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Article

Existence Results of a Coupled System of Caputo Fractional Hahn Difference Equations with Nonlocal Fractional Hahn Integral Boundary Value Conditions

Thongchai Dumrongpokaphan [1], Nichaphat Patanarapeelert [2,*] and Thanin Sitthiwirattham [3,*]

1 Center of Excellence in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand; thongchai.d@cmu.ac.th
2 Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
3 Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok 10700, Thailand
* Correspondence: nichaphat.p@sci.kmutnb.ac.th (N.P.); thanin_sit@dusit.ac.th (T.S.)

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Abstract: In this article, we propose a coupled system of Caputo fractional Hahn difference equations with nonlocal fractional Hahn integral boundary conditions. The existence and uniqueness result of solution for the problem is studied by using the Banach's fixed point theorem. Furthermore, the existence of at least one solution is presented by using the Schauder fixed point theorem.

Keywords: fractional Hahn integral; Caputo fractional Hahn difference; boundary value problems; existence

MSC: 39A10; 39A13; 39A70

1. Introduction

Quantum calculus is the study of calculus without limits. There are several types of quantum difference operators. Used in problems of mathematical areas for instance, orthogonal polynomials, combinatorics, arithmetics, particle physics, quantum mechanics, the theory of relativity and variational calculus [1–9]. In addition, the applications of the integral boundary equations and boundary element methods development could be found in [10–14].

Jackson q-difference operator, the forward (delta) difference operator and the backward (nabla) difference operator are the well-known operators used in many research works. In 1949, the Hahn difference operator developed from the forward difference operator and the Jackson q-difference operator was proposed by Hahn [15] as

$$D_{q,\omega}f(t) := \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, \;\; t \neq \omega_0 := \frac{\omega}{1-q}.$$

We note that

$$D_{q,\omega}f(t) = \Delta_{\omega}f(t) := \frac{f(t+\omega)-f(t)}{\omega} \text{ whenever } q=1,$$

$$D_{q,\omega}f(t) = D_q f(t) := \frac{f(qt) - f(t)}{t(q-1)} \text{ whenever } \omega = 0,$$

$$\text{and} \quad D_{q,\omega}f(t) = f'(t) \text{ whenever } q = 1, \omega \to 0.$$

Hahn's operator has been used in the determination of new families of orthogonal polynomials and in approximation problems (see [16–18]). Recently, the Hahn difference calculus has become a favourite topic for analysis.

Later, the right inverse of Hahn difference operator was introduced by Aldwoah [19,20]. This operator is formulated from Nörlund sum and Jackson q-integral [21]. In 2010, Malinowska and Torres proposed Hahn variational calculus [22,23]. In 2013, Hamza et al. [24,25] proved the existence and uniqueness of solution for the initial value problems for Hahn difference equations. In addition, they obtained a mean value theorems for this calculus, and established Gronwall's and Bernoulli's inequalities with respect to the Hahn difference operator. In the same year, Malinowska and Martins [26] proposed Hahn variational problem to generalize the Hahn calculus of variations. They obtained transversality conditions.

In 2016, Sitthiwirattham [27] studied the nonlocal boundary value problem for a second-order Hahn difference equation, their problem contains two Hahn difference operators with different numbers of q and ω, the existence and uniqueness result was proved by using the Banach fixed point theorem, and the existence of a positive solution was established by using the Krasnoselskii fixed point theorem. In 2017, Sriphanomwan et al. [28] developed the above problem by including an Hahn integro-difference term and integral boundary condition, the existence and uniqueness of solutions was obtained by using the Banach fixed point theorem, and the existence of at least one solution was established by using the Leray–Schauder nonlinear alternative and Krasnoselskii's fixed point theorem.

Meanwhile, there were some research works related to fractional (q,h)-difference operator for $q > 1$ (see [29–33]). However, the fractional Hahn operators must be satisfied with $0 < q < 1$. Presently, the fractional Hahn difference operators was introduced by Brikshavana and Sitthiwirattham [34]. In addition, boundary value problems of fractional Hahn difference equations have been studied (see [35–37]).

Since the boundary value problem for systems of fractional Hahn difference equations have never been presented before, we devote our attention to study this kind of problem. In this paper, we consider the boundary value problem for the system of Caputo fractional Hahn difference equations of the form

$$\begin{aligned} {}^{C}D^{\alpha_1}_{q,\omega}u_1(t) &= F_1\left(t, {}^{C}D^{\beta_1}_{q,\omega}u_1(t), u_2(t)\right), \\ {}^{C}D^{\alpha_2}_{q,\omega}u_2(t) &= F_2\left(t, {}^{C}D^{\beta_2}_{q,\omega}u_2(t), u_1(t)\right), \quad t \in I^T_{q,\omega}, \end{aligned} \tag{1}$$

with the nonlocal three-point fractional Hahn integral boundary value conditions

$$\begin{aligned} u_1(\omega_0) &= \phi_1(u_1, u_2), \quad u_1(t) = \lambda_2 \mathcal{I}^{\theta_2}_{q,\omega} g_2(\eta_2) u_2(\eta_2), \\ u_2(\omega_0) &= \phi_2(u_1, u_2), \quad u_2(t) = \lambda_1 \mathcal{I}^{\theta_1}_{q,\omega} g_1(\eta_1) u_1(\eta_1), \end{aligned} \tag{2}$$

where $I^T_{q,\omega} := \{q^k T + \omega[k]_q : k \in \mathbb{N}_0\} \cup \{\omega_0\}$; $\eta_1, \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\}$; for $i = 1,2$ $\alpha_i \in (1,2]$, $\beta_i, \theta_i \in (0,1]$, $\omega > 0$, $q \in (0,1)$, $\lambda_i \in \mathbb{R}^+$, $F_i \in C(I^T_{q,\omega} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $g_i \in C(I^T_{q,\omega}, \mathbb{R})$ are given functions, and $\phi_i : C\left(I^T_{q,\omega}, \mathbb{R}\right) \times C\left(I^T_{q,\omega}, \mathbb{R}\right) \to \mathbb{R}$ are given functionals.

We organize the paper as follows. We provide some definitions and lemmas in Section 2. We present the existence and uniqueness of a solution for system (1) in Section 3. The Banach fixed point theorem is the tool to get the result. In addition, we prove the existence of at least one solution for system (1) by employing the Schauder's fixed point theorem in Section 4. Finally, we present some examples of the main results.

2. Preliminaries

In this section, we introduce notations, definitions, and lemmas which are used in the main results [15,19,27,28,34]. Let $q \in (0,1)$, $\omega > 0$ and define

$$[n]_q := \frac{1-q^n}{1-q} = q^{n-1} + \ldots + q + 1 \qquad \text{and} \qquad [n]_q! := \prod_{k=1}^{n} \frac{1-q^k}{1-q}, \qquad n \in \mathbb{N}.$$

We define the q-analogue of the power function $(a-b)_q^{\underline{n}}$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$; $a, b \in \mathbb{R}$ as

$$\begin{aligned}
(a-b)_q^{\underline{0}} &:= 1, \\
(a-b)_q^{\underline{n}} &:= \prod_{k=0}^{n-1}(a-bq^k) = a^n \prod_{k=0}^{n-1}\left(1-\frac{b}{a}q^k\right) = a^n \prod_{k=0}^{n-1}\left(1-\frac{b}{a}q^k\right) \frac{\prod_{h=n}^{\infty}\left(1-\frac{b}{a}q^h\right)}{\prod_{h=n}^{\infty}\left(1-\frac{b}{a}q^h\right)} \\
&= a^n \frac{\prod_{k=0}^{\infty}\left(1-\frac{b}{a}q^k\right)}{\prod_{k=0}^{\infty}\left(1-\frac{b}{a}q^{k+n}\right)}.
\end{aligned}$$

The q,ω-analogue of the power function $(a-b)_{q,\omega}^{\underline{n}}$ with $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$; $a, b \in \mathbb{R}$ is defined by

$$\begin{aligned}
(a-b)_{q,\omega}^{\underline{0}} := 1, \quad (a-b)_{q,\omega}^{\underline{n}} &:= \prod_{k=0}^{n-1}\left[a-(bq^k+\omega[k]_q)\right] = \prod_{k=0}^{n-1}\left[(a-\omega_0)-(a-\omega_0)q^k\right] \\
&= ((a-\omega_0)-(a-\omega_0))_q^{\underline{n}}.
\end{aligned}$$

Generally, for $\alpha \in \mathbb{R}$,

$$(a-b)_q^{\underline{\alpha}} = a^\alpha \prod_{n=0}^{\infty} \frac{1-\left(\frac{b}{a}\right)q^n}{1-\left(\frac{b}{a}\right)q^{\alpha+n}}, \ a \neq 0,$$

$$(a-b)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha \prod_{n=0}^{\infty} \frac{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^n}{1-\left(\frac{b-\omega_0}{a-\omega_0}\right)q^{\alpha+n}} = \Big((a-\omega_0)-(b-\omega_0)\Big)_q^{\underline{\alpha}}, \ a \neq \omega_0.$$

Note that $a_q^{\underline{\alpha}} = a^\alpha$ and $(a-\omega_0)_{q,\omega}^{\underline{\alpha}} = (a-\omega_0)^\alpha$. In addition, we use the notation $(0)_q^{\underline{\alpha}} = (\omega_0)_{q,\omega}^{\underline{\alpha}} = 0$ for $\alpha > 0$. The q-gamma and q-beta functions are defined by

$$\begin{aligned}
\Gamma_q(x) &:= \frac{(1-q)_q^{\underline{x-1}}}{(1-q)^{x-1}}, \qquad x \in \mathbb{R} \setminus \{0,-1,-2,\ldots\}, \\
B_q(x,s) &:= \int_0^1 t^{x-1}(1-qt)_q^{\underline{s-1}} d_q t = \frac{\Gamma_q(x)\Gamma_q(s)}{\Gamma_q(x+s)},
\end{aligned}$$

respectively.

Definition 1. *Letting $q \in (0,1)$, $\omega > 0$ and f be defined on an interval $I \subseteq \mathbb{R}$ which contain $\omega_0 := \frac{\omega}{1-q}$, the Hahn difference of f is defined by*

$$D_{q,\omega} f(t) = \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega} \quad \textit{for} \ \ t \neq \omega_0,$$

and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$ provided that f is differentiable at ω_0. We call $D_{q,\omega}f$ the q,ω-derivative of f, and say that f is q,ω-differentiable on I.

Remark 1. *The following are some properties of the Hahn difference operator:*

(1) $D_{q,\omega}[f(t) + g(t)] = D_{q,\omega}f(t) + D_{q,\omega}g(t)$,
(2) $D_{q,\omega}[\alpha f(t)] = \alpha D_{q,\omega}f(t)$,
(3) $D_{q,\omega}[f(t)g(t)] = f(t)D_{q,\omega}g(t) + g(qt+\omega)D_{q,\omega}f(t)$,
(4) $D_{q,\omega}\left[\dfrac{f(t)}{g(t)}\right] = \dfrac{g(t)D_{q,\omega}f(t) - f(t)D_{q,\omega}g(t)}{g(t)g(qt+\omega)}$.

Let $a, b \in I \subseteq \mathbb{R}$ where $a < \omega_0 < b$ and $[k]_q = \frac{1-q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We define the q,ω-interval by

$$\begin{aligned} [a,b]_{q,\omega} &:= \left\{q^k a + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \left\{q^k b + \omega[k]_q : k \in \mathbb{N}_0\right\} \cup \{\omega_0\} \\ &= [a,\omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega} \\ &= (a,b)_{q,\omega} \cup \{a,b\} \;=\; [a,b)_{q,\omega} \cup \{b\} \;=\; (a,b]_{q,\omega} \cup \{a\}. \end{aligned}$$

Observe that, for each $s \in [a,b]_{q,\omega}$, the sequence $\{\sigma^k_{q,\omega}(s)\}_{k=0}^{\infty} = \{q^k s + \omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to ω_0. We next define the forward jump operator $\sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q$ and the backward jump operator $\rho^k_{q,\omega}(t) := \frac{t-\omega[k]_q}{q^k}$ for $k \in \mathbb{N}$.

Definition 2. *Let I be any closed interval of $\mathbb{R}$ which contain a, b and ω_0. Assume that $f : I \to \mathbb{R}$ is a given function. q,ω-integral of f from a to b is defined by*

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := \left[x(1-q) - \omega\right] \sum_{k=0}^{\infty} q^k f\left(xq^k + \omega[k]_q\right), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$. We call f q,ω-integrable on $[a,b]$. The above summation is called the Jackson–Nörlund sum.

We note that f is defined on $[a,b]_{q,\omega} \subset I$. We next provide the following lemma introducing the fundamental theorem of Hahn calculus.

Lemma 1. *Let $f : I \to \mathbb{R}$ be continuous at ω_0. Define*

$$F(x) := \int_{\omega_0}^x f(t)d_{q,\omega}t, \quad x \in I.$$

Then, F is continuous at ω_0. Furthermore, $D_{q,\omega_0}F(x)$ exists for every $x \in I$ and

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}F(t)d_{q,\omega}t = F(b) - F(a) \text{ for all } a, b \in I.$$

Lemma 2. *Let $q \in (0,1)$, $\omega > 0$ and $h : I \to \mathbb{R}$ be continuous at ω_0. Then,*

$$\int_{\omega_0}^{t}\int_{\omega_0}^{r} h(s)\, d_{q,\omega}s\, d_{q,\omega}r \quad = \quad \int_{\omega_0}^{t}\int_{qs+\omega}^{t} h(s)\, d_{q,\omega}r\, d_{q,\omega}s.$$

Lemma 3. *Let $q \in (0,1)$ and $\omega > 0$. Then,*

$$\int_{\omega_0}^{t} d_{q,\omega}s = t-\omega_0 \quad \textit{and} \quad \int_{\omega_0}^{t} [t-\sigma_{q,\omega}(s)]\, d_{q,\omega}s = \frac{(t-\omega_0)^2}{1+q}.$$

Next, fractional Hahn integral, fractional Hahn difference of Riemann–Liouville and Caputo types are introduced.

Definition 3. *Letting $\alpha,\omega > 0$, $q \in (0,1)$ and f be defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn integral is defined by*

$$\begin{aligned}\mathcal{I}_{q,\omega}^{\alpha} f(t) \quad &:= \quad \frac{1}{\Gamma_q(\alpha)}\int_{\omega_0}^{t} \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} f(s) d_{q,\omega}s \\ &= \quad \frac{[t(1-q)-\omega]}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \left(t-\sigma_{q,\omega}^{n+1}(t)\right)_{q,\omega}^{\underline{\alpha-1}} f\left(\sigma_{q,\omega}^{n}(t)\right),\end{aligned}$$

and $(\mathcal{I}_{q,\omega}^{0} f)(t) = f(t)$.

Definition 4. *Letting $\alpha,\omega > 0$, $q \in (0,1)$ and f be defined on $[\omega_0, T]_{q,\omega}$, the fractional Hahn difference of the Caputo type of order α is defined by*

$$\begin{aligned}{}^{C}D_{q,\omega}^{\alpha} f(t) \quad &:= \quad (\mathcal{I}_{q,\omega}^{N-\alpha} D_{q,\omega}^{N} f)(t) \\ &= \quad \frac{1}{\Gamma_q(N-\alpha)}\int_{\omega_0}^{t} \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{N-\alpha-1}} D_{q,\omega}^{N} f(s) d_{q,\omega}s,\end{aligned}$$

and ${}^{C}D_{q,\omega}^{0} f(t) = f(t)$, where $N-1 < \alpha \le N$, $N \in \mathbb{N}$.

Lemma 4. *Let $\alpha > 0, q \in (0,1), \omega > 0$ and $f : I_{q,\omega}^{T} \to \mathbb{R}$. Then,*

$$\mathcal{I}_{q,\omega}^{\alpha}\, {}^{C}D_{q,\omega}^{\alpha} f(t) = f(t) + C_0 + C_1(t-\omega_0) + ... + C_{N-1}(t-\omega_0)^{N-1},$$

for some $C_i \in \mathbb{R}, i \in \{0,1,...,N-1\}$ and $N-1 < \alpha \le N, N \in \mathbb{N}$.

We provide the next lemma for simplify calculating the result.

Lemma 5. *Letting $\alpha,\beta > 0$, $p,q \in (0,1)$ and $\omega > 0$,*

$$\begin{aligned}\int_{\omega_0}^{t} \left(t-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\alpha-1}} (s-\omega_0)_{q,\omega}^{\underline{\beta}}\, d_{q,\omega}s \quad &= \quad (t-\omega_0)^{\alpha+\beta} B_q(\beta+1,\alpha), \\ \int_{\omega_0}^{t}\int_{\omega_0}^{x} \left(t-\sigma_{p,\omega}(x)\right)_{p,\omega}^{\underline{\alpha-1}} \left(x-\sigma_{q,\omega}(s)\right)_{q,\omega}^{\underline{\beta-1}} d_{q,\omega}s\, d_{p,\omega}x \quad &= \quad \frac{(t-\omega_0)^{\alpha+\beta}}{[\beta]_q} B_p(\beta+1,\alpha).\end{aligned}$$

In order to study the existence and uniqueness results of solution of the nonlinear problem (1), we first consider the linear variant of problem (1) and its solution in the following lemma.

Lemma 6. *Let* $\omega > 0$, $q \in (0,1)$, *for* $i = 1,2$ $\alpha_i \in (1,2]$, $\theta_i \in (0,1]$, $\lambda_i \in \mathbb{R}^+$, $h_i, g_i \in C(I^T_{q,\omega}, \mathbb{R})$ *be given functions;* $\phi_i : C\left(I^T_{q,\omega}, \mathbb{R}\right) \times C\left(I^T_{q,\omega}, \mathbb{R}\right) \to \mathbb{R}$ *be given functionals. Then, the problem*

$$\begin{aligned} {}^C D^{\alpha_1}_{q,\omega} u_1(t) &= h_1(t), \\ {}^C D^{\alpha_2}_{q,\omega} u_2(t) &= h_2(t), \quad t \in I^T_{q,\omega}, \\ u_1(\omega_0) = \phi_1(u_1, u_2), &\quad u_1(T) = \lambda_2 \mathcal{I}^{\theta_2}_{q,\omega} g_2(\eta_2) u_2(\eta_2), \; \eta_2 \in I^T_{q,\omega} - \{\omega_0, T\}, \\ u_2(\omega_0) = \phi_2(u_1, u_2), &\quad u_2(T) = \lambda_1 \mathcal{I}^{\theta_1}_{q,\omega} g_1(\eta_1) u_2(\eta_1), \; \eta_1 \in I^T_{q,\omega} - \{\omega_0, T\}, \end{aligned} \tag{3}$$

has the unique solution

$$\begin{aligned} u_1(t) &= (t-\omega_0)\Bigg\{ \frac{\lambda_1}{\Lambda \Gamma_q(\theta_1)} \int_{\omega_0}^{\eta_1} (\eta_1 - \sigma_{q,\omega}(s))^{\underline{\theta_1 - 1}}_{q,\omega} g_1(s)(s-\omega_0)^{\alpha_1 - 1} \times \\ &\quad \mathcal{P}(h_1,h_2)\, d_{q,\omega}s - \frac{\lambda_2}{\Lambda \Gamma_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\eta_2 - \sigma_{q,\omega}(s))^{\underline{\theta_2 - 1}}_{q,\omega} g_2(s) \times \\ &\quad (s-\omega_0)^{\alpha_2 - 1} \mathcal{Q}(h_1,h_2)\, d_{q,\omega}s \Bigg\} + \phi_1(u_1,u_2) \\ &\quad + \frac{1}{\Gamma_q(\alpha_1)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))^{\underline{\alpha_1 - 1}}_{q,\omega} h_1(s)\, d_{q,\omega}s, \end{aligned} \tag{4}$$

$$\begin{aligned} u_2(t) &= (t-\omega_0)\left\{ \frac{(T-\omega_0)^{\alpha_2-1}}{\Lambda} \mathcal{P}(h_1,h_2) - \frac{(T-\omega_0)^{\alpha_1-1}}{\Lambda} \mathcal{Q}(h_1,h_2) \right\} \\ &\quad + \phi_2(u_1,u_2) + \frac{1}{\Gamma_q(\alpha_2)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))^{\underline{\alpha_2 - 1}}_{q,\omega} h_2(s)\, d_{q,\omega}s, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \Lambda &= \frac{\lambda_2(T-\omega_0)}{\Gamma_q(\alpha_2)} \int_{\omega_0}^{\eta_2} (\eta_2 - \sigma_{q,\omega}(s))^{\underline{\theta_2 - 1}}_{q,\omega} g_2(s)\,(s-\omega_0)\, d_{q,\omega}s \\ &\quad - \frac{\lambda_1(T-\omega_0)}{\Gamma_q(\alpha_1)} \int_{\omega_0}^{\eta_1} (\eta_1 - \sigma_{q,\omega}(s))^{\underline{\theta_1 - 1}}_{q,\omega} g_1(s)\,(s-\omega_0)\, d_{q,\omega}s, \end{aligned} \tag{6}$$

and

$$\begin{aligned} \mathcal{P}(h_1,h_2) &= \phi_1(u_1,u_2) - \frac{\lambda_2 \phi_2(u_1,u_2)}{\Gamma_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\eta_2 - \sigma_{q,\omega}(s))^{\underline{\theta_2 - 1}}_{q,\omega} g_2(s)\, d_{q,\omega}s \\ &\quad + \frac{1}{\Gamma_q(\alpha_1)} \int_{\omega_0}^{T} (T - \sigma_{q,\omega}(s))^{\underline{\alpha_1 - 1}}_{q,\omega} h_1(s)\, d_{q,\omega}s - \frac{\lambda_2}{\Gamma_q(\alpha_2)\Gamma_q(\theta_2)} \times \\ &\quad \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{\xi} (\eta_2 - \sigma_{q,\omega}(\xi))^{\underline{\theta_2 - 1}}_{q,\omega} (\xi - \sigma_{q,\omega}(s))^{\underline{\alpha_2 - 1}}_{q,\omega} g_2(s) h_2(s)\, d_{q,\omega}s\, d_{q,\omega}\xi, \end{aligned} \tag{7}$$

$$\begin{aligned} \mathcal{Q}(h_1,h_2) &= \phi_2(u_1,u_2) - \frac{\lambda_1 \phi_1(u_1,u_2)}{\Gamma_q(\theta_1)} \int_{\omega_0}^{\eta_1} (\eta_1 - \sigma_{q,\omega}(s))^{\underline{\theta_1 - 1}}_{q,\omega} g_1(s)\, d_{q,\omega}s \\ &\quad + \frac{1}{\Gamma_q(\alpha_2)} \int_{\omega_0}^{T} (T - \sigma_{q,\omega}(s))^{\underline{\alpha_2 - 1}}_{q,\omega} h_2(s)\, d_{q,\omega}s - \frac{\lambda_1}{\Gamma_q(\alpha_1)\Gamma_q(\theta_1)} \times \\ &\quad \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{\xi} (\eta_1 - \sigma_{q,\omega}(\xi))^{\underline{\theta_1 - 1}}_{q,\omega} (\xi - \sigma_{q,\omega}(s))^{\underline{\alpha_1 - 1}}_{q,\omega} g_1(s) h_1(s)\, d_{q,\omega}s\, d_{q,\omega}\xi. \end{aligned} \tag{8}$$

Proof. For $i, j \in \{1,2\}$ and $i \neq j$, by using Lemma 4 and the fractional Hahn integral of order α for (3), we have

$$\begin{aligned} u_i(t) &= C_{1i}(t-\omega_0) + C_{2i} \\ &\quad + \frac{1}{\Gamma_q(\alpha_i)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_i - 1}} h_i(s)\, d_{q,\omega}s, \quad t \in I_{q,\omega}^{T}. \end{aligned} \tag{9}$$

Using the boundary condition (3), we find that

$$C_{2i} = \phi_i(u_1, u_2). \tag{10}$$

Therefore,

$$\begin{aligned} u_i(t) &= C_{1i}(t-\omega_0) + \phi_i(u_1, u_2) \\ &\quad + \frac{1}{\Gamma_q(\alpha_i)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_i - 1}} h_i(s)\, d_{q,\omega}s. \end{aligned} \tag{11}$$

Taking the fractional Hahn integral of order $0 < \theta_i \leq 1$ for (11), we get

$$\begin{aligned} &\mathcal{I}_{q,\omega}^{\theta_i} u_i(t) \\ = &\frac{C_{1i}}{\Gamma_q(\theta_i)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_i - 1}} (s-\omega_0)\, d_{q,\omega}s + \frac{\phi_i(u_1,u_2)}{\Gamma_q(\theta_i)} \int_{\omega_0}^{t} (t - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_i - 1}}\, d_{q,\omega}s \\ &+ \frac{1}{\Gamma_q(\theta_i)\Gamma_q(\alpha_i)} \int_{\omega_0}^{t} \int_{\omega_0}^{\xi} (t - \sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_i - 1}} (\xi - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_i - 1}} h(s)\, d_{q,\omega}s\, d_{q,\omega}\xi \end{aligned} \tag{12}$$

for $t \in I_{q,\omega}^{T}$. From the boundary condition (3), we have

$$\begin{aligned} &C_{11}(T-\omega_0) + \phi_1(u_1,u_2) + \frac{1}{\Gamma_q(\alpha_1)} \int_{\omega_0}^{T} (T - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1 - 1}} h_1(s)\, d_{q,\omega}s \qquad (13) \\ = &\frac{\lambda_2 C_{12}}{\Gamma_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\eta_2 - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2 - 1}} g_2(s)\, (s-\omega_0)\, d_{q,\omega}s \\ &+ \frac{\lambda_2 \phi_2(u_1,u_2)}{\Gamma_q(\theta_2)} \int_{\omega_0}^{\eta_2} (\eta_2 - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2 - 1}} g_2(s)\, d_{q,\omega}s \\ &+ \frac{\lambda_2}{\Gamma_q(\alpha_2)\Gamma_q(\theta_2)} \int_{\omega_0}^{\eta_2} \int_{\omega_0}^{\xi} (\eta_2 - \sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_2 - 1}} (\xi - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2 - 1}} g_2(s) h(s)\, d_{q,\omega}s\, d_{q,\omega}\xi, \end{aligned}$$

and

$$\begin{aligned} &C_{12}(T-\omega_0) + \phi_2(u_1,u_2) + \frac{1}{\Gamma_q(\alpha_2)} \int_{\omega_0}^{T} (T - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2 - 1}} h_2(s)\, d_{q,\omega}s \qquad (14) \\ = &\frac{\lambda_1 C_{11}}{\Gamma_q(\theta_1)} \int_{\omega_0}^{\eta_1} (\eta_1 - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1 - 1}} g_1(s)\, (s-\omega_0)\, d_{q,\omega}s \\ &+ \frac{\lambda_1 \phi_1(u_1,u_2)}{\Gamma_q(\theta_1)} \int_{\omega_0}^{\eta_1} (\eta_1 - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1 - 1}} g_1(s)\, d_{q,\omega}s \\ &+ \frac{\lambda_1}{\Gamma_q(\alpha_1)\Gamma_q(\theta_1)} \int_{\omega_0}^{\eta_1} \int_{\omega_0}^{\xi} (\eta_1 - \sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_1 - 1}} (\xi - \sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1 - 1}} g_1(s) h(s)\, d_{q,\omega}s\, d_{q,\omega}\xi. \end{aligned}$$

Finally, the constants C_{11} and C_{12} are investigated by solving the system of Equations (13) and (14) as

$$\begin{aligned} C_{11} &= \frac{\lambda_1}{\Lambda\Gamma_q(\theta_1)}\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}g_1(s)(s-\omega_0)\,\mathcal{P}(h_1,h_2)\,d_{q,\omega}s \\ &\quad -\frac{\lambda_2}{\Lambda\Gamma_q(\theta_2)}\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}g_2(s)(s-\omega_0)\,\mathcal{Q}(h_1,h_2)\,d_{q,\omega}s, \end{aligned}$$

and

$$C_{12} = \frac{(T-\omega_0)}{\Lambda}\,\mathcal{P}(h_1,h_2)-\frac{(T-\omega_0)}{\Lambda}\,\mathcal{Q}(h_1,h_2),$$

where $\Lambda, \mathcal{P}(h_1,h_2)$ and $\mathcal{Q}(h_1,h_2)$ are defined as Equations (8)–(10), respectively.

Substituting C_{11} and C_{12} into (11), we then obtain (4) and (5). □

3. Existence and Uniqueness Result

In this section, we aim to prove the existence result for Problems (1) and (2). Here, we let $E : C\left(I_{q,\omega}^T,\mathbb{R}\right)$ be the Banach space for all continuous functions on $I_{q,\omega}^T$, and clearly that the product space $\mathcal{C} = E\times E$ is the Banach space. We set the spaces

$$\mathcal{C}_i = \left\{(u_1,u_2)\in\mathcal{C} : {}^{C}D_{q,\omega}^{\beta_i}u_i(t)\in E,\ t\in I_{q,\omega}^T\right\}, i\in\{1,2\}.$$

Define the norm as follows:

$$\|(u_1,u_2)\|_{\mathcal{C}_i} = \|{}^{C}D_{q,\omega}^{\beta_i}u_i\| + \|u_j\|; i,j\in\{1,2\}, i\neq j,$$

where $\|{}^{C}D_{q,\omega}^{\beta_i}u_i\| = \max_{t\in I_{q,\omega}^T}\left|{}^{C}D_{q,\omega}^{\beta_i}u_i(t)\right|$ and $\|u_j\| = \max_{t\in I_{q,\omega}^T}|u_j(t)|$.

Obviously, the space $\left(\mathcal{C}_1\cap\mathcal{C}_2, \|(u_1,u_2)\|_{\mathcal{C}_1\cap\mathcal{C}_2}\right)$ is also the Banach space with the norm

$$\|(u_1,u_2)\|_{\mathcal{C}_1\cap\mathcal{C}_2} = \max\left\{\|(u_1,u_2)\|_{\mathcal{C}_1}, \|(u_1,u_2)\|_{\mathcal{C}_2}\right\}.$$

Next, we let $\mathcal{U} = \mathcal{C}_1\cap\mathcal{C}_2$ and we define the operator $\mathcal{T} : \mathcal{U}\to\mathcal{U}$ by

$$(\mathcal{T}(u_1,u_2))(t) = \Big((\mathcal{T}_1(u_1,u_2))(t), (\mathcal{T}_2(u_1,u_2))(t)\Big), \tag{15}$$

and

$$\begin{aligned} &(\mathcal{T}_1(u_1,u_2))(t) \\ &\quad = \frac{(t-\omega_0)}{\Lambda}\left\{\frac{\lambda_1}{\Gamma_q(\theta_1)}\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}g_1(s)(s-\omega_0)\mathcal{P}_u(F_1^*,F_2^*)\,d_{q,\omega}s\right. \\ &\quad \left. -\frac{\lambda_2}{\Gamma_q(\theta_2)}\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}g_2(s)(s-\omega_0)\,\mathcal{Q}_u(F_1^*,F_2^*)\,d_{q,\omega}s\right\} + \phi_1(u_1,u_2) \\ &\quad +\frac{1}{\Gamma_q(\alpha_1)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1-1}}F_1^*(s,u)\,d_{q,\omega}s, \qquad (16) \\ &(\mathcal{T}_2(u_1,u_2))(t) \\ &\quad = \frac{(t-\omega_0)}{\Lambda}\left\{(T-\omega_0)\,\mathcal{P}_u(F_1^*,F_2^*) - (T-\omega_0)\,\mathcal{Q}_u(F_1^*,F_2^*)\right\} + \phi_2(u_1,u_2) \end{aligned}$$

$$+\frac{1}{\Gamma_q(\alpha_2)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2-1}}F_2^*(s,u)\,d_{q,\omega}s, \tag{17}$$

where Λ is defined as (7), and the functionals $\mathcal{P}_u(F_1^*,F_2^*)$, $\mathcal{Q}_u(F_1^*,F_2^*)$ defined by

$$\begin{aligned}\mathcal{P}_u(F_1^*,F_2^*) =\ & \phi_1(u_1,u_2)-\frac{\lambda_2\phi_2(u_1,u_2)}{\Gamma_q(\theta_2)}\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}g_2(s)\,d_{q,\omega}s \\ & +\frac{1}{\Gamma_q(\alpha_1)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1-1}}F_1^*(s,u)\,d_{q,\omega}s-\frac{\lambda_2}{\Gamma_q(\alpha_2)\Gamma_q(\theta_2)}\times \\ & \int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\xi}(\eta_2-\sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_2-1}}(\xi-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2-1}}g_2(s)F_2^*(s,u)\,d_{q,\omega}s\,d_{q,\omega}\xi,\end{aligned} \tag{18}$$

$$\begin{aligned}\mathcal{Q}_u(F_1^*,F_2^*) =\ & \phi_2(u_1,u_2)-\frac{\lambda_1\phi_1(u_1,u_2)}{\Gamma_q(\theta_1)}\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}g_1(s)\,d_{q,\omega}s \\ & +\frac{1}{\Gamma_q(\alpha_2)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2-1}}F_2^*(s,u)\,d_{q,\omega}s-\frac{\lambda_1}{\Gamma_q(\alpha_1)\Gamma_q(\theta_1)}\times \\ & \int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\xi}(\eta_1-\sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_1-1}}(\xi-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1-1}}g_1(s)F_1^*(s,u)\,d_{q,\omega}s\,d_{q,\omega}\xi,\end{aligned} \tag{19}$$

with

$$F_1^*(s,u)=F_1\Big(s,{}^{C}D_{q,\omega}^{\beta_1}u_1(s),u_2(t)\Big)\quad\text{and}\quad F_2^*(s,u)=F_2\Big(s,u_1(t),{}^{C}D_{q,\omega}^{\beta_2}u_2(s)\Big).$$

We note that Problems (1) and (2) have solutions if and only if the operator $\mathcal{T}$ has fixed points.

Theorem 1. *For each $i,j\in\{1,2\}$; $i\neq j$, we assume that $F_i\in C(I_{q,\omega}^T\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $\phi_i : C\left(I_{q,\omega}^T,\mathbb{R}\right)\times C\left(I_{q,\omega}^T,\mathbb{R}\right)\to\mathbb{R}$ are given functionals. Suppose*

(H_1) *There exist constants $M_1,M_2,N_1,N_2>0$ such that, for each $t\in I_{q,\omega}^T$,*

$$\begin{aligned}&\left|F_i(t,{}^{C}D_{q,\omega}^{\beta_i}u_i,u_j)-F_i(t,{}^{C}D_{q,\omega}^{\beta_i}v_i,v_j)\right|\\ \le\ & M_i\left|{}^{C}D_{q,\omega}^{\beta_i}u_i-{}^{C}D_{q,\omega}^{\beta_i}v_i\right|+N_j\left|u_j-v_j\right|.\end{aligned}$$

(H_2) *There exist constants $K_1,K_2,L_1,L_2>0$ such that, for each $(u_1,u_2),(v_1,v_2)\in\mathcal{U}$,*

$$\begin{aligned}|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)| &\le K_1\|u_1-v_1\|+K_2\|u_2-v_2\|,\\ \text{and}\quad |\phi_2(u_1,u_2)-\phi_2(v_1,v_2)| &\le L_1\|u_1-v_1\|+L_2\|u_2-v_2\|.\end{aligned}$$

(H_3) *$g_i<g_i(t)<G_i$ for each $t\in I_{q,\omega}^T$.*

Then, Problems (1) and (2) have a unique solution provided that

$$\begin{aligned}\chi :=\ & \max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,M_2\Theta_3\right\}+\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,M_1\Theta_4\right\}\\ & +\max\left\{K_1\tilde{\Theta}_1+L_1\tilde{\Theta}_2+N_1\tilde{\Theta}_4,M_2\tilde{\Theta}_4\right\}+\max\left\{K_2\tilde{\Theta}_1+L_2\tilde{\Theta}_2+N_2\tilde{\Theta}_3,M_1\tilde{\Theta}_3\right\}\\ <\ & 1,\end{aligned}$$

where

$$|\Lambda| = \min\{g_1,g_2\}(T-\omega_0)\left|\frac{\lambda_2(\eta_2-\omega_0)^{\theta_2+1}\Gamma_q(\theta_2)}{\Gamma_q(\alpha_2)\Gamma_q(\theta_2+2)}-\frac{\lambda_1((\eta_1-\omega_0)^{\theta_1+1}\Gamma_q(\theta_1)}{\Gamma_q(\alpha_1)\Gamma_q(\theta_1+2)}\right|, \tag{20}$$

$$\Theta_1 = \frac{\lambda_2 G_2 (T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\Gamma_q(\theta_2+1)}\left\{\frac{\eta_2-\omega_0}{[\theta_2+1]_q}+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}}{\Gamma_q(\theta_1+2)}\right\}, \tag{21}$$

$$\Theta_2 = 1+\frac{\lambda_1 G_1 (T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\Gamma_q(\theta_1+1)}\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}}{\Gamma_q(\theta_2+2)}+\frac{\eta_1-\omega_0}{[\theta_1+1]_q}\right], \tag{22}$$

$$\Theta_3 = \frac{\lambda_2 G_2 (T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|}\times \left\{\frac{(T-\omega_0)^{\alpha_2}(\eta_2-\omega_0)}{|\Lambda|\Gamma_q(\alpha_2+1)\Gamma_q(\theta_2+2)}+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}(\eta_2-\omega_0)^{\alpha_2}}{\Gamma_q(\theta_1+2)\Gamma_q(\alpha_2+\theta_2+1)}\right\}, \tag{23}$$

$$\Theta_4 = \frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+\frac{\lambda_1 G_1 (T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|}\times \left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}(\eta_1-\omega_0)^{\alpha_1}}{\Gamma_q(\theta_2+2)\Gamma_q(\alpha_1+\theta_1+1)}+\frac{(T-\omega_0)^{\alpha_1}(\eta_1-\omega_0)}{\Gamma_q(\alpha_1+1)\Gamma_q(\theta_1+2)}\right], \tag{24}$$

$$\tilde{\Theta}_1 = \frac{(T-\omega_0)^2}{|\Lambda|}\left\{1+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1}}{\Gamma_q(\theta_1+1)}\right\}, \tag{25}$$

$$\tilde{\Theta}_2 = 1+\frac{(T-\omega_0)^2}{|\Lambda|}\left[1+\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2}}{\Gamma_q(\theta_2+1)}\right], \tag{26}$$

$$\tilde{\Theta}_3 = \frac{(T-\omega_0)^2}{|\Lambda|}\left\{\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\alpha_1+\theta_1}}{\Gamma_q(\alpha_1+\theta_1+1)}\right\}, \tag{27}$$

$$\tilde{\Theta}_4 = \frac{(T-\omega_0)^{\alpha_2}}{\Gamma_q(\alpha_2+1)}+\frac{(T-\omega_0)^2}{|\Lambda|}\left[\frac{(T-\omega_0)^{\alpha_2}}{\Gamma_q(\alpha_2+1)\Gamma_q(\theta_1+2)}+\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\alpha_2+\theta_2}}{\Gamma_q(\alpha_2+\theta_2+1)}\right]. \tag{28}$$

Proof. The goal is to prove that $\mathcal{T}$ is a contraction mapping. Letting $t \in I^T_{q,\omega}$ and $(u_1,u_2),(v_1,v_2) \in \mathcal{U}$, we obtain

$$\begin{aligned}
&\left|\mathcal{P}_u(F_1^*,F_2^*)-\mathcal{P}_v(F_1^*,F_2^*)\right| \\
&\leq \left|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right|+\frac{\lambda_2}{\Gamma_q(\theta_2)}\left|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right|\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))^{\underline{\theta_2-1}}_{q,\omega}\, g_2(s)\, d_{q,\omega}s \\
&\quad +\frac{1}{\Gamma_q(\alpha_1)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))^{\underline{\alpha_1-1}}_{q,\omega}\left|F_1^*(s,u)-F_1^*(s,v)\right| d_{q,\omega}s+\frac{\lambda_2}{\Gamma_q(\alpha_2)\Gamma_q(\theta_2)}\times \\
&\quad \int_{\omega_0}^{\eta_2}\int_{\omega_0}^{\xi}(\eta_2-\sigma_{q,\omega}(\xi))^{\underline{\theta_2-1}}_{q,\omega}(\xi-\sigma_{q,\omega}(s))^{\underline{\alpha_2-1}}_{q,\omega}\, g_2(s)\left|F_2^*(s,u)-F_2^*(s,v)\right| d_{q,\omega}s\, d_{q,\omega}\xi, \\
&\leq \Big(K_1\left\|u_1-v_1\right\|+K_2\left\|u_2-v_2\right\|\Big)+\Big(L_1\left\|u_1-v_1\right\|+L_2\left\|u_2-v_2\right\|\Big)\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2}}{\Gamma_q(\theta_2+1)} \\
&\quad +\Big(M_1\left|{}^C D^{\beta_1}_{q,\omega}u_1-{}^C D^{\beta_1}_{q,\omega}v_1\right|+N_2\left|u_2-v_2\right|\Big)\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)} \\
&\quad +\Big(M_2\left|{}^C D^{\beta_2}_{q,\omega}u_2-{}^C D^{\beta_2}_{q,\omega}v_2\right|+N_1\left|u_1-v_1\right|\Big)\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\alpha_2+\theta_2}}{\Gamma_q(\alpha_2+\theta_2+1)},
\end{aligned} \tag{29}$$

and

$$\left|\mathcal{Q}_u(F_1^*,F_2^*)-\mathcal{Q}_v(F_1^*,F_2^*)\right|$$

$$
\begin{aligned}
&\leq \left|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\right| + \frac{\lambda_1}{\Gamma_q(\theta_1)}\left|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right| \int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}} g_1(s)\, d_{q,\omega}s \\
&\quad + \frac{1}{\Gamma_q(\alpha_2)}\int_{\omega_0}^{T}(T-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_2-1}}\left|F_2^*(s,u)-F_2^*(s,v)\right| d_{q,\omega}s + \frac{\lambda_1}{\Gamma_q(\alpha_1)\Gamma_q(\theta_1)}\times \\
&\qquad \int_{\omega_0}^{\eta_1}\int_{\omega_0}^{\xi}(\eta_1-\sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\theta_1-1}}(\xi-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1-1}} g_1(s)\left|F_1^*(s,u)-F_1^*(s,v)\right| d_{q,\omega}s\, d_{q,\omega}\xi \\
&\leq \Big(L_1\|u_1-v_1\|+L_2\|u_2-v_2\|\Big) + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1}}{\Gamma_q(\theta_1+1)}\Big(K_1\|u_1-v_1\|+K_2\|u_2-v_2\|\Big) \\
&\quad + \Big(M_2\left|{}^C D_{q,\omega}^{\beta_2}u_2-{}^C D_{q,\omega}^{\beta_2}v_2\right| + N_1|u_1-v_1|\Big)\frac{(T-\omega_0)^{\alpha_2}}{\Gamma_q(\alpha_2+1)} \\
&\quad + \Big(M_1\left|{}^C D_{q,\omega}^{\beta_1}u_1-{}^C D_{q,\omega}^{\beta_1}v_1\right| + N_2|u_2-v_2|\Big)\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\alpha_1+\theta_1}}{\Gamma_q(\alpha_1+\theta_1+1)}. \qquad (30)
\end{aligned}
$$

From (29) and (30), we find that

$$
\begin{aligned}
&\Big|\left(\mathcal{T}_1(u_1,u_2)\right)(t)-\left(\mathcal{T}_1(v_1,v_2)\right)(t)\Big| \\
&\leq \frac{(t-\omega_0)}{|\Lambda|}\left\{\frac{\lambda_1 G_1}{\Gamma_q(\theta_1)}\left|\mathcal{P}_u(F_1^*,F_2^*)-\mathcal{P}_v(F_1^*,F_2^*)\right|\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}(s-\omega_0)\, d_{q,\omega}s\right. \\
&\qquad \left. + \frac{\lambda_2 G_2}{\Gamma_q(\theta_2)}\left|\mathcal{Q}_u(F_1^*,F_2^*)-\mathcal{Q}_v(F_1^*,F_2^*)\right|\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}(s-\omega_0)\, d_{q,\omega}s\right\} \\
&\quad + \Big(K_1\|u_1-v_1\|+K_2\|u_2-v_2\|\Big) + \Big(M_1\left|{}^C D_{q,\omega}^{\beta_1}u_1-{}^C D_{q,\omega}^{\beta_1}v_1\right| + N_2|u_2-v_2|\Big)\frac{(t-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)} \\
&\leq \Big(L_1\|u_1-v_1\|+L_2\|u_2-v_2\|\Big)\frac{\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\Gamma_q(\theta_2+1)}\left\{\frac{\eta_2-\omega_0}{[\theta_2+1]_q}+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}}{\Gamma_q(\theta_1+2)}\right\} \\
&\quad + \Big(K_1\|u_1-v_1\|+K_2\|u_2-v_2\|\Big)\times \\
&\quad \left\{1+\frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\Gamma_q(\theta_1+1)}\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}}{\Gamma_q(\theta_2+2)}+\frac{\eta_1-\omega_0}{[\theta_1+1]_q}\right]\right\} \\
&\quad + \Big(M_2\left|{}^C D_{q,\omega}^{\beta_2}u_2-{}^C D_{q,\omega}^{\beta_2}v_2\right| + N_1|u_1-v_1|\Big)\frac{\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|}\times \\
&\quad \left\{\frac{(T-\omega_0)^{\alpha_2}(\eta_2-\omega_0)}{|\Lambda|\Gamma_q(\alpha_2+1)\Gamma_q(\theta_2+2)}+\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}(\eta_2-\omega_0)^{\alpha_2}}{\Gamma_q(\theta_1+2)\Gamma_q(\alpha_2+\theta_2+1)}\right\} \\
&\quad + \Big(M_1\left|{}^C D_{q,\omega}^{\beta_1}u_1-{}^C D_{q,\omega}^{\beta_1}v_1\right| + N_2|u_2-v_2|\Big)\left\{\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)}+\frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|}\times\right. \\
&\quad \left.\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}(\eta_1-\omega_0)^{\alpha_1}}{\Gamma_q(\theta_2+2)\Gamma_q(\alpha_1+\theta_1+1)}+\frac{(T-\omega_0)^{\alpha_1}(\eta_1-\omega_0)}{\Gamma_q(\alpha_1+1)\Gamma_q(\theta_1+2)}\right]\right\}. \\
&= \|u_1-v_1\|\left[L_1\Theta_1+K_1\Theta_2+N_1\Theta_3\right] + \left|{}^C D_{q,\omega}^{\beta_2}u_2-{}^C D_{q,\omega}^{\beta_2}v_2\right| M_2\Theta_3 \\
&\quad + \|u_2-v_2\|\left[L_2\Theta_1+K_2\Theta_2+N_2\Theta_4\right] + \left|{}^C D_{q,\omega}^{\beta_1}u_1-{}^C D_{q,\omega}^{\beta_1}v_1\right| M_1\Theta_4 \\
&\leq \left(\|u_1-v_1\|+\left|{}^C D_{q,\omega}^{\beta_2}u_2-{}^C D_{q,\omega}^{\beta_2}v_2\right|\right)\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,\ M_2\Theta_3\right\} \\
&\quad + \left(\|u_2-v_2\|+\left|{}^C D_{q,\omega}^{\beta_1}u_1-{}^C D_{q,\omega}^{\beta_1}v_1\right|\right)\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,\ M_1\Theta_4\right\} \\
&\leq \|u_2-v_2\|_{\mathcal{C}_2}\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,\ M_2\Theta_3\right\}
\end{aligned}
$$

$$+\left\|u_1-v_1\right\|_{\mathcal{C}_1}\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,\ M_1\Theta_4\right\}. \tag{31}$$

Therefore, it implies that

$$\begin{aligned}&\left\|\mathcal{T}_1(u_1,u_2)-\mathcal{T}_1(v_1,v_2)\right\|\\ \leq\ &\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}\Big[\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,M_2\Theta_3\right\}\\ &+\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,M_1\Theta_4\right\}\Big].\end{aligned} \tag{32}$$

Next, taking the Caputo fractional Hahn difference of order $0<\beta_1\leq 1$ for (16), we have

$$\begin{aligned}&{}^{C}D^{\beta_1}_{q,\omega}(\mathcal{T}_1(u_1,u_2))(t) \qquad (33)\\ &=\frac{1}{|\Lambda|}\left\{\frac{\lambda_1}{\Gamma_q(\theta_1)}\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}g_1(s)(s-\omega_0)\mathcal{P}_u(F_1^*,F_2^*)\,d_{q,\omega}s\right.\\ &\left.-\frac{\lambda_2}{\Gamma_q(\theta_2)}\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}g_2(s)(s-\omega_0)\,\mathcal{Q}_u(F_1^*,F_2^*)\,d_{q,\omega}s\right\}+\phi_1(u_1,u_2)\\ &+\frac{1}{\Gamma_q(1-\beta_1)\Gamma_q(\alpha_1)}\int_{\omega_0}^{t}(t-\sigma_{q,\omega}(\xi))_{q,\omega}^{\underline{\beta_1}}\,{}^{C}D^{\beta_1}_{q,\omega}\left\{\int_{\omega_0}^{\xi}(\xi-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\alpha_1-1}}F_1^*(s,u)\,d_{q,\omega}s\right\}d_{q,\omega}\xi.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|{}^{C}D^{\beta_1}_{q,\omega}(\mathcal{T}_1(u_1,u_2))(t)-{}^{C}D^{\beta_1}_{q,\omega}(\mathcal{T}_1(v_1,v_2))(t)\right|\\ &<\ \left\|u_2-v_2\right\|_{\mathcal{C}_2}\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,\ M_2\Theta_3\right\}\\ &\quad+\left\|u_1-v_1\right\|_{\mathcal{C}_1}\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,\ M_1\Theta_4\right\}.\end{aligned} \tag{34}$$

It implies that

$$\begin{aligned}&\left\|{}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(u_1,u_2)-{}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(v_1,v_2)\right\|\\ <\ &\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}\Big[\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3,M_2\Theta_3\right\}\\ &+\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4,M_1\Theta_4\right\}\Big].\end{aligned} \tag{35}$$

Similarly, we obtain

$$\begin{aligned}&\left\|\mathcal{T}_2(u_1,u_2)-\mathcal{T}_2(v_1,v_2)\right\|\\ \leq\ &\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}\Big[\max\left\{K_1\tilde{\Theta}_1+L_1\tilde{\Theta}_2+N_1\tilde{\Theta}_4,M_2\tilde{\Theta}_4\right\}\\ &+\max\left\{K_2\tilde{\Theta}_1+L_2\tilde{\Theta}_2+N_2\tilde{\Theta}_3,M_1\tilde{\Theta}_3\right\}\Big]\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\left\|{}^{C}D^{\beta_2}_{q,\omega}\mathcal{T}_2(u_1,u_2)-{}^{C}D^{\beta_2}_{q,\omega}\mathcal{T}_2(v_1,v_2)\right\|\\ <\ &\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}\Big[\max\left\{K_1\tilde{\Theta}_1+L_1\tilde{\Theta}_2+N_1\tilde{\Theta}_4,M_2\tilde{\Theta}_4\right\}\\ &+\max\left\{K_2\tilde{\Theta}_1+L_2\tilde{\Theta}_2+N_2\tilde{\Theta}_3,M_1\tilde{\Theta}_3\right\}\Big].\end{aligned} \tag{37}$$

From (35) and (36), we can state that

$$\begin{aligned}
&\left\|\mathcal{T}_1(u_1,u_2)-\mathcal{T}_1(v_1,v_2)\right\|_{\mathcal{C}_1} \\
<\ & \|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}\Big[\max\left\{L_1\Theta_1+K_1\Theta_2+N_1\Theta_3, M_2\Theta_3\right\} \\
&\qquad+\max\left\{L_2\Theta_1+K_2\Theta_2+N_2\Theta_4, M_1\Theta_4\right\} \\
&\qquad+\max\left\{K_1\tilde{\Theta}_1+L_1\tilde{\Theta}_2+N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\} \\
&\qquad+\max\left\{K_2\tilde{\Theta}_1+L_2\tilde{\Theta}_2+N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\Big] \\
<\ & \chi\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}.
\end{aligned} \tag{38}$$

Similarly, by (32) and (37), we have

$$\left\|\mathcal{T}_1(u_1,u_2)-\mathcal{T}_1(v_1,v_2)\right\|_{\mathcal{C}_2} < \chi\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}. \tag{39}$$

Therefore, by (38) and (39), we can conclude that

$$\left\|\mathcal{T}(u_1,u_2)-\mathcal{T}(v_1,v_2)\right\|_{\mathcal{U}} < \chi\|(u_1-v_1,u_2-v_2)\|_{\mathcal{U}}. \tag{40}$$

Since $\chi<1$, $\mathcal{T}$ is a contraction mapping, from the Banach fixed point theorem, we can conclude that the operator $\mathcal{T}$ has a fixed point. Therefore, Problems (1) and (2) have a unique solution. □

4. Existence of at Least One Solution

In this section, we further present the existence of at least one solution to (1) and (2) by using Schauder's fixed point theorem.

Theorem 2. *Suppose that* (H_1)–(H_3) *hold. Then, Problem* (2) *has at least one solution on* $I^T_{q,\omega}$.

Proof. We divide the proof into three steps as follows
Step I. Verify that $\mathcal{T}$ map bounded sets into bounded sets in $B_R=\{(u_1,u_2)\in\mathcal{U}:\|(u_1,u_2)\|_{\mathcal{U}}\le R\}$. We let $\max\limits_{t\in I^T_{q,\omega}}|F_i(t,0,0)|=A_i$, $\sup\limits_{(u_1,u_2)\in\mathcal{U}}|\phi_i(u_1,u_2)|=B_i$ for $i=1,2$ and choose a constant

$$R\ge\frac{B_1(\Theta_1+\Theta_2)+B_2(\Theta_1+\Theta_2)}{1-\Phi}, \tag{41}$$

where $\Theta_i,\tilde{\Theta}_i,\ i=1,2,3,4$ are defined as (21)–(28), respectively, and Φ is defined by

$$\begin{aligned}
\Phi:=\ &\max\left\{N_1\Theta_3, M_2\Theta_3\right\}+\max\left\{N_2\Theta_4, M_1\Theta_4\right\} \\
&+\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\}+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}.
\end{aligned} \tag{42}$$

Here, we assume that

$$\begin{aligned}
|F_1^{**}(s,0)| &= \left|F_1\left(s,{}^CD^{\beta_1}_{q,\omega}u_1(s),u_2(t)\right)-F_1(s,0,0)\right|+|F_1(s,0,0)| \\
\text{and}\quad |F_2^{**}(s,0)| &= \left|F_2\Big(s,u_1(t),{}^CD^{\beta_2}_{q,\omega}u_2(s)\Big)-F_2(s,0,0)\right|+|F_2(s,0,0)|.
\end{aligned}$$

For each $t\in I^T_{q,\omega}$ and $(u_1,u_2)\in B_R$, we obtain

$$\left|\mathcal{P}_u(F_1^*,F_2^*)\right|$$

$$\leq B_1 + \frac{B_2\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2}}{\Gamma_q(\theta_2+1)} + \Big(M_1\left|{}^C D^{\beta_1}_{q,\omega}u_1\right| + N_2|u_2| + A_1\Big)\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)}$$
$$+\Big(M_2\left|{}^C D^{\beta_2}_{q,\omega}u_2\right| + N_1|u_1| + A_2\Big)\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\alpha_2+\theta_2}}{\Gamma_q(\alpha_2+\theta_2+1)}, \tag{43}$$

and

$$\Big|\mathcal{Q}_u(F_1^*,F_2^*) - \mathcal{Q}_v(F_1^*,F_2^*)\Big|$$
$$\leq B_2 + \frac{B_1\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1}}{\Gamma_q(\theta_1+1)} + \Big(M_2\left|{}^C D^{\beta_2}_{q,\omega}u_2\right| + N_1|u_1| + A_2\Big)\frac{(T-\omega_0)^{\alpha_2}}{\Gamma_q(\alpha_2+1)}$$
$$+\Big(M_1\left|{}^C D^{\beta_1}_{q,\omega}u_1\right| + N_2|u_2| + A_1\Big)\frac{\lambda_1 G_1(\eta_1-\omega_0)^{\alpha_1+\theta_1}}{\Gamma_q(\alpha_1+\theta_1+1)}. \tag{44}$$

From (43) and (44), we find that

$$\Big|\left(\mathcal{T}_1(u_1,u_2)\right)(t)\Big|$$
$$\leq \frac{B_2\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\Gamma_q(\theta_2+1)}\left\{\frac{(\eta_2-\omega_0)}{[\theta_2+1]_q} + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}}{\Gamma_q(\theta_1+2)}\right\}$$
$$+B_1\left\{1 + \frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\Gamma_q(\theta_1+1)}\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}}{\Gamma_q(\theta_2+2)} + \frac{\eta_1-\omega_0}{[\theta_1+1]_q}\right]\right\}$$
$$+\Big(M_2\left|{}^C D^{\beta_2}_{q,\omega}u_2\right| + N_1|u_1| + A_2\Big)\frac{\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|}\times$$
$$\left\{\frac{(T-\omega_0)^{\alpha_2}(\eta_2-\omega_0)}{|\Lambda|\Gamma_q(\alpha_2+1)\Gamma_q(\theta_2+2)} + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}(\eta_2-\omega_0)^{\alpha_2}}{\Gamma_q(\theta_1+2)\Gamma_q(\alpha_2+\theta_2+1)}\right\}$$
$$+\Big(M_1\left|{}^C D^{\beta_1}_{q,\omega}u_1\right| + N_2|u_2| + A_1\Big)\left\{\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)} + \frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|}\times\right.$$
$$\left.\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}(\eta_1-\omega_0)^{\alpha_1}}{\Gamma_q(\theta_2+2)\Gamma_q(\alpha_1+\theta_1+1)} + \frac{(T-\omega_0)^{\alpha_1}(\eta_1-\omega_0)}{\Gamma_q(\alpha_1+1)\Gamma_q(\theta_1+2)}\right]\right\}.$$
$$= \Big(|u_1|N_1\Theta_3 + \left|{}^C D^{\beta_2}_{q,\omega}u_2\right|M_2\Theta_3\Big) + \Big(|u_2|N_2\Theta_4 + \left|{}^C D^{\beta_1}_{q,\omega}u_1\right|M_1\Theta_4\Big)$$
$$+B_2\Theta_1 + B_1\Theta_2$$
$$\leq \left(|u_1| + \left|{}^C D^{\beta_2}_{q,\omega}u_2\right|\right)\max\left\{N_1\Theta_3, M_2\Theta_3\right\} + \left(|u_2| + \left|{}^C D^{\beta_1}_{q,\omega}u_1\right|\right)\max\left\{N_2\Theta_4,\ M_1\Theta_4\right\}$$
$$+B_2\Theta_1 + B_1\Theta_2$$
$$\leq \|u_2\|_{\mathcal{C}_2}\max\left\{N_1\Theta_3, M_2\Theta_3\right\} + \|u_1\|_{\mathcal{C}_1}\max\left\{N_2\Theta_4, M_1\Theta_4\right\} + B_2\Theta_1 + B_1\Theta_2$$
$$\leq \|(u_1,u_2)\|_{\mathcal{U}}\Big[\max\left\{N_1\Theta_3, M_2\Theta_3\right\} + \max\left\{N_2\Theta_4, M_1\Theta_4\right\}\Big] + B_2\Theta_1 + B_1\Theta_2. \tag{45}$$

Similarly to Theorem 1, we obtain

$$\left|{}^C D^{\beta_1}_{q,\omega}\left(\mathcal{T}_1(u_1,u_2)\right)(t)\right|$$
$$< \|(u_1,u_2)\|_{\mathcal{U}}\Big[\max\left\{N_1\Theta_3, M_2\Theta_3\right\} + \max\left\{N_2\Theta_4, M_1\Theta_4\right\}\Big] + B_2\Theta_1 + B_1\Theta_2. \tag{46}$$

Furthermore, we have

$$\begin{aligned}&\left\|\mathcal{T}_2(u_1,u_2)\right\| \\ \leq\ & \|(u_1,u_2)\|_{\mathcal{U}}\left[\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\}+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\right]+B_1\Theta_1+B_2\Theta_2,\end{aligned} \tag{47}$$

and

$$\begin{aligned}&\left\|{}^CD^{\beta_2}_{q,\omega}\mathcal{T}_2(u_1,u_2)\right\| \\ <\ & \|(u_1,u_2)\|_{\mathcal{U}}\left[\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\}+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\right]+B_1\Theta_1+B_2\Theta_2.\end{aligned} \tag{48}$$

From (46) and (47), we can show that

$$\begin{aligned}&\left\|\mathcal{T}_1(u_1,u_2)\right\|_{\mathcal{C}_1} \\ <\ & \|(u_1,u_2)\|_{\mathcal{U}}\Big[\max\left\{N_1\Theta_3, M_2\Theta_3\right\}+\max\left\{N_2\Theta_4, M_1\Theta_4\right\}+\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\} \\ &+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\Big]+B_1(\Theta_1+\Theta_2)+B_2(\Theta_1+\Theta_2).\end{aligned} \tag{49}$$

Similarly, from (45) and (48), we have

$$\begin{aligned}&\left\|\mathcal{T}_2(u_1,u_2)\right\|_{\mathcal{C}_2} \\ <\ & \|(u_1,u_2)\|_{\mathcal{U}}\Big[\max\left\{N_1\Theta_3, M_2\Theta_3\right\}+\max\left\{N_2\Theta_4, M_1\Theta_4\right\}+\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\} \\ &+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\Big]+B_1(\Theta_1+\Theta_2)+B_2(\Theta_1+\Theta_2).\end{aligned} \tag{50}$$

Therefore, from (49) and (50), we can conclude that

$$\begin{aligned}&\left\|\mathcal{T}(u_1,u_2)\right\|_{\mathcal{U}} \\ <\ & \|(u_1,u_2)\|_{\mathcal{U}}\Big[\max\left\{N_1\Theta_3, M_2\Theta_3\right\}+\max\left\{N_2\Theta_4, M_1\Theta_4\right\}+\max\left\{N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\} \\ &+\max\left\{N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\}\Big]+B_1(\Theta_1+\Theta_2)+B_2(\Theta_1+\Theta_2) \\ =\ & \|(u_1,u_2)\|_{\mathcal{U}}\,\Phi+B_1(\Theta_1+\Theta_2)+B_2(\Theta_1+\Theta_2).\end{aligned} \tag{51}$$

From (51), we get $\left\|\mathcal{T}(u_1,u_2)\right\|_{\mathcal{U}} \leq R$. Therefore, $\mathcal{T}$ is uniformly bounded.
Step II. Show that $\mathcal{T}$ is continuous on B_R.
Letting $\epsilon > 0$, there exists $\delta = \max\{\delta_1,\delta_2,\delta_3,\delta_4\} > 0$ such that, for each $t \in I^T_{q,\omega}$ and $(u_1,u_2),(v_1,v_2) \in B_R$ with

$$\left|F_1^*(t,u)-F_1^*(t,v)\right| < \min\left\{\frac{\epsilon}{8\Theta_4},\frac{\epsilon}{8\tilde{\Theta}_4}\right\},$$

whenever $|u_2 = v_2| + \left|{}^CD^{\beta_1}_{q,\omega}u_1 - {}^CD^{\beta_1}_{q,\omega}v_1\right| < \delta_1$,

$$\left|F_2^*(t,u)-F_2^*(t,v)\right| < \min\left\{\frac{\epsilon}{8\Theta_3},\frac{\epsilon}{8\tilde{\Theta}_3}\right\},$$

whenever $|u_1 = v_1| + \left|{}^CD^{\beta_1}_{q,\omega}u_2 - {}^CD^{\beta_1}_{q,\omega}v_2\right| < \delta_2$,

$$\left|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\right| < \min\left\{\frac{\epsilon}{8\Theta_2},\frac{\epsilon}{8\tilde{\Theta}_2}\right\},$$

whenever $\max\{|u_1 - u_2|, |v_1 - v_2|\} < \delta_3$,

$$|\phi_2(u_1,u_2) - \phi_2(v_1,v_2)| < \min\left\{\frac{\epsilon}{8\Theta_1}, \frac{\epsilon}{8\tilde{\Theta}_1}\right\},$$

whenever $\max\{|u_1 - u_2|, |v_1 - v_2|\} < \delta_4$.
Consider

$$\begin{aligned}
&\left|\mathcal{P}_u(F_1^*, F_2^*) - \mathcal{P}_v(F_1^*, F_2^*)\right| \\
&\le \|\phi_1(u_1,u_2) - \phi_1(v_1,v_2)\| + \|\phi_2(u_1,u_2) - \phi_2(v_1,v_2)\| \frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2}}{\Gamma_q(\theta_2+1)} \\
&+ \|F_1^*(t,u) - F_1^*(t,v)\| \frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)} + \|F_2^*(t,u) - F_2^*(t,v)\| \frac{\lambda_2 G_2(\eta_2-\omega_0)^{\alpha_2+\theta_2}}{\Gamma_q(\alpha_2+\theta_2+1)},
\end{aligned} \tag{52}$$

and

$$\begin{aligned}
&\left|\mathcal{Q}_u(F_1^*, F_2^*) - \mathcal{Q}_v(F_1^*, F_2^*)\right| \\
&\le \|\phi_2(u_1,u_2) - \phi_2(v_1,v_2)\| + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1}}{\Gamma_q(\theta_1+1)} \|\phi_1(u_1,u_2) - \phi_1(v_1,v_2)\| \\
&+ \|F_2^*(t,u) - F_2^*(t,v)\| \frac{(T-\omega_0)^{\alpha_2}}{\Gamma_q(\alpha_2+1)} + \|F_1^*(t,u) - F_1^*(t,v)\| \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\alpha_1+\theta_1}}{\Gamma_q(\alpha_1+\theta_1+1)}.
\end{aligned} \tag{53}$$

From (52) and (53), we find that

$$\begin{aligned}
&\left|(\mathcal{T}_1(u_1,u_2))(t) - (\mathcal{T}_1(v_1,v_2))(t)\right| \\
\le\ & \|\phi_2(u_1,u_2) - \phi_2(v_1,v_2)\| \frac{\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|\Gamma_q(\theta_2+1)} \left\{\frac{\eta_2-\omega_0}{[\theta_2+1]_q} + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}}{\Gamma_q(\theta_1+2)}\right\} \\
&+ \|\phi_1(u_1,u_2) - \phi_1(v_1,v_2)\| \times \\
&\left\{1 + \frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|\Gamma_q(\theta_1+1)}\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}}{\Gamma_q(\theta_2+2)} + \frac{\eta_1-\omega_0}{[\theta_1+1]_q}\right]\right\} \\
&+ \|F_2^*(t,u) - F_2^*(t,v)\| \frac{\lambda_2 G_2(T-\omega_0)(\eta_2-\omega_0)^{\theta_2}}{|\Lambda|} \times \\
&\left\{\frac{(T-\omega_0)^{\alpha_2}(\eta_2-\omega_0)}{|\Lambda|\Gamma_q(\alpha_2+1)\Gamma_q(\theta_2+2)} + \frac{\lambda_1 G_1(\eta_1-\omega_0)^{\theta_1+1}(\eta_2-\omega_0)^{\alpha_2}}{\Gamma_q(\theta_1+2)\Gamma_q(\alpha_2+\theta_2+1)}\right\} \\
&+ \|F_1^*(t,u) - F_1^*(t,v)\| \left\{\frac{(T-\omega_0)^{\alpha_1}}{\Gamma_q(\alpha_1+1)} + \frac{\lambda_1 G_1(T-\omega_0)(\eta_1-\omega_0)^{\theta_1}}{|\Lambda|} \times \right. \\
&\left.\left[\frac{\lambda_2 G_2(\eta_2-\omega_0)^{\theta_2+1}(\eta_1-\omega_0)^{\alpha_1}}{\Gamma_q(\theta_2+2)\Gamma_q(\alpha_1+\theta_1+1)} + \frac{(T-\omega_0)^{\alpha_1}(\eta_1-\omega_0)}{\Gamma_q(\alpha_1+1)\Gamma_q(\theta_1+2)}\right]\right\}.
\end{aligned} \tag{54}$$

Therefore, it implies that

$$\begin{aligned}
\|\mathcal{T}_1(u_1,u_2) - \mathcal{T}_1(v_1,v_2)\| \le\ & \|\phi_2(u_1,u_2) - \phi_2(v_1,v_2)\|\,\Theta_1 + \|\phi_1(u_1,u_2) - \phi_1(v_1,v_2)\|\,\Theta_2 + \\
& \|F_2^*(t,u) - F_2^*(t,v)\|\,\Theta_3 + \|F_1^*(t,u) - F_1^*(t,v)\|\,\Theta_4 \\
<\ & \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{2}.
\end{aligned} \tag{55}$$

Similarly to the above proof and Theorem 1, we obtain

$$\begin{aligned}\Big\|{}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(u_1,u_2) - {}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(v_1,v_2)\Big\| & \\ &< \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|\,\Theta_1 + \|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|\,\Theta_2 + \\ &\quad \|F_2^*(t,u)-F_2^*(t,v)\|\,\Theta_3 + \|F_1^*(t,u)-F_1^*(t,v)\|\,\Theta_4 \\ &< \frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8} = \frac{\epsilon}{2},\end{aligned} \tag{56}$$

$$\begin{aligned}\Big\|\mathcal{T}_2(u_1,u_2)-\mathcal{T}_2(v_1,v_2)\Big\| &\le \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|\,\tilde{\Theta}_1 + \|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|\,\tilde{\Theta}_2 + \\ &\quad \|F_2^*(t,u)-F_2^*(t,v)\|\,\tilde{\Theta}_3 + \|F_1^*(t,u)-F_1^*(t,v)\|\,\tilde{\Theta}_4 \\ &< \frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8} = \frac{\epsilon}{2},\end{aligned} \tag{57}$$

and

$$\begin{aligned}\Big\|{}^{C}D^{\beta_2}_{q,\omega}\mathcal{T}_2(u_1,u_2) - {}^{C}D^{\beta_1 2}_{q,\omega}\mathcal{T}_2(v_1,v_2)\Big\| & \\ &< \|\phi_2(u_1,u_2)-\phi_2(v_1,v_2)\|\,\tilde{\Theta}_1 + \|\phi_1(u_1,u_2)-\phi_1(v_1,v_2)\|\,\tilde{\Theta}_2 + \\ &\quad \|F_2^*(t,u)-F_2^*(t,v)\|\,\tilde{\Theta}_3 + \|F_1^*(t,u)-F_1^*(t,v)\|\,\tilde{\Theta}_4 \\ &< \frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8}+\frac{\epsilon}{8} = \frac{\epsilon}{2}.\end{aligned} \tag{58}$$

From (56) and (57), we can show that

$$\begin{aligned}\Big\|\mathcal{T}_1(u_1,u_2)-\mathcal{T}_1(v_1,v_2)\Big\|_{\mathcal{C}_1} & \\ &= \Big\|{}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(u_1,u_2) - {}^{C}D^{\beta_1}_{q,\omega}\mathcal{T}_1(v_1,v_2)\Big\| + \Big\|\mathcal{T}_2(u_1,u_2)-\mathcal{T}_2(v_1,v_2)\Big\| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{59}$$

Similarly, from (55) and (58), we have

$$\begin{aligned}\Big\|\mathcal{T}_2(u_1,u_2)-\mathcal{T}_2(v_1,v_2)\Big\|_{\mathcal{C}_1} & \\ &= \Big\|{}^{C}D^{\beta_2}_{q,\omega}\mathcal{T}_2(u_1,u_2) - {}^{C}D^{\beta_2}_{q,\omega}\mathcal{T}_2(v_1,v_2)\Big\| + \Big\|\mathcal{T}_1(u_1,u_2)-\mathcal{T}_1(v_1,v_2)\Big\| \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.\end{aligned} \tag{60}$$

Therefore, from (59) and (60), we can conclude that $\Big\|\mathcal{T}(u_1,u_2)\mathcal{T}(v_1,v_2)\Big\|_{\mathcal{U}} < \epsilon$. This means that $\mathcal{T}$ is continuous on B_R.

Step III. For this step, we prove that $\mathcal{T}$ is equicontinuous with B_R. For any $t_1, t_2 \in I^T_{q,\omega}$ with $t_1 < t_2$, we have

$$\begin{aligned}\Big|(\mathcal{T}_1(u_1,u_2))(t_2) - (\mathcal{T}_1(u_1,u_2))(t_1)\Big| & \\ &\le \frac{|t_2-t_1|}{|\Lambda|}\Bigg\{\frac{\lambda_1}{\Gamma_q(\theta_1)}\int_{\omega_0}^{\eta_1}(\eta_1-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_1-1}}\, g_1(s)(s-\omega_0)\mathcal{P}_u(F_1^*,F_2^*)\,d_{q,\omega}s \\ &\quad + \frac{\lambda_2}{\Gamma_q(\theta_2)}\int_{\omega_0}^{\eta_2}(\eta_2-\sigma_{q,\omega}(s))_{q,\omega}^{\underline{\theta_2-1}}\, g_2(s)(s-\omega_0)\,\mathcal{Q}_u(F_1^*,F_2^*)\,d_{q,\omega}s\Bigg\} \\ &\quad + \frac{\|F_1^*\|}{\Gamma_q(\alpha_1+1)}\Big|(t_2-\omega_0)^{\alpha_1} - (t_1-\omega_0)^{\alpha_1}\Big|,\end{aligned} \tag{61}$$

and

$$\begin{aligned}&\left|{}^{C}D_{q,\omega}^{\beta_1}(\mathcal{T}_1(u_1,u_2))(t_2)-{}^{C}D_{q,\omega}^{\beta_1}(\mathcal{T}_1(v_1,v_2))(t_1)\right| \\ &\qquad\leq \frac{\|F_1^*\|\Gamma_q(\beta_1)}{\Gamma_q(1-\beta_1)\Gamma_q(\alpha_1+\beta_1)}\left|(t_2-\omega_0)^{\alpha_1+\beta_1}-(t_1-\omega_0)^{\alpha_1+\beta_1}\right|.\end{aligned} \tag{62}$$

Furthermore, by (61) and (62), we have

$$\begin{aligned}&\left|(\mathcal{T}_2(u_1,u_2))(t_2)-(\mathcal{T}_2(u_1,u_2))(t_1)\right| \\ &\quad= \frac{|t_2-t_1|}{|\Lambda|}\left\{(T-\omega_0)\,\mathcal{P}_u(F_1^*,F_2^*)-(T-\omega_0)\,\mathcal{Q}_u(F_1^*,F_2^*)\right\}+\phi_2(u_1,u_2) \\ &\quad+\frac{\|F_2^*\|}{\Gamma_q(\alpha_2+1)}\left|(t_2-\omega_0)^{\alpha_2}-(t_1-\omega_0)^{\alpha_2}\right|,\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\left|{}^{C}D_{q,\omega}^{\beta_1}(\mathcal{T}_2(u_1,u_2))(t_2)-{}^{C}D_{q,\omega}^{\beta_1}(\mathcal{T}_2(v_1,v_2))(t_1)\right| \\ &\qquad\leq \frac{\|F_2^*\|\Gamma_q(\beta_2)}{\Gamma_q(1-\beta_12)\Gamma_q(\alpha_2+\beta_2)}\left|(t_2-\omega_0)^{\alpha_2+\beta_2}-(t_1-\omega_0)^{\alpha_2+\beta_2}\right|.\end{aligned} \tag{64}$$

When $|t_2-t_1| \to 0$, the right-hand side of inequalities (61) and (64) tends be zero. Thus, $\mathcal{T}$ is relatively compact on B_R.

Therefore, $\mathcal{G}(B_R)$ is an equicontinuous set. From the result of Steps I to III together with the Arzelá–Ascoli theorem, we find that $\mathcal{T}:\mathcal{U}\to\mathcal{U}$ is completely continuous. Therefore, we can conclude from the Schauder fixed point theorem that Problems (1) and (2) have at least one solution. □

5. Example

In this section, we provide some examples to show the applicability of our results. Consider the system of fractional Hahn difference equations

$$\begin{aligned}{}^{C}D_{\frac{1}{3},2}^{\frac{3}{2}}u_1(t) &= \frac{\sin^2\pi t}{(100+t)^3}\cdot\frac{|u_2(t)|}{2+|u_2(t)|}+\frac{e^{-2t}}{(20+t)^3}\,{}^{C}D_{\frac{1}{3},2}^{\frac{1}{4}}u_1(t), \\ {}^{C}D_{\frac{1}{3},2}^{\frac{4}{3}}u_2(t) &= \frac{\cos^2\pi t}{(200+t)^2}\cdot\frac{|u_1(t)|}{3+|u_1(t)|}+\frac{e^{-2|\sin\pi t+10|}}{(10+t)^3}\,{}^{C}D_{\frac{1}{3},2}^{\frac{2}{3}}u_2(t),\quad t\in[3,10]_{\frac{1}{3},2}, \\ u_1(3) = \frac{\cos^2|\pi u_1|}{(100\pi)^3}&(|u_1|+e|u_2|) = \frac{1}{2}\mathcal{I}_{\frac{1}{3},2}^{\frac{1}{3}}[200\pi+20\sin^2\pi t]\,u_2\left(\frac{250}{81}\right), \\ u_2(3) = \frac{\sin^2|\pi u_2|}{(100e)^3}&(|u_1|+\pi|u_2|) = \frac{2}{3}\mathcal{I}_{\frac{1}{3},2}^{\frac{3}{4}}[100e+10\cos^2\pi t]\,u_1\left(\frac{34}{9}\right).\end{aligned} \tag{65}$$

Here, we set $q=\frac{1}{3}$, $\omega=2$, $\omega_0=\frac{\omega}{1-q}=3$, $\alpha_1=\frac{3}{2}$, $\alpha_2=\frac{4}{3}$, $\beta_1=\frac{1}{4}$, $\beta_2=\frac{2}{3}$, $\gamma_1=\frac{2}{3}$, $\gamma_2=\frac{1}{2}$, $\theta_1=\frac{3}{4}$, $\theta_2=\frac{1}{3}$, $T=10$, $\eta_1=10\left(\frac{1}{3}\right)^2+2[2]_{\frac{1}{3}}=\frac{34}{9}$, $\eta_2=10\left(\frac{1}{3}\right)^4+2[4]_{\frac{1}{3}}=\frac{250}{81}$, $g_1(t)=$

$200\pi + 20\sin^2 \pi t$, $g_2(t) = 100e + 10\cos^2 \pi t$, $\phi_1(u_1, u_2) = \frac{\cos^2 |\pi u_1|}{(100\pi)^3}(|u_1| + e|u_2|)$, $\phi_2(u_1, u_2) = \frac{\sin^2 |\pi u_2|}{(100e)^3}(|u_1| + \pi|u_2|)$ and

$$F_1\left(t, u_2(t), {}^{C}D^{\beta_1}_{q,\omega}u_1(t)\right) = \frac{\sin^2 \pi t}{(100+t)^3} \cdot \frac{|u_2(t)|}{2 + |u_2(t)|} + \frac{e^{-2t}}{(20+t)^3}{}^{C}D^{\frac{1}{4}}_{\frac{1}{3},2}u_1(t),$$
$$F_2\left(t, u_1(t), {}^{C}D^{\beta_2}_{q,\omega}u_2(t)\right) = \frac{\cos^2 \pi t}{(200+t)^2} \cdot \frac{|u_1(t)|}{3 + |u_1(t)|} + \frac{e^{-2|\sin \pi t + 10|}}{(10+t)^3}{}^{C}D^{\frac{2}{3}}_{\frac{1}{3},2}u_2(t).$$

For all $t \in [3, 10]_{\frac{1}{3},2}$ and $u, v \in \mathbb{R}$, it is clear that

$$\left|F_1\left(t, u_2, {}^{C}D^{\beta_1}_{q,\omega}u_1\right) - F_1\left(t, v_2, {}^{C}D^{\beta_1}_{q,\omega}v_1\right)\right| \le \frac{1}{e^9 23^3}|{}^{C}D^{\beta_1}_{q,\omega}u_1 - {}^{C}D^{\beta_1}_{q,\omega}v_1| + \frac{1}{103^3}|u_2 - v_2|,$$
$$\left|F_2\left(t, u_1, {}^{C}D^{\beta_2}_{q,\omega}u_2\right) - F_2\left(t, u_1, {}^{C}D^{\beta_2}_{q,\omega}v_2\right)\right| \le \frac{1}{e^{11} 13^2}|{}^{C}D^{\beta_2}_{q,\omega}u_2 - {}^{C}D^{\beta_2}_{q,\omega}v_2| + \frac{1}{203^2}|u_1 - v_1|.$$

Thus, $(H1)$ holds with $M_1 = 1.014 \times 10^{-8}$, $M_2 = 9.883 \times 10^{-8}$ and $N_1 = 9.151 \times 10^{-7}$, $N_2 = 0.0000243$.
For all $u, v \in \mathcal{C}$, we have

$$|\phi_1(u_1, u_2) - \phi_1(v_1, v_2)| \le \frac{1}{(100\pi)^3}\|u_1 - v_1\| + \frac{e}{(100\pi)^3}\|u_2 - v_2\|,$$
$$|\phi_2(u_1, u_2) - \phi_2(v_1, v_2)| \le \frac{1}{(100e)^3}\|u_1 - v_1\| + \frac{\pi}{(100e)^3}\|u_2 - v_2\|.$$

Thus, $(H2)$ holds with $K_1 = 3.225 \times 10^{-8}$, $K_2 = 8.767 \times 10^{-8}$ and $L_1 = 1.564 \times 10^{-7}$, $L_2 = 4.979 \times 10^{-8}$.
For all $t \in [3, 10]_{\frac{1}{3},2}$, we have

$$g_1 = 100e \le g_1(t) \le 100e + 10 = G_1,$$
$$\text{and} \quad g_2 = 200\pi \le g_2(t) \le 200\pi + 20 = G_2.$$

Thus, $(H3)$ holds.
In addition, we find that

$$|\Lambda| = 2.18417.011, \quad \Theta_1 = 3.377, \quad \Theta_2 = 1.6079, \quad \Theta_3 = 0.108, \quad \Theta_4 = 20.038,$$
$$\tilde{\Theta}_1 = 0.255, \quad \tilde{\Theta}_2 = 1.023, \quad \tilde{\Theta}_3 = 0.119, \quad \tilde{\Theta}_4 = 12.363.$$

Then, we find that

$$\begin{aligned}\chi = & \max\left\{L_1\Theta_1 + K_1\Theta_2 + N_1\Theta_3, M_2\Theta_3\right\} + \max\left\{L_2\Theta_1 + K_2\Theta_2 + N_2\Theta_4, M_1\Theta_4\right\} \\ & + \max\left\{K_1\tilde{\Theta}_1 + L_1\tilde{\Theta}_2 + N_1\tilde{\Theta}_4, M_2\tilde{\Theta}_4\right\} + \max\left\{K_2\tilde{\Theta}_1 + L_2\tilde{\Theta}_2 + N_2\tilde{\Theta}_3, M_1\tilde{\Theta}_3\right\} \\ = & \ 0.00489 < 1.\end{aligned}$$

Therefore, we can conclude from Theorem 1 that Problem (65) has a unique solution. □

6. Conclusions

We initiate the study of the existence and a unique result of the solution for a Caputo fractional Hahn difference equations with nonlocal fractional Hahn integral boundary conditions. Some conditions are obtained when Banach's fixed point theorem is used as a tool. In addition, the conditions for the case of at least one solution is obtained by using the Schauder fixed point theorem.

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