mathematics

MDPI

Article

Representing Sums of Finite Products of Chebyshev Polynomials of the First Kind and Lucas Polynomials by Chebyshev Polynomials

Taekyun Kim [1,2], Dae San Kim [3], Dmitry V. Dolgy [4] and Jongkyum Kwon [5,*]

1 Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China
2 Department of Mathematics, Kwangwoon University, Seoul 139-701, Korea; tkkim@kw.ac.kr
3 Department of Mathematics, Sogang University, Seoul 121-742, Korea; dskim@sogang.ac.kr
4 Hanrimwon, Kwangwoon University, Seoul 139-701, Korea; d_dol@mail.ru
5 Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo 52828, Korea
* Correspondence: mathkjk26@gnu.ac.kr; Tel.: +82-055-772-2252

Received: 21 November 2018; Accepted: 21 December 2018; Published: 27 December 2018

Abstract: In this paper, we study sums of finite products of Chebyshev polynomials of the first kind and Lucas polynomials and represent each of them in terms of Chebyshev polynomials of all kinds. Here, the coefficients involve terminating hypergeometric functions ${}_2F_1$ and these representations are obtained by explicit computations.

Keywords: sums of finite products; Chebyshev polynomials of the first kind; Lucas polynomials; Chebyshev polynomials of all kinds

1. Introduction and Preliminaries

1.1. Introduction

In this article, we will consider the two sums of finite products

$$\begin{aligned}\alpha_{m,r}(x) = &\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\ &- \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x), \\ &(m \geq 2, r \geq 1),\end{aligned} \tag{1}$$

in terms of Chebyshev polynomials of the first kind and

$$\begin{aligned}\beta_{m,r}(x) = &\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x) \\ &+ \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x), \\ &(m \geq 2, r \geq 1),\end{aligned} \tag{2}$$

in terms of Lucas polynomials. More precisely, we will express these polynomials $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ of degree m as linear combinations of Chebyshev polynomials of the first, second, third and fourth kinds. Here, the coefficients involve some terminating hypergeometric functions ${}_2F_1$. Before we state our main result at the end of this Section 1.1, we will mention some of the previous works that are related to our contribution.

Along the same line as the present paper, various sums of finite products of several non-Appell polynomials, namely Chebyshev polynomials of the second, third and fourth kinds and Fibonacci, Legendre, Laguerre polynomials, have been represented by Chebyshev polynomials of all kinds (see [1–3]). For sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials, they are also represented by Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials in [4].

In addition, representations by Bernoulli polynomials have been done for certain sums of finite products of some Appell and non-Appell polynomials. Indeed, certain sums of finite products of Bernoulli and Euler polynomials were studied in [5,6], and such sums of finite products were expressed as linear combinations of Bernoulli polynomials. These were done by deriving Fourier series expansions for functions closely related to those sums of finite products. The same had been done also for quite a few non-Appell polynomials in [7–10], namely Chebyshev polynomials of the first, second, third and fourth kinds, and Legendre, Laguerre, Fibonacci and Lucas polynomials.

Our pursuit of this line of research can be justified by the following. Firstly, the present research can be viewed as a generalization of the classical linearization problems. Indeed, the linearization problem is concerned with determining the coefficients in the expansion of the product $u_m(x)v_n(x)$ of two polynomials $u_m(x)$ and $v_n(x)$ in terms of an arbitrary polynomial sequence $\{w_k(x)\}_{k\geq 0}$:

$$u_m(x)v_n(x) = \sum_{k=0}^{m+n} c_k(mn)w_k(x).$$

Secondly, our problem has to do with the famous Faber–Pandharipande–Zagier and Miki's identities. Namely, it is possible to express the sums of products of two Bernoulli polynomials $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x)B_{m-k}(x)$, $(m \geq 2)$, as linear combinations of Bernoulli polynomials. Namely, we can show that

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{2k\,(2m-2k)} B_{2k}\,(x)\,B_{2m-2k}\,(x) + \frac{2}{2m-1} B_1\,(x)\,B_{2m-1}\,(x) \\
=&\frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k}B_{2m-2k}\,(x) + \frac{1}{m}H_{2m-1}B_{2m}\,(x) + \frac{2}{2m-1}B_{2m-1}B_1\,(x)\,,
\end{aligned} \tag{3}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers. This follows, for example, from the Fourier series expansion of $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(\langle x\rangle)B_{m-k}(\langle x\rangle)$, where $\langle x\rangle = x - [x]$ denotes the fractional part of any real number x. Then, the Miki's and Faber–Pandharipande–Zagier identities respectively follow by letting $x = 0$ and $x = \frac{1}{2}$ in (3). This approach via Fourier series expansions is simple compared to other much more involved ones. For some details on these, we let the reader refer to Introduction in [11] and the papers therein.

As we said in the above, we are going to investigate the sums of finite products of Chebyshev polynomials of the first kind in (1) and those of Lucas polynomials in (2). Then, we will represent $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ in terms of Chebyshev polynomials of the four kinds $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$. These will be done by explicit computations, using the general formulas in Proposition 1 and Proposition 2. We note here that the results in Proposition 1 can be derived by making use of orthogonalities, Rodrigues' formulas and integration by parts. The next two theorems are the main results of this paper.

Theorem 1. *Let m, r be any integers with $m \geq 2, r \geq 1$. Then, we have the following:*

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&- \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&= \binom{m+r}{r} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{j} \mathcal{E}_{m-2j\,2}F_1(-j, j-m; 1-m-r; 1) T_{m-2j}(x) && (4) \\
&= \frac{(m+r)!}{(m+1)!} \sum_{j=0}^{[\frac{m}{2}]} \binom{m+1}{j} (m-2j+1)\,{}_2F_1(-j, j-m-1; 1-m-r; 1) U_{m-2j}(x) && (5) \\
&= \binom{m+r}{r} \sum_{j=0}^{m} \binom{m}{[\frac{j}{2}]} {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; 1) V_{m-j}(x) && (6) \\
&= \binom{m+r}{r} \sum_{j=0}^{m} (-1)^j \binom{m}{[\frac{j}{2}]} {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; 1) W_{m-j}(x). && (7)
\end{aligned}$$

Here, $[x]$ denotes the greatest integer $\leq x$.

Theorem 2. *Let m, r be integers with $m \geq 2$, $r \geq 1$. Then, we have the following identities:*

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x) \\
&+ \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} (\frac{x}{2})^l L_{i_1}(x) L_{i_2}(x) \cdots L_{i_{r+1}}(x) \\
&= 2^{r+1-m} \binom{m+r}{r} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{j} \mathcal{E}_{m-2j} \\
&\quad \times {}_2F_1(-j, j-m; 1-m-r; -4) T_{m-2j}(x) && (8) \\
&= \frac{2^{r+1-m}}{r} \binom{m+r}{r-1} \sum_{j=0}^{[\frac{m}{2}]} (m-2j+1) \binom{m+1}{j} \\
&\quad \times {}_2F_1(-j, j-m-1; 1-m-r; -4) U_{m-2j}(x) && (9) \\
&= 2^{r+1-m} \binom{m+r}{r} \sum_{j=0}^{m} \binom{m}{[\frac{j}{2}]} \\
&\quad \times {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; -4) V_{m-j}(x) && (10) \\
&= 2^{r+1-m} \binom{m+r}{r} \sum_{j=0}^{m} (-1)^j \binom{m}{[\frac{j}{2}]} \\
&\quad \times {}_2F_1(-[\frac{j}{2}], [\frac{j}{2}]-m; 1-m-r; -4) W_{m-j}(x). && (11)
\end{aligned}$$

For related papers on Chebyshev polynomials, we let the reader refer to [12,13].

1.2. Preliminaries

In this section, after fixing some notations, we will recall some basic facts that are needed in this paper.

For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $\langle x\rangle_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{12}$$

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ \langle x\rangle_0 = 1. \tag{13}$$

The two factorial polynomials are related by

$$(-1)^n (x)_n = \langle -x\rangle_n,\ (-1)^n \langle x\rangle_n = (-x)_n. \tag{14}$$

The hypergeometric function ${}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x)$ is defined by

$$\begin{aligned} &{}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) \\ &= \sum_{n=0}^{\infty} \frac{\langle a_1\rangle_n \cdots \langle a_p\rangle_n}{\langle b_1\rangle_n \cdots \langle b_q\rangle_n} \frac{x^n}{n!}. \end{aligned} \tag{15}$$

Below, we are going to recall some very basic facts about Chebyshev polynomials of the first, second, third and fourth kinds, and Lucas polynomials. The Chebyshev polynomials belong to the family of classical orthogonal polynomials. For full accounts of this fascinating area of mathematics, we let the reader refer to [14–16].

In terms of generating functions, the Lucas polynomials and Chebyshev polynomials of the first, second, third and fourth kinds are respectively given by

$$F(t,x) = \frac{2-xt}{1-xt-t^2} = \sum_{n=0}^{\infty} L_n(x)t^n, \tag{16}$$

$$G(t,x) = \frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{17}$$

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{18}$$

$$\frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{19}$$

$$\frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty} W_n(x)t^n. \tag{20}$$

In addition, the Lucas polynomials and Chebyshev polynomials of the first, second, third and fourth kinds are respectively given by the following explicit expressions:

$$L_n(x) = n \sum_{l=0}^{[\frac{n}{2}]} \frac{1}{n-l} \binom{n-l}{l} x^{n-2l},\ (n \geq 1), \tag{21}$$

$$\begin{aligned} T_n(x) &= {}_2F_1(-n, n; \tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \frac{n}{2} \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \frac{1}{n-l} \binom{n-l}{l} (2x)^{n-2l},\ (n \geq 1), \end{aligned} \tag{22}$$

$$\begin{aligned} U_n(x) &= (n+1){}_2F_1(-n, n+2; \tfrac{3}{2}; \tfrac{1-x}{2}) \\ &= \sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n-l}{l} (2x)^{n-2l},\ (n \geq 0), \end{aligned} \tag{23}$$

$$V_n(x) = {}_2F_1(-n, n+1; \tfrac{1}{2}; \tfrac{1-x}{2})$$
$$= \sum_{l=0}^{n} \binom{2n-l}{l} 2^{n-l}(x-1)^{n-l}, \ (n \geq 0), \tag{24}$$
$$W_n(x) = (2n+1)\,{}_2F_1(-n, n+1; \tfrac{3}{2}; \tfrac{1-x}{2})$$
$$= (2n+1) \sum_{l=0}^{n} \frac{2^{n-l}}{2n-2l+1} \binom{2n-l}{l} (x-1)^{n-l}, \ (n \geq 0). \tag{25}$$

The Chebyshev polynomials of the first, second, third and fourth kinds are given by Rodrigues' formulas:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}} \frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{26}$$
$$U_n(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-\frac{1}{2}} \frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}, \tag{27}$$
$$(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}} V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n}(1-x)^{n-\frac{1}{2}}(1+x)^{n+\frac{1}{2}}, \tag{28}$$
$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}} W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!} \frac{d^n}{dx^n}(1-x)^{n+\frac{1}{2}}(1+x)^{n-\frac{1}{2}}. \tag{29}$$

As is well known, the Chebyshev polynomials satisfy orthogonalities with respect to various weight functions as in the following:

$$\int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} T_n(x) T_m(x)\, dx = \frac{\pi}{\mathcal{E}_n} \delta_{n,m}, \tag{30}$$
$$where \ \ \delta_{n,m} = \begin{cases} 0, \text{if } n \neq m, \\ 1, \text{if } n = m, \end{cases} \quad \mathcal{E}_n = \begin{cases} 1, \text{if } n = 0, \\ 2, \text{if } n \geq 1, \end{cases} \tag{31}$$
$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}} U_n(x) U_m(x)\, dx = \frac{\pi}{2} \delta_{n,m}, \tag{32}$$
$$\int_{-1}^{1} (\frac{1+x}{1-x})^{\frac{1}{2}} V_n(x) V_m\, dx = \pi \delta_{n,m}, \tag{33}$$
$$\int_{-1}^{1} (\frac{1-x}{1+x})^{\frac{1}{2}} W_n(x) W_m\, dx = \pi \delta_{n,m}. \tag{34}$$

Our paper is organized as follows. In Section 1, we give an introduction as to our problem of interest, justification of our research, our contributions to the problem and the necessary ingredients on Lucas polynomials and Chebyshev polynomials of all kinds. In Section 2, we prove Theorem 1 by using the general facts in Propositions 1 and 2 and the key Lemma 1. In Section 3, we show Theorem 2 again by exploiting Propositions 1 and 2 and the crucial Lemma 2. In Section 4, we will be able to get identities among eight expressions by using the well known relationship between Lucas polynomials and Chebyshev polynomials of the first kind and combining Theorems 1 and 2. Finally, in Section 5, we present the conclusions of this paper.

2. Proof of Theorem 1

Here, we will prove only (4) and (6) in Theorem 1, leaving (5) and (7) as an exercise for the reader. For this purpose, we first state two results that are needed in showing Theorems 1 and 2.

The formulas (a) and (b) in Proposition 1 are respectively from the Equations (24) and (36) of [17], while (c) and (d) are respectively from (23) and (38) of [18]. All of them follow easily from the Rodrigues' Formulas (26)–(29), and the orthogonalities in (30) and (32)–(34).

Proposition 1. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then, we have the following:*

$$
\begin{aligned}
&(a) \quad q(x) = \sum_{k=0}^{n} C_{k,1} T_k(x), \ \textit{where} \\
&\qquad C_{k,1} = \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx, \\
&(b) \quad q(x) = \sum_{k=0}^{n} C_{k,2} U_k(x), \ \textit{where} \\
&\qquad C_{k,2} = \frac{(-1)^k 2^{k+1} (k+1)!}{(2k+1)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\frac{1}{2}} dx, \\
&(c) \quad q(x) = \sum_{k=0}^{n} C_{k,3} V_k(x), \ \textit{where} \\
&\qquad C_{k,3} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx, \\
&(d) \quad q(x) = \sum_{k=0}^{n} C_{k,4} W_k(x), \ \textit{where} \\
&\qquad C_{k,4} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx.
\end{aligned}
$$

The next Proposition is stated and proved in [1].

Proposition 2. *Let m, k be nonnegative integers. Then, we have the following:*

$$
(a) \quad \int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} x^m dx = \begin{cases} 0, & \textit{if } m \equiv 1 \ (\textit{mod } 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \textit{if } m \equiv 0 \ (\textit{mod } 2), \end{cases}
$$

$$
(b) \quad \int_{-1}^{1} (1-x^2)^{k+\frac{1}{2}} x^m dx = \begin{cases} 0, & \textit{if } m \equiv 1 \ (\textit{mod } 2), \\ \dfrac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & \textit{if } m \equiv 0 \ (\textit{mod } 2), \end{cases}
$$

$$
\begin{aligned}
(c) \quad & \int_{-1}^{1} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} x^m dx \\
&= \begin{cases} \dfrac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \textit{if } m \equiv 1 \ (\textit{mod } 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \textit{if } m \equiv 0 \ (\textit{mod } 2), \end{cases} \\
(d) \quad & \int_{-1}^{1} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} x^m dx \\
&= \begin{cases} -\dfrac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \textit{if } m \equiv 1 \ (\textit{mod } 2), \\ \dfrac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \textit{if } m \equiv 0 \ (\textit{mod } 2). \end{cases}
\end{aligned}
$$

The following lemma was stated and proved in [10].

Lemma 1. *Let m, r be integers with $m \geq 2, r \geq 1$. Then, we have the following identity:*

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&- \sum_{l=0}^{m-2} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} \binom{r+l}{r} x^l T_{i_1}(x) T_{i_2}(x) \cdots T_{i_{r+1}}(x) \\
&= \frac{1}{2^{r-1} r!} T_{m+r}^{(r)}(x),
\end{aligned} \tag{35}$$

where the first and second inner sums on the left-hand side are respectively over all nonnegative integers $i_1, i_2, \cdots i_{r+1}$, with $i_1 + i_2 + \cdots + i_{r+1} = m - l$ and $i_1 + i_2 + \cdots + i_{r+1} = m - l - 2$.

From (22), the rth derivative of $T_n(x)$ is given by

$$T_n^{(r)}(x) = \frac{n}{2} \sum_{l=0}^{[\frac{n-r}{2}]} (-1)^l \frac{1}{n-l} \binom{n-l}{l} 2^{n-2l} (n-2l)_r x^{n-2l-r}. \tag{36}$$

Thus, in particular, we have

$$\begin{aligned}
&T_{m+r}^{(r+k)}(x) \\
&= \frac{m+r}{2} \sum_{l=0}^{[\frac{m-k}{2}]} (-1)^l \frac{1}{m+r-l} \binom{m+r-l}{l} 2^{m+r-2l} (m+r-2l)_{r+k} x^{m-k-2l}.
\end{aligned} \tag{37}$$

With $\alpha_{m,r}(x)$ as in (1), we let

$$\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,1} T_k(x). \tag{38}$$

Then, from (a) of Proposition 1, (35), (37), and integration by parts k times, we have

$$\begin{aligned}
C_{k,1} &= \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)! \pi} \int_{-1}^{1} \alpha_{m,r}(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{(-1)^k 2^k k! \mathcal{E}_k}{(2k)! \pi 2^{r-1} r!} \int_{-1}^{1} T_{m+r}^{(r)}(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{2^k k! \mathcal{E}_k}{(2k)! \pi 2^{r-1} r!} \int_{-1}^{1} T_{m+r}^{(r+k)}(x) (1-x^2)^{k-\frac{1}{2}} dx \\
&= \frac{2^k k! \mathcal{E}_k}{(2k)! \pi 2^{r-1} r!} \frac{m+r}{2} \sum_{l=0}^{[\frac{m-k}{2}]} (-1)^l \frac{1}{m+r-l} \binom{m+r-l}{l} \\
&\quad \times 2^{m+r-2l} (m+r-2l)_{r+k} \int_{-1}^{1} x^{m-k-2l} (1-x^2)^{k-\frac{1}{2}} dx.
\end{aligned} \tag{39}$$

We note from (a) in Proposition 2 that

$$\begin{aligned}
&\int_{-1}^{1} x^{m-k-2l} (1-x^2)^{k-\frac{1}{2}} dx \\
&= \begin{cases} 0, & \text{if } k \not\equiv m \pmod 2, \\ \dfrac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l} (\frac{m+k}{2} - l)! (\frac{m-k}{2} - l)! k!}, & \text{if } k \equiv m \pmod 2, \end{cases}
\end{aligned} \tag{40}$$

From (38)–(40), and, after some simplifications, we get

$$
\begin{aligned}
\alpha_{m,r}(x) &= \sum_{\substack{0\le k\le m \\ k\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{\mathcal{E}_k(m+r)(-1)^l(m+r-l)!}{r!(m+r-l)l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!} T_k(x) \\
&= \sum_{j=0}^{[\frac{m}{2}]} \frac{T_{m-2j}(x)\mathcal{E}_{m-2j}(m+r)}{r!} \sum_{l=0}^{j} \frac{(-1)^l(m+r-1-l)!}{l!(m-l-j)!(j-l)!} \\
&= \sum_{j=0}^{[\frac{m}{2}]} \frac{T_{m-2j}(x)\mathcal{E}_{m-2j}(m+r)!}{r!(m-j)!j!} \sum_{l=0}^{j} \frac{\langle -j\rangle_l \langle j-m\rangle_l}{l!\langle 1-m-r\rangle_l} \\
&= \binom{m+r}{r} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{j} \mathcal{E}_{m-2j}\, {}_2F_1(-j,j-m;1-m-r;1) T_{m-2j}(x).
\end{aligned} \tag{41}
$$

This completes the proof for (4) in Theorem 1.
Next, we let

$$
\alpha_{m,r}(x) = \sum_{k=0}^{m} C_{k,3} V_k(x). \tag{42}
$$

Then, from (c) of Proposition 1, (35), (37) and integration by parts k times , we obtain

$$
\begin{aligned}
C_{k,3} = {}& \frac{k!2^k}{(2k)!\pi 2^{r-1} r!} \frac{m+r}{2} \sum_{l=0}^{[\frac{m-k}{2}]} (-1)^l \frac{1}{m+r-l} \binom{m+r-l}{l} \\
& \times 2^{m+r-2l}(m+r-2l)_{r+k} \int_{-1}^{1} x^{m-k-2l}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx.
\end{aligned} \tag{43}
$$

From (c) of Proposition 2, we observe that

$$
\begin{aligned}
& \int_{-1}^{1} x^{m-k-2l}(1-x)^{k-\dfrac{1}{2}}(1+x)^{k+\dfrac{1}{2}} dx \\
= & \begin{cases}
\dfrac{(m-k-2l+1)!(2k)!\pi}{2^{m+k-2l+1}(\dfrac{m+k+1}{2}-l)!(\dfrac{m-k+1}{2}-l)!k!}, & \text{if } , k \not\equiv m \pmod 2, \\
\dfrac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l}(\dfrac{m+k}{2}-l)!(\dfrac{m-k}{2}-l)!k!}, & \text{if } k \equiv m \pmod 2.
\end{cases}
\end{aligned} \tag{44}
$$

By (42)–(44), and, after some simplifications, we get

$$
\begin{aligned}
\alpha_{m,r}(x) = {}& \frac{(m+r)}{2r!} \sum_{\substack{0\le k\le m \\ k\not\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} V_k(x) \frac{(-1)^l(m+r-1-l)!(m-k-2l+1)}{l!(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!} \\
& + \frac{(m+r)}{r!} \sum_{\substack{0\le k\le m \\ k\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} V_k(x) \frac{(-1)^l(m+r-1-l)!}{l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!} \\
= {}& \frac{(m+r)}{r!} \sum_{j=0}^{[\frac{m-1}{2}]} \sum_{l=0}^{j} V_{m-2j-1}(x) \frac{(-1)^l(m+r-1-l)!}{l!(m-j-l)!(j-l)!} \\
& + \frac{(m+r)}{r!} \sum_{j=0}^{[\frac{m}{2}]} \sum_{l=0}^{j} V_{m-2j}(x) \frac{(-1)^l(m+r-1-l)!}{l!(m-j-l)!(j-l)!}.
\end{aligned} \tag{45}
$$

Further modifications of (45) give us

$$\begin{aligned}\alpha_{m,r}(x) &= \frac{(m+r)}{r!}\sum_{j=0}^{[\frac{m-1}{2}]}\sum_{l=0}^{j}\frac{1}{(m-j)!j!}V_{m-2j-1}(x)\frac{\langle -j\rangle_l\langle j-m\rangle_l}{l!\langle 1-m-r\rangle_l}\\ &+\frac{(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}\frac{1}{(m-j)!j!}V_{m-2j}(x)\frac{\langle -j\rangle_l\langle j-m\rangle_l}{l!\langle 1-m-r\rangle_l}\\ &=\binom{m+r}{r}\sum_{j=0}^{[\frac{m-1}{2}]}\binom{m}{j}{}_2F_1(-j,j-m;1-m-r;1)V_{m-2j-1}(x)\\ &+\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}{}_2F_1(-j,j-m;1-m-r;1)V_{m-2j}(x)\\ &=\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}{}_2F_1(-[\frac{j}{2}],[\frac{j}{2}]-m;1-m-r;1)V_{m-j}(x).\end{aligned} \tag{46}$$

This finishes up the proof for (6) in Theorem 1.

3. Proof of Theorem 2

Here, we will show only (9) and (11) in Theorem 2, leaving the proofs for (8) and (10) as an exercise for the reader. The following lemma is crucial to our discussion in this section. As it is stated in [10] but not proved, we are going to show this.

Lemma 2. *Let m,r be integers with $m \geq 2, r \geq 1$. Then, the following identity holds:*

$$\begin{aligned}&\sum_{l=0}^{m}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l}\binom{r+l}{r}(\frac{x}{2})^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)\\ &+\sum_{l=0}^{m-2}\sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2}\binom{r+l}{r}(\frac{x}{2})^l L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)\\ &=\frac{2^{r+1}}{r!}L_{m+r}^{(r)}(x),\end{aligned} \tag{47}$$

where the first and second inner sums on the left-hand side are respectively over all nonnegative integers $i_1, i_2, \cdots i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1} = m-l$ and $i_1+i_2+\cdots+i_{r+1} = m-l-2$.

Proof. By differentiating (16) r times, we have

$$\frac{\partial^r}{\partial x^r}F(t,x) = t^r(1+t^2)r!(1-xt-t^2)^{-(r+1)}, \quad (r \geq 1), \tag{48}$$

$$\frac{\partial^r}{\partial x^r}F(t,x) = \sum_{m=r}^{\infty}L_m^{(r)}(x)t^m = \sum_{m=0}^{\infty}L_{m+r}^{(r)}(x)t^{m+r}. \tag{49}$$

Equating (48) and (49), we obtain

$$\left(\frac{1}{1-xt-t^2}\right)^{r+1} = \frac{1}{r!(1+t^2)}\sum_{m=0}^{\infty}L_{m+r}^{(r)}(x)t^m. \tag{50}$$

On the other hand, from (16) and (50), we note that

$$\begin{aligned}
&\sum_{l=0}^{\infty} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l \\
=&\left(\sum_{l=0}^{\infty} L_l(x)t^l\right)^{r+1} \\
=&(2-xt)^{r+1}\left(\frac{1}{1-xt-t^2}\right)^{r+1} \\
=&(2-xt)^{r+1}(1+t^2)^{-1}\frac{1}{r!}\sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^m.
\end{aligned} \tag{51}$$

Thus, from (51), we have

$$\begin{aligned}
&\frac{1}{r!}\sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^m \\
&= 2^{-(r+1)}(1+t^2)(1-\frac{xt}{2})^{-(r+1)}\sum_{l=0}^{\infty} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l \\
&= 2^{-(r+1)}(1+t^2)\sum_{j=0}^{\infty}\binom{r+j}{r}(\frac{x}{2})^j t^j \sum_{l=0}^{\infty} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l.
\end{aligned} \tag{52}$$

In turn, from (52), we get

$$\begin{aligned}
&\frac{2^{r+1}}{r!}\sum_{m=0}^{\infty} L_{m+r}^{(r)}(x)t^m \\
&= \left(\sum_{j=0}^{\infty}\binom{r+j}{r}(\frac{x}{2})^j t^j + \sum_{j=2}^{\infty}\binom{r+j-2}{r}(\frac{x}{2})^{j-2}t^j\right) \\
&\times \sum_{l=0}^{\infty} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^l \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{r+m-l}{r}(\frac{x}{2})^{m-l} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&+ \sum_{m=2}^{\infty}\sum_{l=0}^{m-2}\binom{r+m-l-2}{r}(\frac{x}{2})^{m-l-2} \sum_{i_1+i_2+\cdots+i_{r+1}=l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&= \sum_{m=0}^{\infty}\sum_{l=0}^{m}\binom{r+l}{r}(\frac{x}{2})^{l} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m \\
&+ \sum_{m=2}^{\infty}\sum_{l=0}^{m-2}\binom{r+l}{r}(\frac{x}{2})^{l} \sum_{i_1+i_2+\cdots+i_{r+1}=m-l-2} L_{i_1}(x)L_{i_2}(x)\cdots L_{i_{r+1}}(x)t^m.
\end{aligned} \tag{53}$$

Comparing the coefficients on both sides of (53), we get the desired result. □

From (10), we note that the rth derivative of $L_n(x)$ is given by

$$L_n^{(r)}(x) = n\sum_{l=0}^{[\frac{n-r}{2}]}\frac{1}{n-l}\binom{n-l}{l}(n-2l)_r x^{n-2l-r}. \tag{54}$$

In particular, we have

$$L_{m+r}^{(r+k)}(x) = (m+r)\sum_{l=0}^{[\frac{m-k}{2}]} \frac{1}{m+r-l}\binom{m+r-l}{l}(m+r-2l)_{r+k}x^{m-k-2l}. \tag{55}$$

With $\beta_{m,r}(x)$ as in (2), we put

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,2}U_k(x). \tag{56}$$

Then, from (b) of Proposition 2, (47), (55), and integration by parts k times, we have

$$\begin{aligned}
C_{k,2} &= \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\beta_{m,r}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{(-1)^k 2^{k+1}(k+1)!2^{r+1}}{(2k+1)!\pi r!}\int_{-1}^{1}L_{m+r}^{(r)}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{2^{k+1}(k+1)!2^{r+1}}{(2k+1)!\pi r!}\int_{-1}^{1}L_{m+r}^{(r+k)}(x)(1-x^2)^{k+\frac{1}{2}}dx \\
&= \frac{2^{k+1}(k+1)!2^{r+1}(m+r)}{(2k+1)!\pi r!}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{1}{m+r-l}\binom{m+r-l}{l}(m+r-2l)_{r+k} \\
&\quad \times \int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k+\frac{1}{2}}dx.
\end{aligned} \tag{57}$$

We observe from (b) in Proposition 2 that

$$\begin{aligned}
&\int_{-1}^{1}x^{m-k-2l}(1-x^2)^{k+\frac{1}{2}}dx \\
&= \begin{cases} 0, & \text{if } k \not\equiv m \pmod 2, \\ \dfrac{(m-k-2l)!(2k+2)!\pi}{2^{m+k-2l+2}(\frac{m+k}{2}-l+1)!(\frac{m-k}{2}-l)!(k+1)!}, & \text{if } k \equiv m \pmod 2. \end{cases}
\end{aligned} \tag{58}$$

Now, from (56)–(58), and, after some simplifications, we get

$$\begin{aligned}
&\beta_{m,r}(x) \\
&= \frac{2^{r+1-m}(m+r)}{r!}\sum_{\substack{0\le k\le m \\ k\equiv m \pmod 2}}\sum_{l=0}^{[\frac{m-k}{2}]}\frac{(k+1)4^l(m+r-1-l)!}{l!(\frac{m+k}{2}-l+1)!(\frac{m-k}{2}-l)!}U_k(x) \\
&= \frac{2^{r+1-m}(m+r)}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}(m-2j+1)U_{m-2j}(x)\frac{4^l(m+r-1-l)!}{l!(m-j-l+1)!(j-l)!} \\
&= \frac{2^{r+1-m}(m+r)!}{r!}\sum_{j=0}^{[\frac{m}{2}]}\sum_{l=0}^{j}\frac{(m-2j+1)U_{m-2j}(x)}{(m-j+1)!j!}\frac{(-4)^l\langle -j\rangle_l\langle j-m-1\rangle_l}{l!\langle 1-m-r\rangle_l} \\
&= \frac{2^{r+1-m}}{r}\binom{m+r}{r-1}\sum_{j=0}^{[\frac{m}{2}]}(m-2j+1)\binom{m+1}{j} \\
&\times {}_2F_1(-j, j-m-1; 1-m-r; -4)U_{m-2j}(x).
\end{aligned} \tag{59}$$

This completes the proof for (9) in Theorem 2.

Next, we let

$$\beta_{m,r}(x) = \sum_{k=0}^{m} C_{k,4} W_k(x). \tag{60}$$

Then, from (d) of Proposition 1, (47), (55) and integration by parts k times, we have

$$\begin{aligned} C_{k,4} = \frac{k! 2^{k+r+1}(m+r)}{(2k)!\pi r!} \sum_{l=0}^{[\frac{m-k}{2}]} \frac{1}{m+r-l} \binom{m+r-l}{l} \\ \times (m+r-2l)_{r+k} \int_{-1}^{1} x^{m-k-2l} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx. \end{aligned} \tag{61}$$

From (d) of Proposition 2, we note that

$$\begin{aligned} &\int_{-1}^{1} x^{m-k-2l} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx \\ &= \begin{cases} -\dfrac{(m-k-2l+1)!(2k)!\pi}{2^{m+k-2l+1}(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!k!}, & \text{if } , k \not\equiv m \pmod 2, \\ \dfrac{(m-k-2l)!(2k)!\pi}{2^{m+k-2l}(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!k!}, & \text{if } k \equiv m \pmod 2. \end{cases} \end{aligned} \tag{62}$$

By (60)–(62), and after some simplifications, we obtain

$$\begin{aligned} &\beta_{m,r}(x) \\ &= -\frac{2^{r-m}(m+r)}{r!} \sum_{\substack{0 \le k \le m \\ k \not\equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} W_k(x) \frac{4^l (m+r-1-l)!(m-k-2l+1)}{l!(\frac{m+k+1}{2}-l)!(\frac{m-k+1}{2}-l)!} \\ &+ \frac{2^{r+1-m}(m+r)}{r!} \sum_{\substack{0 \le k \le m \\ k \equiv m \pmod 2}} \sum_{l=0}^{[\frac{m-k}{2}]} W_k(x) \frac{4^l (m+r-1-l)!}{l!(\frac{m+k}{2}-l)!(\frac{m-k}{2}-l)!} \\ &= -\frac{2^{r-m}(m+r)}{r!} \sum_{j=0}^{[\frac{m-1}{2}]} \sum_{l=0}^{j} W_{m-2j-1}(x) \frac{4^l (m+r-1-l)!(2j-2l+2)}{l!(m-j-l)!(j-l+1)!} \\ &+ \frac{2^{r+1-m}(m+r)}{r!} \sum_{j=0}^{[\frac{m}{2}]} \sum_{l=0}^{j} W_{m-2j}(x) \frac{4^l (m+r-1-l)!}{l!(m-j-l)!(j-l)!}. \end{aligned} \tag{63}$$

After further modifications of (63), we get

$$\begin{aligned} &\beta_{m,r}(x) \\ &= -\frac{2^{r+1-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m-1}{2}]} \frac{W_{m-2j-1}(x)}{(m-j)!j!} \sum_{l=0}^{j} \frac{(-4)^l \langle -j \rangle_l \langle j-m \rangle_l}{l! \langle 1-m-r \rangle_l} \\ &+ \frac{2^{r+1-m}(m+r)!}{r!} \sum_{j=0}^{[\frac{m}{2}]} \frac{W_{m-2j}(x)}{(m-j)!j!} \sum_{l=0}^{j} \frac{(-4)^l \langle -j \rangle_l \langle j-m \rangle_l}{l! \langle 1-m-r \rangle_l} \end{aligned}$$

$$
\begin{aligned}
&= -2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m-1}{2}]}\binom{m}{j}W_{m-2j-1}(x)\,{}_2F_1(-j,j-m;1-m-r;-4)\\
&+ 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}W_{m-2j}(x)\,{}_2F_1(-j,j-m;1-m-r;-4)\\
&= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}\,{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-m;1-m-r;-4)W_{m-j}(x).
\end{aligned}
\tag{64}
$$

This finishes up the proof for (11) in Theorem 2.

4. Further Remarks

It is well known that the Lucas polynomials $L_n(x)$ and the Chebyshev polynomials of the first kind $T_n(x)$ are related by

$$L_n(x) = 2i^{-n}T_n(\frac{ix}{2}). \tag{65}$$

Then, it is immediate to see from (1), (2) and (65) that the following identity holds:

$$2^{r+1}i^{-m}\alpha_{m,r}(\frac{ix}{2}) = \beta_{m,r}(x). \tag{66}$$

Now, the following theorem follows from Theorem 1, Theorem 2, and (66).

Theorem 3. *Let m, r be integers with $m \geq 2, r \geq 1$. Then, the following identities hold true:*

$$
\begin{aligned}
&i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}\mathcal{E}_{m-2j}\,{}_2F_1(-j,j-m;1-m-r;1)T_{m-2j}(\frac{ix}{2})\\
&= i^{-m}2^{r+1}\frac{(m+r)!}{(m+1)!}\sum_{j=0}^{[\frac{m}{2}]}\binom{m+1}{j}(m-2j+1)\\
&\quad\times {}_2F_1(-j,j-m-1;1-m-r;1)U_{m-2j}(\frac{ix}{2})\\
&= i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}\,{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-m;1-m-r;1)V_{m-j}(\frac{ix}{2})\\
&= i^{-m}2^{r+1}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}\,{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-m;1-m-r;1)W_{m-j}(\frac{ix}{2})\\
\\
&= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{j}\mathcal{E}_{m-2j}\,{}_2F_1(-j,j-m;1-m-r;-4)T_{m-2j}(x)\\
&= \frac{2^{r+1-m}}{r}\binom{m+r}{r-1}\sum_{j=0}^{[\frac{m}{2}]}(m-2j+1)\binom{m+1}{j}\\
&\quad\times {}_2F_1(-j,j-m-1;1-m-r;-4)U_{m-2j}(x)\\
&= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}\binom{m}{[\frac{j}{2}]}\,{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-m;1-m-r;-4)V_{m-j}(x)\\
&= 2^{r+1-m}\binom{m+r}{r}\sum_{j=0}^{m}(-1)^j\binom{m}{[\frac{j}{2}]}\,{}_2F_1(-[\tfrac{j}{2}],[\tfrac{j}{2}]-m;1-m-r;-4)W_{m-j}(x).
\end{aligned}
\tag{67}
$$

5. Conclusions

In the present paper, we considered the sums of finite products of Chebyshev polynomials $\alpha_{m,r}(x)$ of the first kind in (1) and those of Lucas polynomials $\beta_{m,r}(x)$ in (2), and expressed each of them as linear combinations of $T_n(x)$, $U_n(x)$, and $V_n(x)$, and $W_n(x)$, to find that all the coefficients involve terminating hypergeometric functions ${}_2F_1$. Here, we remark that Lemmas 1 and 2 have been crucial to our discussion, which say that $\alpha_{m,r}(x)$ and $\beta_{m,r}(x)$ are respectively equal to $\frac{1}{2^{r-1}r!}T^{(r)}_{m+r}(x)$ and $\frac{2^{r+1}}{r!}L^{(r)}_{m+r}(x)$ by using the generating functions in (16) and (17). Then, our results were obtained by using those Lemmas 1 and 2, Propositions 1 and 2, and integration by parts. Consequently, by using a well known relation between Chebyshev polynomials of the first kind and Lucas polynomials and combining Theorems 1 and 2, we were able to discover the amusing identities in (67) among all kinds of Chebyshev polynomials. It is certainly possible to express such sums of finite products in terms of other orthogonal polynomials, which is one of our ongoing projects. More generally, along the same line as the present paper, we are planning to study various sums of finite products of certain special polynomials and want to discover many applications of them.

Author Contributions: T.K. and D.S.K. conceived the framework and structured the whole paper; D.S.K. wrote the paper; J.K., T.K. and D.V.D. checked the results of the paper; D.S.K., J.K., T.K. and D.V.D. completed the revision of the article.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

Acknowledgments: The first author has been appointed a chair professor at Tianjin Polytechnic University by Tianjin City in China from August 2015 to August 2019. We would like to thank the referees for their valuable comments and suggestions, which greatly improved the presentation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kim, T.; Dolgy, D.V.; Kim, D.S. Representing sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials in terms of Chebyshev polynomials. *Adv. Stud. Contemp. Math.* **2018**, *28*, 321–335.
2. Kim, T.; Kim, D.S.; Dolgy, D.V.; Ryoo, C.-S. Representing sums of finite products of Chebyshev polynomials of third and fourth kinds by Chebyshev polynomials. *Symmetry* **2018**, *10*, 258, [CrossRef]
3. Kim, T.; Kim, D.S.; Kwon, H.-I. Representing sums of finite products of Legendre and Laguerre polynomials by Chebyshev polynomials. *Adv. Stud. Contemp. Math. (Kyungshang)* **2018**, submitted.
4. Kim, T.; Kim, D.S.; Kwon, J.; Dolgy, D.V. Expressing sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials by several orthogonal polynomials. *Mathematics* **2018**, *6*, 210. [CrossRef]
5. Agarwal, R.P.; Kim, D.S.; Kim, T.; Kwon, J. Sums of finite products of Bernoulli functions. *Adv. Differ. Equ.* **2017**, *2017*, 237. [CrossRef]
6. Kim, T.; Kim, D.S.; Jang, G.-W.; Kwon, J. Sums of finite products of Euler functions. In *Advances in Real and Complex Analysis with Applications*; Trends in Mathematics; Springer: Singapore, 2017; pp. 243–260,
7. Kim, T.; Kim, D.S.; Dolgy, D.V.; Kwon, J. Sums of finite products of Chebyshev polynomials of the third and fourth kinds. *Adv. Differ. Equ.* **2018**, *2018*, 283. [CrossRef]
8. Kim, T.; Kim, D.S.; Dolgy, D.V.; Park, J.-W. Sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials. *J. Inequal. Appl.* **2018**, *2018*, 148. [CrossRef] [PubMed]
9. Kim, T.; Kim, D.S.; Dolgy, D.V.; Park, J.-W. Sums of finite products of Legendre and Laguerre polynomials. *Adv. Differ. Equ.* **2018**, *2018*, 277. [CrossRef]
10. Kim, T.; Kim, D.S.; Jang, L.C.; Jang, G.-W. Fourier Series for Functions Related to Chebyshev Polynomials of the First Kind and Lucas Polynomials. *Mathematics* **2018**, *6*, 276. [CrossRef]
11. Kim, T.; Kim, D.S.; Jang, L.C.; Jang, G.-W. Sums of finite products of Genocchi functions. *Adv. Differ. Equ.* **2017**, *2017*, 268. [CrossRef]

12. Doha, E.H.; Abd-Elhameed, W.M.; Alsuyuti, M.M. On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems. *J. Egypt. Math. Soc.* **2015**, *23*, 397–405. [CrossRef]
13. Mason, J.C. Chebyshev polynomials of the second, third and fourth kinds in approximation, indefinite integration, and integral transforms. *J. Comput. Appl. Math.* **1993**, *49*, 169–178. [CrossRef]
14. Andrews, G.E.; Askey, R.; Roy, R. Special functions. In *Encyclopedia of Mathematics and its Applications 71*; Cambridge University Press: Cambridge, UK, 1999.
15. Beals, R.; Wong, R. Special functions and orthogonal polynomials. In *Cambridge Studies in Advanced Mathematics 153*; Cambridge University Press: Cambridge, UK, 2016.
16. Wang, Z.X.; Guo, D.R. *Special Functions*; Translated from the Chinese by Guo and X. J. Xia; World Scientific Publishing Co., Inc.: Teaneck, NJ, USA, 1989.
17. Kim, D.S.; Kim, T.; Lee, S.-H. Some identities for Bernoulli polynomials involving Chebyshev polynomials. *J. comput. Anal. Appl.* **2014**, *16*, 172–180.
18. Kim, D.S.; Dolgy, D.V.; Kim, T.; Rim, S.-H. Identities involving Bernoulli and Euler polynomials arising from Chebyshev polynomials. *Proc. Jangjeon Math. Soc.* **2012**, *15*, 361–370.