p-j} \cdot S(p,j) \cdot j! \cdot \frac{(h-1)! \cdot (2h-2)!}{(j+1-h)! \cdot (2h-2-j)!} \\ = \; & H^{(2h-2)}(0) = \begin{cases} 0 & \text{if } h < \frac{p+3}{2}, \\ -p! \cdot \dfrac{(h-1)! \cdot (2h-2)!}{(p+1-h)! \cdot (2h-2-p)!} & \text{if } h \geq \frac{p+3}{2}. \end{cases} \end{aligned} \tag{16}$$

Combining (16) and the induction hypothesis (13), we can deduce that:

$$S(p, h-1) \equiv 0 \bmod p(p-1).$$

That is to say, Theorem 2 is also correct for $i = h - 1$. Now, Theorem 2 follows from the mathematical induction.

This completes the proofs of all our theorems.

4. Conclusions

The main results of this paper are Theorems 1 and 2. The feature of Theorem 1 is to express the complex sums (2) as a simple combination of $F_n(x)$, and its right side is very easy to calculate. In addition, it also profoundly reveals the structural properties of the Fibonacci polynomials. That is, the convolution (2) is composed of $F_i(x)$ and some recursive sequences $S(h, j)$. Actually, $S(h, j)$ in Theorem 1 can be calculated immediately by computer, which is very practical and easy.

Theorem 2 proves interesting congruence properties of the sequence $S(p, i)$. In particular, its proving method is also very unique, and the mathematical induction is used upside down. What surprised us is that $p(p-1)$ divides $S(p, i)$ for all $1 \leq i \leq p - 1$.

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