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Role of Media and Effects of Infodemics and Escapes in the Spatial Spread of Epidemics: A Stochastic Multi-Region Model with Optimal Control Approach

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Abstract: Mass vaccination campaigns play major roles in the war against epidemics. Such prevention strategies cannot always reach their goals significantly without the help of media and awareness campaigns used to prevent contacts between susceptible and infected people. Feelings of fear, infodemics, and misconception could lead to some fluctuations of such policies. In addition to the vaccination strategy, the movement restriction approach is essential because of the factor of mobility or travel. However, anti-epidemic border measures may also be disturbed if some infected travelers manage to escape and infiltrate into a safer region. In this paper, we aim to study infection dynamics related to the spatial spread of an epidemic in interconnected regions in the presence of random perturbations caused by the three above-mentioned reasons. Therefore, we devise a stochastic multi-region epidemic model in which contacts between susceptible and infected populations, vaccination-based and movement restriction optimal control approaches are all assumed to be unpredictable, and then, we discuss the effectiveness of such policies. In order to reach our goal, we employ a stochastic maximum principle version for noised systems, state and prove the sufficient and necessary conditions of optimality, and finally provide the numerical results obtained using a stochastic progressive-regressive schemes method.

Keywords: multi-region epidemic model; stochastic model; media coverage; infodemics; misconception; vaccination; stochastic optimal control; stochastic multi-points boundary value problems

1. Introduction

Media plays a tremendous role in mounting awareness among susceptible populations in an attempt to reduce their contact with infection. In fact, it has the potential of generating a psychological impact on the social conduct, as explained in [1–3]. Then, many modelers of epidemics saw it was also important to introduce and discuss the effect of awareness through media in the outbreaks of diseases; see studies in [4–8].

On the other hand, in times of recently serious epidemic outbreaks, infodemics or epidemics of rumors may appear very quickly in the virtual world. Because of contradictory views and unreliable and misleading information diffused by some Internet users, people become confused and fearful. In such circumstances, the role of media is essential, and journalists with scientists are obliged to report and exhibit concrete and convincing proof = about the nature of the epidemic and should explain the reasons for control interventions led by health authorities [9]. Here, we try to focus on rumors that can prevent a large portion of the population from being informed about the necessity of following an urgent anti-epidemic measure, namely vaccination.

Recently, multi-region epidemic models as in [10,11] have been interested in the study of the spatial spread of some epidemics using metapopulation-like differential equations. Other papers as in [12–21] chose to model the same phenomenon, namely the regional spread of epidemics, using the framework of difference equations. All these last references treated dynamics of epidemics in the presence of a so-called travel-blocking control strategy. In fact, the mobility factor is very important in such considered systems, and then, it is important to discuss the effectiveness of movement restrictions between regions; otherwise, a vaccination control policy alone would not seem sufficient, especially when we talk about an infection spreading in or around a large geographical territory. In order to discuss other missing considerations on this subject, we try to propose here a stochastic version of those epidemic modeling approaches based on a susceptible-infected-removed (SIR) multi-region stochastic model that describes the spatial-temporal spread of an epidemic in the presence of optimal vaccination and movement restriction strategies under perturbations. In the analysis of the optimization of these control interventions, we focus in this paper on providing existence results of the sought controls in a proposition and theorem of sufficient and necessary conditions, and thereafter, we discuss the numerical results.

In the last decade, many applied mathematicians have been interested in the study of infection dynamics under perturbations based on stochastic compartmental models; see for example [22–37]. The main idea here is that contacts between susceptible and infected populations are unpredictable. As an example, media coverage alone could lead to random spread of an epidemic because some susceptible individuals would avoid meeting infected people at any time once they receive alerts. We investigate in this case the stochastic dynamics of infection when it occurs in regions that are neighbors and interconnected by any kind of anthropological movement. The model is devised here for the study of the spread of an epidemic in a domain Ω and that has the form of a stochastic control differential state equation written at a time t as

$$\dot{x}^{\Omega}(t) = f(t, x^{\Omega}(t), u^{\Omega}(t)) + g(t, x^{\Omega}(t), u^{\Omega}(t))\frac{dW^{\Omega}(t)}{dt}$$

with $t \in [0, T]$.

In the following sections, we define our stochastic multi-region SIR model and apply thereafter a stochastic maximum principle for characterizing the sought optimal control functions and that is associated with the mass vaccination strategy and movement restriction policies.

2. Model Description and Definitions

Presentation of the Stochastic Model without Control

First of all and based on the assumptions of the deterministic modeling approach proposed in [10,11], we assume that there are p geographical regions denoted Ω_j (domains) of the domain studied. $\Omega = \bigcup_{j=1}^{p} \Omega_j$. Let $N^{\Omega_j}(t)$ be the population of domain Ω_j at time t, presenting the number of individuals who are physically present in Ω_j, both residents and travelers, and let the host population of Ω_j be grouped into three epidemiological compartments. Let $S^{\Omega_j}(t)$, $I^{\Omega_j}(t)$, and $R^{\Omega_j}(t)$ be the number of individuals in the susceptible, infective, and removed compartments of Ω_j at time t, respectively.

The stochastic disease transmission in a given domain Ω_j at time t is modeled using a perturbed standard incidence, which we present by:

$$\sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)$$

where the stochastic disease transmission coefficient $\rho^{jk}(t)$ is the stochastic proportion of adequate contacts in domain Ω_j between a susceptible from Ω_j $(j = 1, ..., p)$ and an infective from another domain Ω_k at a time t, and which we define by:

$$\rho^{jk}(t) = \beta_{jk} + \sigma_j \frac{dW^{\Omega_j}(t)}{dt}$$

where $\beta_{jk} > 0$ is the equivalent disease transmission coefficient to ρ^{jk}, but in the deterministic case, which had been defined also as the proportion of adequate contacts in the study cases of [10,11], σ_j $(j = 1, ..., p)$ are real constants and represent the intensities of fluctuations caused by media, and $\{W^{\Omega_j}(t)\}_{t\in[0,T]}$ is an independent random variable composed of continuous white noises independent of $\mathcal{F}_t \in \mathcal{F}$ and that is a standard Brownian motion supposed to be caused in this part without control, by media coverage diffused in region Ω_j.

When there is no control introduced yet in Ω_j, the stochastic multi-regional continuous-time SIR model associated with Ω_j is presented as follows:

$$\dot{S}^{\Omega_j}(t) = -\sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j \quad (1)$$

$$\dot{I}^{\Omega_j}(t) = \sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \quad (2)$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) \quad (3)$$

where d_j is the birth and death rate and γ_j is the recovery rate. The biological background requires that all parameters be non-negative.

$N^{\Omega_j}(t) = S^{\Omega_j}(t) + I^{\Omega_j}(t) + R^{\Omega_j}(t)$ is the population size corresponding to domain Ω_j at time t. The population size remains constant for all $t \in [0, T]$; in fact:

$$\dot{N}^{\Omega_j}(t) = \dot{S}^{\Omega_j}(t) + \dot{I}^{\Omega_j}(t) + \dot{R}^{\Omega_j}(t) = 0$$

Therefore, as a function of the deterministic proportion of adequate contacts β_{jk} and the continuous Wiener process $W^{\Omega_j}(t)$, the stochastic system (6) becomes the Itô multi-region stochastic differential equations (SDEs) model:

$$\dot{S}^{\Omega_j}(t) = -\sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - \sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)\frac{dW^{\Omega_j}(t)}{dt} \quad (4)$$

$$\dot{I}^{\Omega_j}(t) = \sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) + \sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)\frac{dW^{\Omega_j}(t)}{dt} \quad (5)$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) \quad (6)$$

3. The Model with Vaccination

3.1. Presentation of the Control Model

In this section, we introduce a function $\theta^{\Omega_j}(t)$ as a perturbation of the control variable denoted by $u^{\Omega_j}(t)$ and that characterizes the effectiveness of vaccination in the above-mentioned model (4)–(6) as in [11]. This perturbation with disturbances can be caused by infodemics and rumors and also ideas from feelings of fear and misconception. Then, for a given domain Ω_j targeted by vaccination, the model is given by the following equations:

$$\begin{aligned}
\dot{S}^{\Omega_j}(t) &= -\sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j \\
&\quad -\theta^{\Omega_j}(t)S^{\Omega_j}(t) && (7)\\
\dot{I}^{\Omega_j}(t) &= \sum_{k=1}^{p} \rho^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) && (8)\\
\dot{R}^{\Omega_j}(t) &= \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) + \theta^{\Omega_j}(t)S^{\Omega_j}(t) && (9)
\end{aligned}$$

with:

$$\theta^{\Omega_j}(t) = u^{\Omega_j}(t) + \delta_j \frac{dW^{\Omega_j}(t)}{dt}.$$

Thus, in a more general form, it refers to the stochastic control equation written at a time t as

$$\dot{x}^{\Omega}(t) = f(t, x^{\Omega}(t), u^{\Omega}(t)) + g(t, x^{\Omega}(t), u^{\Omega}(t))\frac{dW^{\Omega}(t)}{dt}$$

where at time t and for $j = 1, ..., p$:

$$x^{\Omega}(t) = x^{\Omega_j}(t) = \begin{pmatrix} S^{\Omega_j}(t) \\ I^{\Omega_j}(t) \\ R^{\Omega_j}(t) \end{pmatrix}$$

$u^{\Omega}(t) = u^{\Omega_j}(t),$

$f(t, x^{\Omega}(t), u^{\Omega}(t))$

$$= \begin{pmatrix} -\sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - u^{\Omega_j}(t)S^{\Omega_j}(t) \\ \sum_{k=1}^{p} \beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \\ \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) + u^{\Omega_j}(t)S^{\Omega_j}(t) \end{pmatrix}$$

and:

$$g(t, x^{\Omega}(t), u^{\Omega}(t)) = \begin{pmatrix} -\left(\sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \delta_j S^{\Omega_j}(t)\right) \\ \sigma_j \sum_{k=1}^{p} \frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) \\ \delta_j S^{\Omega_j}(t) \end{pmatrix}$$

Our goal is to try to minimize the population of the infected group and the cost of vaccination, while maximizing the number of removed people in all regions. Our control functions take values between $u_{min}^{\Omega_j}$ and $u_{max}^{\Omega_j}$, where $u_{min}^{\Omega_k}, u_{max}^{\Omega_k} \in [0,1]$, $\forall k = 1, ..., p$.

3.2. A Stochastic Optimal Control Approach

3.2.1. Optimal Control Characterization and Necessary Conditions

We devise in this paper an optimal control approach that aims to minimize the number of the infected people and maximize the ones in the removed category for all regions, while minimizing the cost of vaccination.

The effort expended in preventing the epidemic in each region Ω_j is proportional to u^{Ω_j}, i.e., achieving a higher value of control u^{Ω_j} means that more money, equipment, personnel, and resources must be brought to Ω_j. This implies that the cost of vaccination in region Ω_j, lets say a function $C(u^{\Omega_j})$, may not be linear; that is, doubling u^{Ω_j} may require more than double the expenditure [38].

Then, we are interested in minimizing the functional

$$J(u^{\Omega_j}) = \mathbb{E}\left(\int_0^T f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt\right)$$

with:

$$f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) \quad = \quad \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2\right)$$

where $A_j > 0$, $\alpha_j^I > 0$, $\alpha_j^R > 0$ are the weight constants of control, the infected, and the removed in region Ω_j, respectively.

In [11], the authors studied the special case of the minimization problem of the cost functional J when there were no perturbations.

Here, our goal is to minimize the number of infected people and minimize the systemic costs attempting to increase the number of removed people in each Ω_j. In other words, we are seeking an optimal control $u^{\Omega_j *}$ such that:

$$J(u^{\Omega_j *}) = \min\{J(u^{\Omega_j})/u^{\Omega_j} \in U_j\}$$

where U_j is the control set defined by:

$$U_j([0,T]) = \{u^{\Omega_j}(t)\ \mathcal{F}_t\text{-}progressively\ measurable | u_{min}^{\Omega_j} \leq u^{\Omega_j}(t) \leq u_{max}^{\Omega_j}, t \in [0,T]\}$$

Let us define the Hamiltonian function H by:

$$\begin{aligned} H(x^{\Omega}, u^{\Omega}, \mu^{\Omega}, \nu^{\Omega}) \qquad &= f_0(x^{\Omega}, u^{\Omega}) + \langle f(x^{\Omega}, u^{\Omega}), \mu^{\Omega}\rangle \\ &+ tr\left[\nu^{\Omega^T} g(x^{\Omega}, u^{\Omega})\right] \end{aligned}$$

At time $t \in [0,T]$ and for $j = 1, ..., p$, it can be rewritten as:

$$\begin{aligned} H(x^{\Omega_j}(t), u^{\Omega_j}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t)) \qquad &= f_0(x^{\Omega_j}(t), u^{\Omega_j}(t)) + \langle f(x^{\Omega_j}(t), u^{\Omega_j}(t)), \mu^{\Omega_j}(t)\rangle \\ &+ \sum_{l=1}^{3} g^{l^T}(x^{\Omega_j}(t), u^{\Omega_j}(t))\nu^{\Omega_j^l}(t) \end{aligned}$$

Here, $.^T$ means the transposition, while in a domain Ω, $(\mu(t), \nu(t)))$ is a pair of adjoint variables satisfying the following adjoint BSDE (backward stochastic differential equation):

$$\left\{\begin{array}{lcl} d\mu^{\Omega}(t) & = & -[f_x^T(t, x^{\Omega}(t), u^{\Omega}(t))\mu^{\Omega}(t) + \sum_{l=1}^{3} g_{x^{\Omega}}^{l^T}(t, x^{\Omega}(t), u^{\Omega}(t))\nu^{\Omega^l}(t) \\ & & + f_{0_{x^{\Omega}}}(t, x^{\Omega}(t), u^{\Omega}(t))]dt + \nu^{\Omega}(t)dW^{\Omega}(t), \\ \mu^{\Omega}(T) & = & 0. \end{array}\right. \qquad (10)$$

Using a stochastic version of Pontryagin's maximum principle [39], we characterize the optimal control u in the following theorem to find its analytical formulation.

Theorem 1. *(Stochastic maximum principle and characterization of* $u^{\Omega *}$*)*
If there exists an optimal pair $(x^{\Omega *}, u^{\Omega *})$ and a pair of processes $(\mu(t), \nu(t))$ satisfying (27), then for $j = 1, ..., p$, we have:

$$H(x^{\Omega_j *}(t), u^{\Omega_j *}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t)) = \min_{u^{\Omega_j} \in U} H(x^{\Omega_j}(t), u^{\Omega_j *}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t)).$$

Moreover, we obtain the bounded stochastic control:

$$u^{\Omega_j *} = min(max(u^{\Omega_j}_{min}, -\frac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j *}}{A_j}), u^{\Omega_j}_{max}),$$

the solution of the FBSDEs (forward-backward stochastic differential equations):

$$\begin{cases} dx^{\Omega_j}(t) & = & f(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt + g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t) \\ d\mu^{\Omega_j}(t) & = & -[f^T_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))\mu^{\Omega_j}(t) + \sum_{l=1}^{3} g^{l^T}_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))\nu^{\Omega_j^l}(t) \\ & & + f_{0_{x^{\Omega_j}}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))]dt + \nu^{\Omega_j}(t)dW^{\Omega_j}(t), \\ x^{\Omega_j}(0) & = & (S_0^{\Omega_j}, I_0^{\Omega_j}, R_0^{\Omega_j}) \\ \mu^{\Omega_j}(T) & = & 0. \end{cases} \tag{11}$$

Proof. Since our control u^{Ω_j} is bounded, we then prove the previous theorem by using the following Lagrangian:

$$\begin{aligned} & L(x^{\Omega_j}(t), u^{\Omega_j}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t), \omega_1(t), \omega_2(t)) \\ = \; & \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \mu^{\Omega_j^T}(t) f(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) \\ + \; & \sum_{l=1}^{3} g^{l^T}(x^{\Omega_j}(t), u^{\Omega_j}(t))\nu^{\Omega_j^l}(t) + \omega_1(t)(u^{\Omega_j}_{max} - u^{\Omega_j}(t)) + \omega_2(t)(u^{\Omega_j}(t) - u^{\Omega_j}_{min}) \end{aligned}$$

where $\omega_1, \omega_2 \geq 0$, verifying at $u^{\Omega_j} = u^{\Omega_j *}$ the two conditions:

$$\omega_1(u^{\Omega_j}_{max} - u^{\Omega_j *}) = 0 \text{ and } \omega_2(u^{\Omega_j *} - u^{\Omega_j}_{min}) = 0.$$

Owing to the condition of minimization, we define by:

$$L(x^{\Omega_j *}(t), u^{\Omega_j *}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t), \omega_1(t), \omega_2(t)) = \min_{u^{\Omega_j} \in U_j} L(x^{\Omega_j}(t), u^{\Omega_j}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t), \omega_1(t), \omega_2(t)).$$

We differentiate the Lagrangian with respect to u^{Ω_j} on the set:

$$\{t | u^{\Omega_j}_{min} \leq u^{\Omega_j}(t) \leq u^{\Omega_j}_{max}\}$$

to obtain the optimality equation

$$\begin{aligned} & \frac{dL}{du^{\Omega_j}}(x^{\Omega_j}(t), u^{\Omega_j}(t), \mu^{\Omega_j}(t), \nu^{\Omega_j}(t), \omega_1(t), \omega_2(t))|_{u^{\Omega_j}} \\ & = u^{\Omega_j *} = A_j u^{\Omega_j}(t) + (\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} - \omega_1(t) + \omega_2(t) = 0. \end{aligned}$$

Furthermore, we find $u^{\Omega_j*}(t) = -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} - \omega_1 + \omega_2}{A_j}$.

- ***If*** $u_{min}^{\Omega_j} < u^{\Omega_j*}(t) < u_{max}^{\Omega_j}$***, then*** $w_1(t) = w_2(t) = 0$*; therefore:*

$$u^{\Omega_j*}(t) = -\frac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j}.$$

- ***If*** $u^{\Omega_j*}(t) = u_{min}^{\Omega_j}$***, then*** $w_1(t) = 0$*;, therefore,* $u_{min}^{\Omega_j} = -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} + w_2(t)}{A_j}$*, implying that* $w_2(t) = -((\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} + A_j u_{min}^{\Omega_j})$.

Due to $w_2(t) \geq 0$ *and* $A_j > 0$*, we obtain* $u^{\Omega_j*}(t) \leq -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j}$

- ***If*** $u^{\Omega_j*}(t) = u_{max}^{\Omega_j}$***, then*** $w_2(t) = 0$*; thus,* $u_{max}^{\Omega_j} = -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} - w_1(t)}{A_j}$ *implying that* $w_1(t) = (\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j} + A_j u_{max}^{\Omega_j}$.

In view of $w_1(t) \geq 0$ *and* $A_j > 0$*, we get* $u^{\Omega_j*}(t) \geq -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j}$.

Using these standard optimality arguments, we characterize the control $u^{\Omega_j*}(t)$ by

$$u^{\Omega_j*}(t) = \begin{cases} -\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j} & if \;\; u_{min}^{\Omega_j} < -\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j} < u_{max}^{\Omega_j} \\ u_{min}^{\Omega_j} & if \;\; -\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j} \leq u_{min}^{\Omega_j} \\ u_{max}^{\Omega_j} & if \;\; -\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j}}{A_j} \geq u_{max}^{\Omega_j} \end{cases}$$

or by a more reduced form, we can rewrite $u^{\Omega_j*}(t) = min(max(u_{min}^{\Omega_j}, -\dfrac{(\mu_3^{\Omega_j}(t) - \mu_1^{\Omega_j}(t))S^{\Omega_j*}}{A_j}), u_{max}^{\Omega_j}))$. □

3.2.2. Existence of Solutions and Sufficient Conditions

Note that $f : [0, T] \times \mathbb{R}^3 \times U_j \to \mathbb{R}^3$, $g : [0, T] \times \mathbb{R}^3 \times U_j \to \mathbb{R}^3 \bigotimes \mathbb{R}^3$ and $f_0 : [0, T] \times \mathbb{R}^3 \times U_j \to \mathbb{R}$ are measurable such that $f(t, x, .) : U_j \to \mathbb{R}^3$ and $f_0(t, x, .) : U_j \to \mathbb{R}^3$ are continuous, f, g, and f_0 are bounded, and there exists a constant $K > 0$ such that for all $t \in [0, T]$ and for all $x^{\Omega_j}, \hat{x}^{\Omega_j} \in \mathbb{R}^3$, the following properties are checked [40,41]:

$$|f(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - f(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j}(t))| + ||g(t, x^{\Omega_j}(t), u^{\Omega_j}) - g(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j})|| \quad \leq K|x^{\Omega_j} - \hat{x}^{\Omega_j}| \tag{12}$$

In fact, if we suppose $x^{\Omega_j} = (S^{\Omega_j}, I^{\Omega_j}, R^{\Omega_j})$, $\hat{x}^{\Omega_j} = (\hat{S}^{\Omega_j}, \hat{I}^{\Omega_j}, \hat{R}^{\Omega_j}) \in \Gamma \subset \mathbb{R}^3$ with $\Gamma =]a_1, a_2[\times]b_1, b_2[\times]c_1, c_2[$ ($a_{i=1,2}, b_{i=1,2}$ and $c_{i=1,2}$ are positive constants). In order to prove (12), we can take constants

$K_{11} = \delta_j + \sigma_j \frac{pb_2}{a_1+b_1+c_1}$,
$K_{21} = \sigma_j \frac{pa_2}{a_1+b_1+c_1}$,
$K_{31} = \delta_j$
and $K_1 = \max(K_{11}, K_{21}, K_{31})$ to obtain:

$$||g(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - g(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j}(t))|| \leq K_1|x^{\Omega_j} - \hat{x}^{\Omega_j}|$$

while taking also
$K_2 = u_{max}^{\Omega_j} + (b_2 + c_2)d_j + \sum_{k=1}^{p} \beta_{jk} \frac{b_2}{a_1+b_1+c_1}$,
$K_3 = d_j + \gamma_j + \sum_{k=1}^{p} \beta_{jk} \frac{a_2}{a_1+b_1+c_1}$

and $K_4 = u^{\Omega_j}_{max} + d_j + b_2\gamma_j$,
Thus, using the maximum norm, we have
$|f_1(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - f_1(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j}(t))| \leq K_2|x^{\Omega_j} - \hat{x}^{\Omega_j}|$,
$|f_2(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - f_2(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j}(t))| \leq K_3|x^{\Omega_j} - \hat{x}^{\Omega_j}|$,
and $|f_3(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - f_3(t, x^{\Omega_j}(t), u^{\Omega_j}(t))| \leq K_4|x^{\Omega_j} - \hat{x}^{\Omega_j}|$

with f_1, f_2, and f_3, the right-hand sides of differential equations in (24), (25), and (26). Thus, by setting $f = (f_1, f_2, f_3)$, the property (12) is checked by considering
$K = K_1 + \max(K_2, K_3, K_4)$,

and we should also have:

$$|f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) - f_0(t, \hat{x}^{\Omega_j}(t), u^{\Omega_j}(t))| \quad \leq K|x^{\Omega_j} - \hat{x}^{\Omega_j}|. \tag{13}$$

which can easily be checked for the particular integrand supposed here.

Proposition 1. *The optimal control problem identified by the objective functional J and linked to the state system (24)–(26) that satisfies the properties (12) and (13) admits an optimal control pair.*

Proof. At first, a backward stochastic differential equation (BSDE) with a terminal condition is introduced:

$$\begin{array}{l} dY(t) = -f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt + Z(t)dW^{\Omega_j}(t),\ t \in [0, T] \\ Y(T) = 0 \quad given; \end{array} \tag{14}$$

where:

$$f_0(t, x^{\Omega_j}(t), u^{\Omega_j}) = \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2$$

Note that under the previous observations, (12) and (13), for $(x_0^{\Omega_j}, u^{\Omega_j}) \in \mathbb{R}^3 \times U$, the backward stochastic differential Equation (14) becomes linear. Therefore, the state system admits a unique strong solution [39] that is written in the following form $x^{\Omega_j}(.) \equiv x^{\Omega_j}(.; x_0^{\Omega_j}, u^{\Omega_j})$.

Moreover, for a given $(x^{\Omega_j}(.), u^{\Omega_j})$, the backward stochastic differential equation (14) admits a unique adapted solution $(Y(.), Z(.)) \equiv (Y(.; x_0^{\Omega_j}, u^{\Omega_j}), Z(.; x_0^{\Omega_j}, u^{\Omega_j}))$, which depends on $(x_0^{\Omega_j}, u^{\Omega_j})$ through $(x^{\Omega_j}(.), u^{\Omega_j})$.

Notice that for $t \in [0, T]$, the process $Y(t)$, the solution of (14) has the general form:

$$Y(t) = Y(T) + \int_t^T f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt - \int_t^T Z(t)dW^{\Omega_j}(t), \tag{15}$$

Denote that $\mathcal{F}_t$ is the natural filtration of the Brownian motion $W(t)$. Here, $Y(t)$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted [39], implying that:

$$Y(t) = \mathbb{E}[Y(t)|\mathcal{F}_t],\ \ t \in [0, T]. \tag{16}$$

Thus,

$$Y(0) = J(u^{\Omega_j}) = \mathbb{E}[\int_0^T (\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2)dt] \tag{17}$$

Now, the objective functional J can be rewritten as:

$$J(u^{\Omega_j}) = Y(0; x_0^{\Omega_j}, u^{\Omega_j}) \tag{18}$$

Considering the new formulation of the objective function (18), a forward backward stochastic differential equations (FBSDE) system is introduced:

$$\begin{array}{ll} dx^{\Omega_j}(t) & = (\hat{f}(t, x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t))dt + g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t), \\ dY(t) & = -(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2)dt + Z(t)dW^{\Omega_j}(t), \\ x^{\Omega_j}(0) & = x_0^{\Omega_j}, \;\; Y(T) = 0 \quad \text{given.} \end{array} \tag{19}$$

In the following, we will use an appropriate approach in order to use a parabolic formulation and to establish the existence result.

In fact, if (x^{Ω_j}, Y, Z) is an adapted solution of (19), then there exists an appropriate function θ such that the following relationship is verified [42]:

$$Y(t) = \theta(t, x^{\Omega_j}(t), u^{\Omega_j}(t)), \; t \in [0, T] \;\; a.s.\mathcal{P}, \tag{20}$$

where the undetermined function θ is assumed to be of $\mathcal{C}^{1,2,0}([0, T] \times R^3 \times U_j)$.

The control u^{Ω_j} is sought to minimize the cost functional $Y(0) = J(u^{\Omega_j})$. From (19), we have:

$$dY(t) = -(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2)dt + Z(t)dW^{\Omega_j}(t).$$

To apply Ito's formula to $\theta(t, x^{\Omega_j}(t), u^{\Omega_j}(t))$, start by using Taylor's polynomial:

$$\begin{array}{rcl} d\theta(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) & = & \theta_t(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt + \theta_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dx^{\Omega_j} \\ & + & \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))(dx^{\Omega_j})^2, \end{array}$$

and recall that:

$$dx^{\Omega_j}(t) = (\hat{f}(t, x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t))dt + g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t).$$

Replacing the formulation of $dx^{\Omega_j}(t)$ in $d\theta(t)$:

$$\begin{array}{rcl} d\theta(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) & = & \theta_t(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt + \theta_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))[\hat{f}(t, x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t)]dt \\ & + & \theta_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t) \\ & + & \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))[((\hat{f}(t, x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t))dt)^2 \\ & + & (g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t))^2 + 2g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))(\hat{f}(t, x^{\Omega_j}(t)) \\ & + & f_1(x^{\Omega_j}(t))u^{\Omega_j}(t))dtdW^{\Omega_j}(t)], \end{array}$$

and using Itô's multiplication, to obtain the final formulation of $d\theta(t)$:

$$\begin{array}{rcl} d\theta(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) & = & [\theta_t(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))g^2(t, x^{\Omega_j}(t), u^{\Omega_j}(t)) \\ & + & \theta_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))(\hat{f}(t, x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t))]dt \\ & + & \theta_{x^{\Omega_j}}(t, x^{\Omega_j}(t), u^{\Omega_j}(t))g(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dW^{\Omega_j}(t). \end{array}$$

According to (19), by equating the corresponding drift terms $dY(t)$ and $d\theta(t)$, the following parabolic PDE is obtained:

$$\begin{aligned} d\theta(t,x^{\Omega_j}(t),u^{\Omega_j}(t)) &= [\theta_t(t,x^{\Omega_j}(t),u^{\Omega_j}(t)) + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))g^2(t,x^{\Omega_j}(t),u^{\Omega_j}(t)) \\ &+ \theta_{x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))(\hat{f}(t,x^{\Omega_j}(t)) + f_1(x^{\Omega_j}(t))u^{\Omega_j}(t)]dt \\ &+ \theta_{x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))g(t,x^{\Omega_j}(t),u^{\Omega_j}(t))dW^{\Omega_j}(t). \\ &= -(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2)dt + Z(t)dW^{\Omega_j}(t). \end{aligned}$$

Substitute $Z(t) = \theta_{x^{\Omega_j}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t))g(t,x^{\Omega_j}(t),u^{\Omega_j}(t))$ into the above parabolic PDE to obtain:

$$\begin{aligned} 0 &= \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}(\hat{f} + f_1 u^{\Omega_j}) + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 \\ &= \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + f_1\theta_{x^{\Omega_j}}u^{\Omega_j} + \frac{A_j}{2}(u^{\Omega_j}(t))^2 \\ &= \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) \\ &+ f_1\theta_{x^{\Omega_j}}u^{\Omega_j} + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \frac{1}{2A_j}|f_1\theta_{x^{\Omega_j}}|^2 - \frac{1}{2A_j}|f_1\theta_{x^{\Omega_j}}|^2 \\ &= \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) \\ &- \frac{1}{2A_j}|f_1\theta_{x^{\Omega_j}}|^2 + \frac{A_j}{2}[{u^{\Omega_j}}^2 + \frac{|f_1\theta_{x^{\Omega_j}}|^2}{A_j^2} + \frac{2f_1\theta_{x^{\Omega_j}}u^{\Omega_j}}{A_j}] \\ &= \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}[u^{\Omega_j} + \frac{f_1\theta_{x^{\Omega_j}}}{A_j}]^2 - \frac{1}{2A_j}|f_1\theta_{x^{\Omega_j}}|^2. \end{aligned}$$

By rearranging terms, the above equation can be written in the form of a backward parabolic PDE with a terminal condition on θ:

$$\begin{aligned} &\theta_t + \tfrac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) - \tfrac{1}{2A_j}|f_1^T\theta_{x^{\Omega_j}}|^2 + \tfrac{A_j}{2}[u^{\Omega_j} + \frac{f_1^T\theta_{x^{\Omega_j}}}{A_j}]^2 = 0 \\ &\theta(T,x^{\Omega_j}(T)) = 0, \quad x^{\Omega_j} \in \mathbb{R}^3. \end{aligned} \tag{21}$$

Using a standard maximum principle for parabolic partial differential equations (PPDE) [43], the smallest θ solution should be the one of the following PPDE problem:

$$\begin{aligned} &\theta_t + \tfrac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) - \tfrac{1}{2A_j}|f_1^T\theta_{x^{\Omega_j}}|^2 = 0 \\ &\theta(T,x^{\Omega_j}(T)) = 0, \quad x^{\Omega_j} \in \mathbb{R}^3. \end{aligned} \tag{22}$$

For the resolution of (22), results from Ladyzenskaja et al. [43] show that there exists a unique classical solution for comparable types of parabolic partial differential equations. The discussion of the solvability of this type of quasi-linear parabolic PDE in the presence of a quadratic gradient term can be found in [44]. Thus, there exists a unique classical solution to this equation. In this case, the control u^{Ω_j} is sought in order to minimize the objective function J such that:

$$\min_{u^{\Omega_j}\in U_j} J(u^{\Omega_j}) = J({u^{\Omega_j}}^*),$$

Thus:

$$\min_{u^{\Omega_j}\in U_j} \theta(0;x^{\Omega_j}(0),u^{\Omega_j}) = \theta(0;x^{\Omega_j}(0),{u^{\Omega_j}}^*),$$

Then, use (15), (20), and (21) to obtain:

$$\begin{aligned}\int_0^T f_0(t, x^{\Omega_j}(t), u^{\Omega_j}(t))dt &= \theta(0, x^{\Omega_j}(0)) - \theta(T, x^{\Omega_j}(T)) + \int_0^T Z(t)dW^{\Omega_j}(t) \\ &+ \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) \\ &- \frac{1}{2A_j}|f_1^T\theta_{x^{\Omega_j}}|^2 + \frac{A_j}{2}[u^{\Omega_j} + \frac{f_1^T\theta_{x^{\Omega_j}}}{A_j}]^2.\end{aligned}$$

Therefore,

$$\begin{aligned}J(u^{\Omega_j}) &= \theta(0, x^{\Omega_j}(0)) - \mathbb{E}\{\theta(T, x^{\Omega_j}(T))\} \\ &+ \mathbb{E}\int_0^T \theta_t + \frac{1}{2}\theta_{x^{\Omega_j}x^{\Omega_j}}g^2 + \theta_{x^{\Omega_j}}\hat{f} + \alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) \\ &- \frac{1}{2A_j}|f_1^T\theta_{x^{\Omega_j}}|^2 + \frac{A_j}{2}[u^{\Omega_j} + \frac{f_1^T\theta_{x^{\Omega_j}}}{A_j}]^2 dt \\ &= \theta(0, x^{\Omega_j}(0), u^{\Omega_j}) + \frac{A_j}{2}\mathbb{E}\int_0^T [u^{\Omega_j} + \frac{f_1^T\theta_{x^{\Omega_j}}}{A_j}]^2 dt \\ &\geq \theta(0, x^{\Omega_j}(0), u^{\Omega_j *}) = J(u^{\Omega_j *}).\end{aligned}$$

Consequently, it is finally concluded that $u^{\Omega_j *} = -f_1^T\theta_{x^{\Omega_j}}$ is an optimal control. □

3.2.3. Numerical Results

In this part, we suppose we have three interconnected regions, and we investigate numerically using the stochastic progressive-regressive schemes method presented in [45] the effectiveness of the optimal control approach presented above. In our simulations, we note that:

- We study three regions, denoted by Ω_1, Ω_2, and Ω_3 and that are all assumed to be infected.
- β_{jk} are replaced by β_k just to avoid more complications in the program code. In other words, we assume that the probability to be infected does not depend on the source location of susceptibility, but on the source location of infectivity only, namely Ω_k. More explicitly, $\beta_{11} = \beta_{21} = \beta_{31}$; they are represented by β_1, $\beta_{12} = \beta_{22} = \beta_{32}$; they are represented by β_2 and, finally, $\beta_{13} = \beta_{23} = \beta_{33} = \beta_3$; and they are represented by β_3.
- The unit of the parameters d_j, γ_j and β_{jk} is $days^{-1}$ $\forall$ fixed j and mobile k.
- $\forall i\ t_{i+1} = t_i + h$, $\Delta W_i \to \sqrt{h}N(0,1)$ with h the time step.
- The coefficients in diffusions δ_j, ς_{jk}, and σ_j are assumed to be all equal to 0.125. Larger values can be considered; however, they only increase the level of stochasticity, and this is not very interesting here.

Figure 1 depicts the dynamics of S^{Ω_j}, I^{Ω_j}, and R^{Ω_j}, $j = 1,2,3$ in the absence and presence of optimal controls $u^{\Omega_1 *}$, $u^{\Omega_2 *}$, and $u^{\Omega_3 *}$. As we can observe in the upper part of this figure and because of a strong immunity of a fraction of the population, there is a natural recovery from the infection and that causes an increase of the number of R^{Ω_j} in all three regions, but without exceeding the number of 40,000 people.

In the same part, we can also see that the number of infected people, namely I^{Ω_j} $\forall j$, exceeds 70,000 people, and this proves the necessity of an urgent control measure to fight against the epidemic. In the lower part of this figure, we reach our goal that is summarized in maximizing the number of removed people while minimizing the number of infected people, and we can observe that the number of removed individuals has reached now a number that is near 90,000 people in all regions, while the number of infected people tended to zero values after 30, 40, and 48 days in regions Ω_1, Ω_2, and Ω_3, respectively. We note that the optimal controls $u^{\Omega_j *}$, $\forall j$ take the same values along the optimal control strategy period and that are all equal to one, and they tend to zero at the final time T. This is also the reason behind our avoidance of exhibiting simulations of the three optimal controls separately.

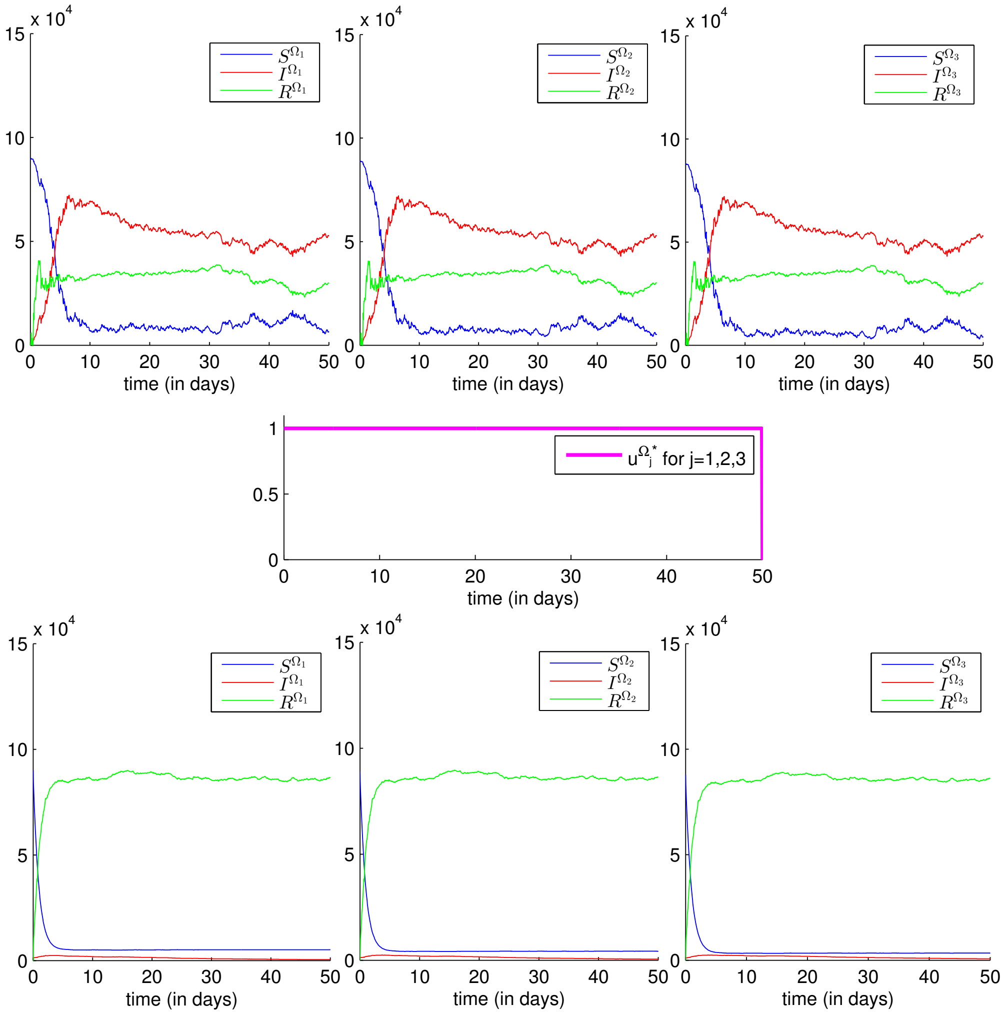

Figure 1. (Top) $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j}$, $j = 1,2,3$ stochastic dynamics without controls. (Bottom) $S^{\Omega_1} I^{\Omega_1} R^{\Omega_1}$ stochastic dynamics in the presence of all optimal controls $u^{\Omega_1 *}$, $u^{\Omega_2 *}$, and $u^{\Omega_3 *}$. $S_0^{\Omega_1}$ =90,000, $I_0^{\Omega_1}$ = 1200, $S_0^{\Omega_2}$ = 89,000, $I_2^{\Omega_2}$ = 1100, $S_0^{\Omega_3}$ = 88,000, $I_0^{\Omega_3}$ = 1000, $R_0^{\Omega_j} = 0 \; \forall \, j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

In Figure 2, we deduce that when we follow the optimal vaccination control strategy in the two regions Ω_2 and Ω_3, the number R^{Ω_j}, $j = 2,3$ exceeded 100,000 individuals in the first five days, but decreased after towards values around 80,000 people, which represents an important value also, but they were smaller values than the ones obtained in the case treated in the first figure. In addition, the number I^{Ω_j}, $j = 2,3$ decreased to values around 2000 people, which shows also the effectiveness of the followed vaccination control strategy, but they were larger values than the ones obtained in the previous case. In parallel, despite the cancellation of vaccination in region Ω_1, we can see that the number of removed people approached values around 50,000, which shows an increase by 10,000 people compared to the case when controls were not introduced in all regions, and this proves some influence of the optimal control policy followed in other regions in a region that is not yet controlled directly. We note also here that the optimal controls $u^{\Omega_j *}$, $j = 2,3$ take the same values along the optimal control strategy period, which were all equal to one with some perturbations after 5 days, and they tended to zero at the final time T.

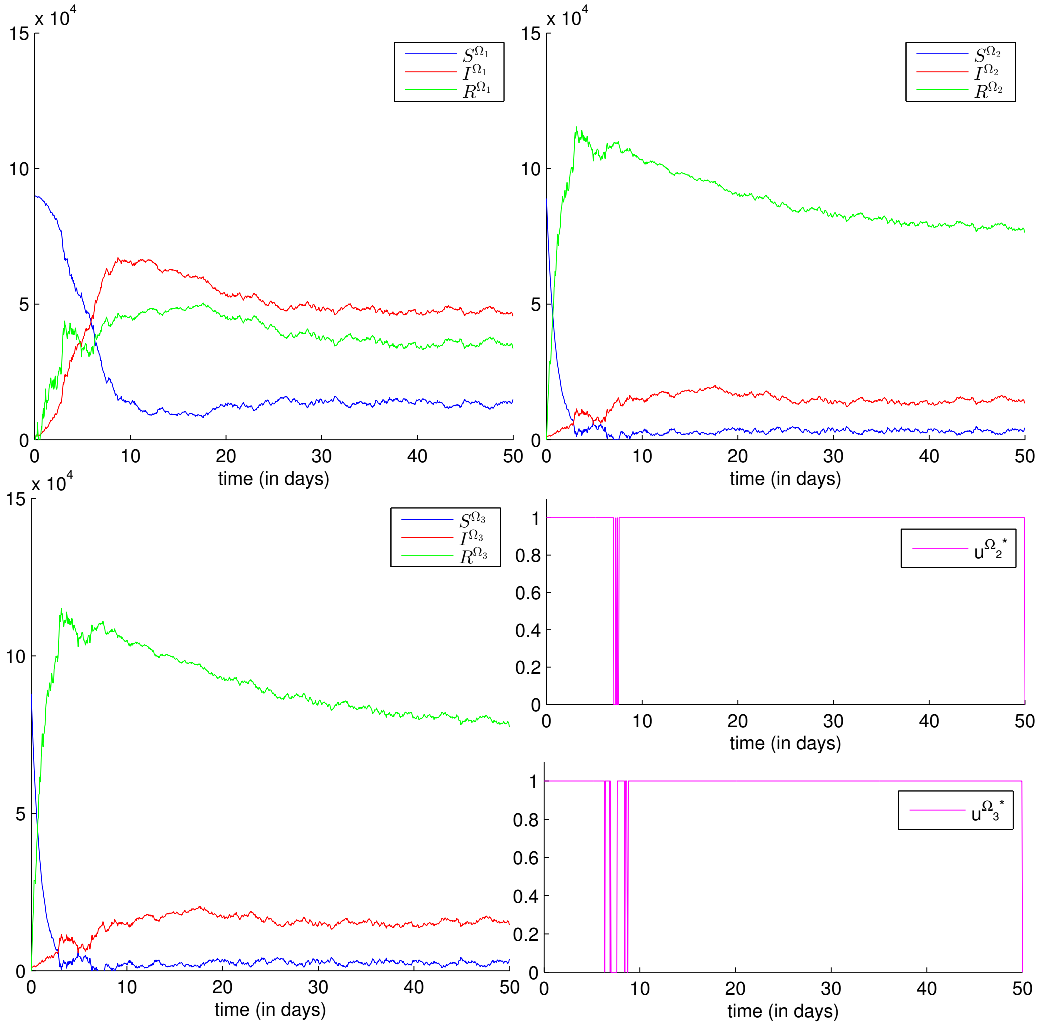

Figure 2. $S^{\Omega_j} I^{\Omega_j} R^{\Omega_j}$, $j = 1, 2, 3$ stochastic dynamics in the absence of optimal control u^{Ω_1*} and the presence of optimal controls u^{Ω_2*} and u^{Ω_3*}. Same parameters and initial conditions as in the first figure.

In Figure 3, we study the dynamics of S^{Ω_j}, I^{Ω_j}, and R^{Ω_j}, $j = 1, 2, 3$ when we introduce an optimal vaccination control in region Ω_1 only, and we can deduce some negative impact of the two regions Ω_2 and Ω_3 on the first region due to the absence of a control strategy in them. In fact, as they were at higher risk of infection than region Ω_1, the optimal control u^{Ω_1*} seemed not sufficient for eradicating infection, as shown in first figure when controls were introduced in all regions, but this can at least increase the number of removed people in this region to an important value that exceeds 90,000 individuals. We note that the optimal control u^{Ω_1*} takes one as the maximal value.

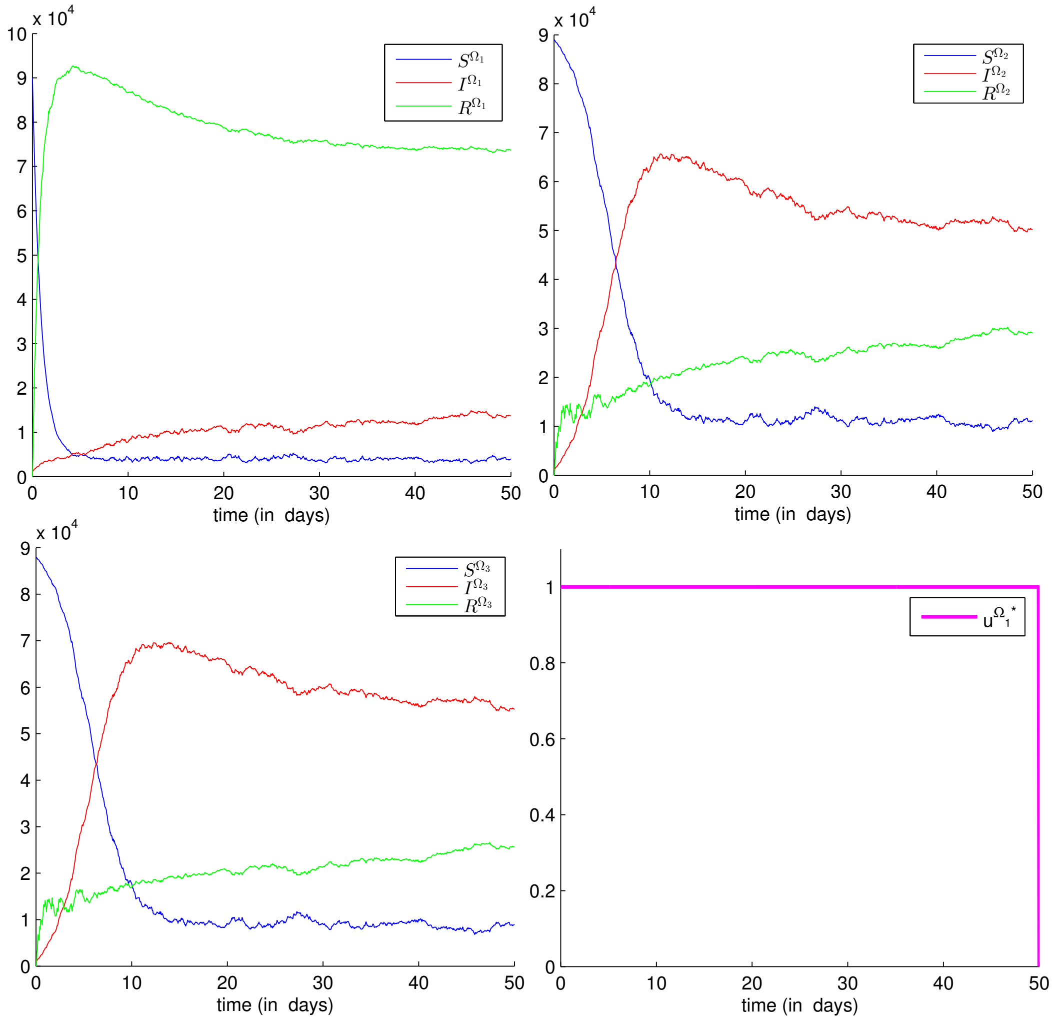

Figure 3. $S^{\Omega_j}I^{\Omega_j}R^{\Omega_j}$, $j = 1,2,3$ stochastic dynamics in the absence of optimal controls $u^{\Omega_2 *}$ and $u^{\Omega_3 *}$ and the presence of optimal control $u^{\Omega_1 *}$ only. $S_0^{\Omega_1} = 90{,}000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89{,}000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88{,}000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0 \; \forall \, j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

4. The Model with Vaccination Plus Movement Restriction

4.1. Presentation of the Control Model

Let $I = \{1, ..., p\}$ and $I_H \subset I$ be the set of indices of regions at high-risk and then having the ability to spread the epidemic to other regions. Here, we study the case when a given region Ω_j is under vaccination control u^{Ω_j} and at the same time under the threat of infection coming from other regions. For this, we add to the vaccination strategy another control denoted as $v^{j\Omega_k}$ to characterize the effectiveness of movement restriction operations, in order to prevent the infected of regions Ω_k, $k \in I_H$ coming to the controlled region Ω_j, where:

$$\begin{cases} v^{j\Omega_k} \neq 0 & \forall k \in I_H \quad k \neq j \\ v^{j\Omega_k} = 0 & \text{elsewhere} \end{cases} \tag{23}$$

Then, the model (24)–(26) in the controlled region Ω_j is rewritten as follows:

$$\dot{S}^{\Omega_j}(t) = -\sum_{k=1}^{p} \vartheta^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - \theta^{\Omega_j}(t)S^{\Omega_j}(t) \quad (24)$$

$$\dot{I}^{\Omega_j}(t) = \sum_{k=1}^{p} \vartheta^{jk}(t)\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \quad (25)$$

$$\dot{R}^{\Omega_j}(t) = \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) + \theta^{\Omega_j}(t)S^{\Omega_j}(t) \quad (26)$$

with the vaccination control defined as:

$$\theta^{\Omega_j}(t) = u^{\Omega_j}(t) + \delta_j \frac{dW^{\Omega_j}(t)}{dt}.$$

and the function $\vartheta^{jk}(t)$ defined as:

$$\vartheta^{jk}(t) = \left(1 - v^{j\Omega_k}\right)\beta_{jk} + (1-\varsigma_{jk})\sigma_j \frac{dW^{\Omega_j}(t)}{dt}$$

where σ_j and ς_{jk}, $k \in I_H$, $(j = 1, ..., p)$ are real constants and representing the intensities of fluctuations caused by media and escapes of infected people through borders between Ω_k and Ω_j, respectively.

4.2. A Stochastic Optimal Control Approach

Now, the problem (Section 3.2.1) is changed in this part to $J(u^{\Omega_j}, v^{j\Omega}) = \sum_{k \in I_H} \mathbb{E}\left(\int_0^T \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \frac{B_j}{2}(v^{j\Omega_k}(t))^2\right) dt\right)$ where $B_j > 0$ is the weight constant of the new control, while $u^{\Omega_j} \in U_j$ and $v^{j\Omega} = (v^{j\Omega_k})_{k \in I_H}$ belong to the control set $V_j^{I_H}$ defined as:

$$V_j^{I_H}([0,T]) = \{v^{j\Omega}(t)\ \mathcal{F}_t\text{-progressively measurable} | v_{min}^{\Omega_j} \leq v^{j\Omega_k}(t) \leq v_{max}^{\Omega_j}, k \in I_H, t \in [0,T]\}$$

The Hamiltonian in this case is defined as:

$$H = \sum_{k \in I_H} \left(\alpha_j^I I^{\Omega_j}(t) - \alpha_j^R R^{\Omega_j}(t) + \frac{A_j}{2}(u^{\Omega_j}(t))^2 + \frac{B_j}{2}(v^{j\Omega_k}(t))^2 + \mu^{\Omega_j^T}(t) f(t, x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t)) + \sum_{l=1}^{3} g^{l^T}(x^{\Omega_j}(t), u^{\Omega_j}(t), v^{j\Omega_k}(t)) \nu_j^{\Omega^l}(t) \right)$$

where the state function f is defined as:
$f(t, x^{\Omega}(t), u^{\Omega}(t), v^{j\Omega}(t))$

$$= \begin{pmatrix} -\sum_{k=1}^{p}\left(1 - v^{j\Omega_k}\right)\beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) + \left(N^{\Omega_j}(t) - S^{\Omega_j}(t)\right)d_j - u^{\Omega_j}(t)S^{\Omega_j}(t) \\ \sum_{k=1}^{p}\left(1 - v^{j\Omega_k}\right)\beta_{jk}\frac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) - \gamma_j I^{\Omega_j}(t) - d_j I^{\Omega_j}(t) \\ \gamma_j I^{\Omega_j}(t) - d_j R^{\Omega_j}(t) + u^{\Omega_j}(t)S^{\Omega_j}(t) \end{pmatrix}$$

and the diffusion matrix g is defined as:

$$g(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t)) = \begin{pmatrix} -\left(\sigma_j\sum_{k=1}^{p}(1-\varsigma_{jk})\dfrac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t)+\delta_j S^{\Omega_j}(t)\right) \\ \sigma_j\sum_{k=1}^{p}(1-\varsigma_{jk})\dfrac{I^{\Omega_k}(t)}{N^{\Omega_j}(t)}S^{\Omega_j}(t) \\ \delta_j S^{\Omega_j}(t) \end{pmatrix}$$

while in a domain Ω, $(\mu(t),\nu(t)))$ is a pair of adjoint variables satisfying the following adjoint BSDE (backward stochastic differential equation):

$$\begin{cases} d\mu^{\Omega}(t) &= -[f_x^T(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))\mu^{\Omega}(t)+\sum_{l=1}^{3}g_{x^{\Omega}}^{l^T}(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))\nu^{\Omega^l}(t) \\ & \quad +f_{0_{x^{\Omega}}}(t,x^{\Omega}(t),u^{\Omega}(t),v^{j\Omega}(t))]dt+\nu^{\Omega}(t)dW^{\Omega}(t), \\ \mu^{\Omega}(T) &= 0. \end{cases} \tag{27}$$

4.2.1. Optimal Control Characterization and Necessary Conditions

Using the stochastic Pontryagin's maximum principle as done for the first control strategy, we obtain the following optimal control characterization and necessary conditions:

Theorem 2. *If there exists an optimal pair $(x^{\Omega*},u^{\Omega*},v^{j\Omega*})$ and a pair of processes $(\mu(t),\nu(t))$ satisfying (27), then for $j=1,...,p$, $k\in I_H$, we have:*

$$H(t,x^{\Omega_j*}(t),u^{\Omega_j*}(t),v^{j\Omega_k*}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t))$$
$$= \min_{(u^{\Omega_j},v^{j\Omega_k}(t))\in U_j\times V_j^{I_H}} H(t,x^{\Omega_j}(t),u^{\Omega_j*}(t),v^{j\Omega_k*}(t),\mu^{\Omega_j}(t),\nu^{\Omega_j}(t)).$$

Moreover, we obtain the bounded stochastic control:

$$u^{\Omega_j*} = min(max(u_{min}^{\Omega_j}, -\frac{(\mu_3^{\Omega_j}(t)-\mu_1^{\Omega_j}(t))S^{\Omega_j*}}{A_j}), u_{max}^{\Omega_j}),$$

$$v^{j\Omega_k*} = min(max(v_{min}^{\Omega_j}, -\frac{(\mu_1^{\Omega_j}(t)-\mu_2^{\Omega_j}(t))\beta_{jk}I^{\Omega_k*}S^{\Omega_j*}}{B_j}), v_{max}^{\Omega_j}),$$

and solutions of the FBSDEs (forward-backward stochastic differential equations):

$$\begin{cases} dx^{\Omega_j}(t) &= f(t,x^{\Omega_j}(t),u^{\Omega_j}(t),v^{j\Omega_k}(t))dt+g(t,x^{\Omega_j}(t),u^{\Omega_j}(t),v^{j\Omega_k}(t))dW^{\Omega_j}(t) \\ d\mu^{\Omega_j}(t) &= -[f_{x^{\Omega_j}}^T(t,x^{\Omega_j}(t),u^{\Omega_j}(t),v^{j\Omega_k}(t))\mu^{\Omega_j}(t)+\sum_{l=1}^{3}g_{x^{\Omega_j}}^{l^T}(t,x^{\Omega_j}(t),u^{\Omega_j}(t),v^{j\Omega_k}(t))\nu^{\Omega_j^l}(t) \\ & \quad +f_{0_{x^{\Omega_j}}}(t,x^{\Omega_j}(t),u^{\Omega_j}(t),v^{j\Omega_k}(t))]dt+\nu^{\Omega_j}(t)dW^{\Omega_j}(t), \\ x^{\Omega_j}(0) &= (S_0^{\Omega_j},I_0^{\Omega_j},R_0^{\Omega_j}) \\ \mu^{\Omega_j}(T) &= 0. \end{cases} \tag{28}$$

4.2.2. Numerical Results

In this part, we show the importance of following an optimal control strategy that is based on movement restrictions along with the presence of the vaccination policy discussed previously. For this, we take the example of preventing the epidemic from arriving at the region Ω_1 through infected travelers who come from other regions. This means we restrict movements of infected people coming from Ω_2 and Ω_3 either using $v^{1\Omega_2*}$ and/or $v^{1\Omega_3*}$, respectively. Figure 4 presents the simulations of I^{Ω_j} and R^{Ω_j}, $j=1,2,3$ in the presence of all optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$, and u^{Ω_3*}, and we can see that when we restrict movement of individuals coming from region Ω_2 and aiming to enter Ω_1, we can reach smaller values of infected people after only 15 days in all regions; at the same time, we deduce that the level of the number of removed people is approximately the same as in the absence of movement restrictions. We note that the optimal controls u^{Ω_j*}, $j = 2,3$ along with the optimal control $v^{1\Omega_2*}$ take the same values along the optimal control strategy period, which are all equal to one, and they tend to zero at the final time T.

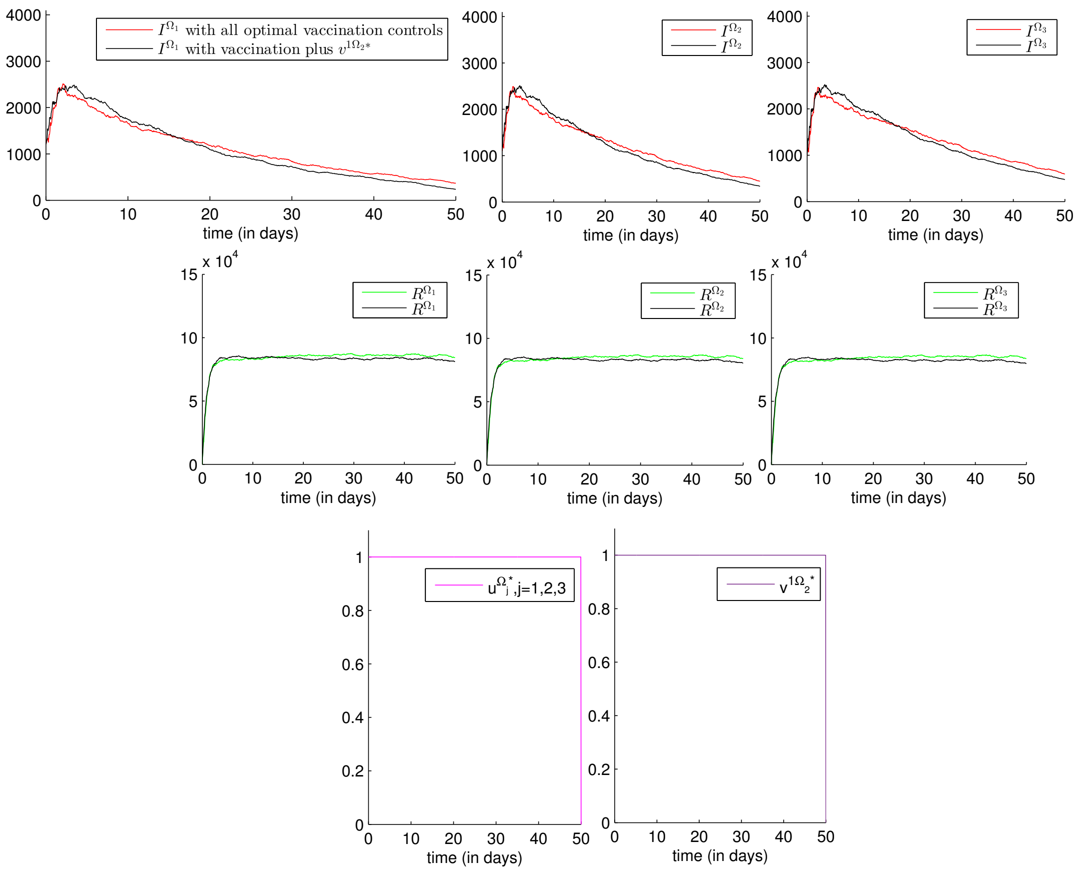

Figure 4. (Red and green curves) I^{Ω_j} and R^{Ω_j}, $j = 1,2,3$ dynamics with optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$ and u^{Ω_3*}. (Black curves) I^{Ω_j} and R^{Ω_j}, $j = 1,2,3$ dynamics with optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$ and u^{Ω_3*} plus $v^{1\Omega_2*}$. $S_0^{\Omega_1} = 90{,}000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89{,}000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88{,}000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0 \, \forall \, j = 1,2,3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

In Figure 5, we can better see the importance of movement restrictions, especially when we prevent also individuals coming from region Ω_3, which is at highest risk of infection. In fact, when we add the optimal control $v^{1\Omega_2*}$ to $v^{1\Omega_3*}$, we can reach better results as in the previous case since the number of infected people has decreased more and the number of removed people has increased more in all regions compared to the case when optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$, and u^{Ω_3*} are followed alone. We note also here that the optimal controls u^{Ω_j*}, $j = 2,3$ along with the optimal controls $v^{1\Omega_k*}, k = 2,3$ take the same values along the optimal control strategy period, which are all equal to one, and they tend to zero at the final time T.

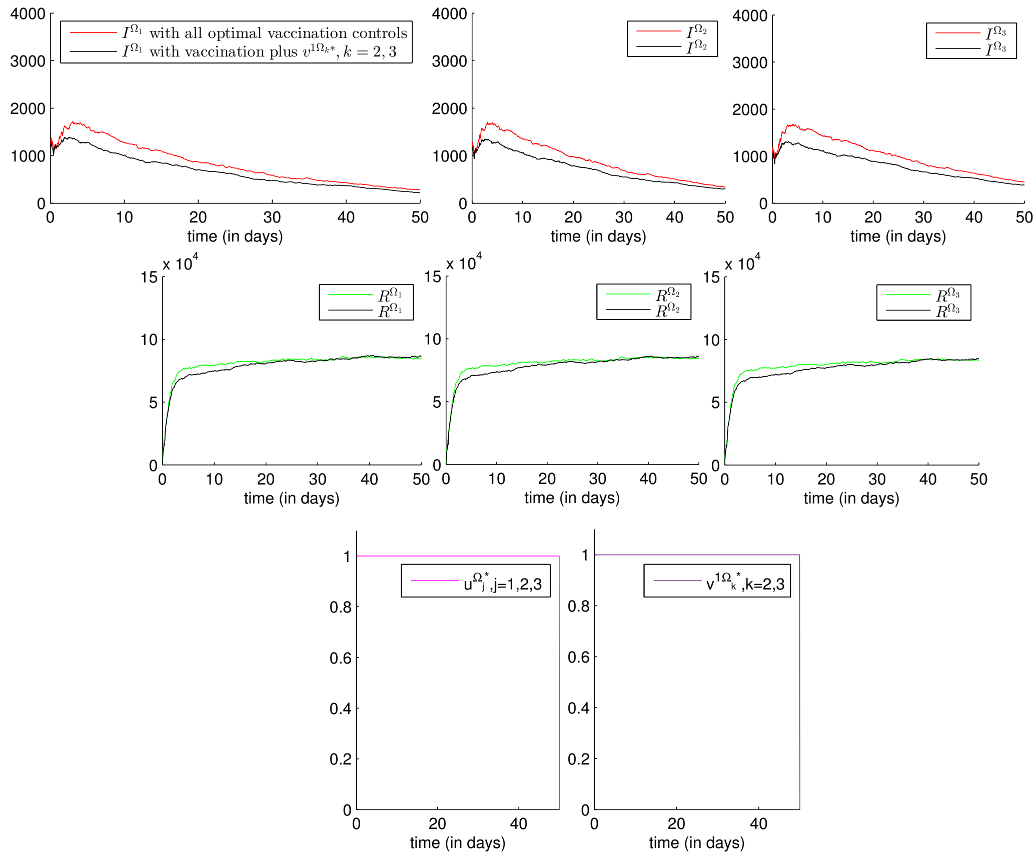

Figure 5. (Red and green curves) I^{Ω_j} and R^{Ω_j}, $j = 1, 2, 3$ dynamics with optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$ and u^{Ω_3*}. (Black curves) I^{Ω_j} and R^{Ω_j}, $j = 1, 2, 3$ dynamics with optimal controls $u^{\Omega_1*}, u^{\Omega_2*}$ and u^{Ω_3*} plus $v^{1\Omega_2*}$. $S_0^{\Omega_1} = 90{,}000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89{,}000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88{,}000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0 \, \forall \, j = 1, 2, 3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

5. Discussion

In this section, we provide other simulations to compare between the results of the optimal control strategy proposed in Section 3 and the other one treated in Section 4, as well as the case when there was no control yet as defined in Section 1. We also discuss the effect of severity controls weight A_j and B_j and the initial conditions of infection on the optimal values of controls and states.

In Figure 6, we treat the case when there is no vaccination in region Ω_1, and we compare it with the case when there is no control policy in all regions and with the case treated in Figure 1. From this figure, we can deduce that even in the absence of vaccination in the first region, there is an advantage of movement restrictions represented by optimal controls $v^{1\Omega_k*}, k = 2, 3$, applied between region Ω_1 and other regions, in decreasing the number of infected people and increasing the number of removed people. This figure also shows the importance of vaccination when it is followed in all regions, and that can be strengthened more if we add $v^{1\Omega_k*}, k = 2, 3$, as concluded in the previous figure. We note also here that the optimal controls u^{Ω_j*}, $j = 1, 2, 3$ along with the optimal controls $v^{1\Omega_k*}, k = 2, 3$ take the same values along the optimal control strategy period, which are all equal to one, with some perturbations after five days, and they tend to zero at the final time T.

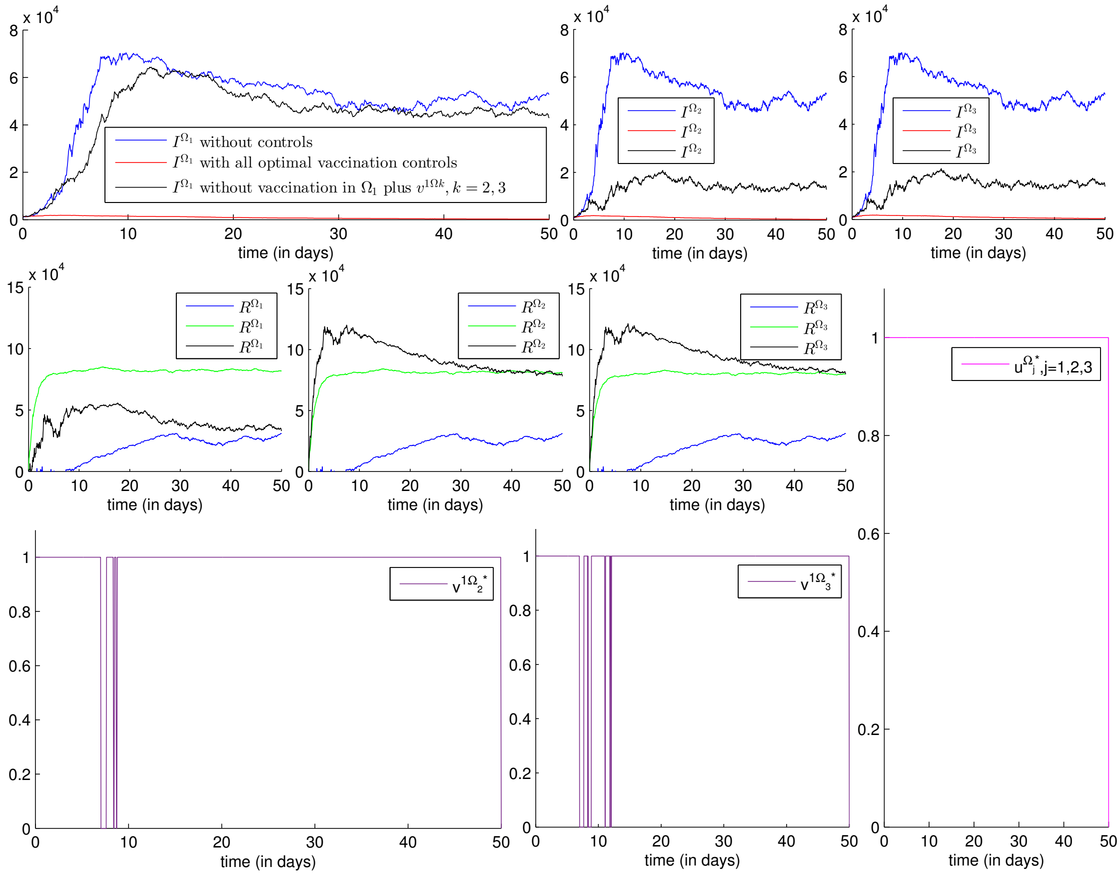

Figure 6. (Blue curves) I^{Ω_j} and R^{Ω_j}, $j = 1, 2, 3$ dynamics without controls. (Red and green curves) I^{Ω_j} and R^{Ω_j}, $j = 1, 2, 3$ dynamics with optimal controls $u^{\Omega_1 *}, u^{\Omega_2 *}$ and $u^{\Omega_3 *}$. (Black curves) I^{Ω_j} and R^{Ω_j}, $j = 1, 2, 3$ dynamics with optimal controls $u^{\Omega_2 *}$ and $u^{\Omega_3 *}$ plus $v^{1\Omega_2 *}$. $S_0^{\Omega_1} = 90{,}000$, $I_0^{\Omega_1} = 1200$, $S_0^{\Omega_2} = 89{,}000$, $I_2^{\Omega_2} = 1100$, $S_0^{\Omega_3} = 88{,}000$, $I_0^{\Omega_3} = 1000$, $R_0^{\Omega_j} = 0 \ \forall\ j = 1, 2, 3$, $d_1 = 0.06$, $\gamma_1 = 0.04$, $\beta_1 = 0.5$, $d_2 = 0.05$, $\gamma_2 = 0.03$, $\beta_2 = 0.4$, $d_3 = 0.04$, $\gamma_3 = 0.02$, $\beta_3 = 0.1$.

As in Figure 7, we present the simulation of the function I^{Ω_1} when there is no control, and we compare it with the case with the control for different values of severity controls weights, namely A_j and B_j associated with $u^{\Omega_j *}$ and $v^{j\Omega_k *}$, respectively, as noted previously. We observe that after introducing the five optimal controls $u^{\Omega_j *}, j = 1, 2, 3$, $v^{1\Omega_2 *}$, and $v^{1\Omega_3 *}$, the number of infected people has decreased towards the value of 400 individuals when $A_j = 1$, referring to the previous cases for obtaining $u^{\Omega_j *}$, but with $B_j = 10^{20}$ now. This number can be decreased earlier, as we can see below in the left-hand side part of this figure when we take $A_j = 10^4$. Defined as the denominator in the characterization of $v^{j\Omega_k *}$, the value $B_j = 10^{20}$ here implies a reduction of the optimal controls' values, $v^{1\Omega_2 *}$ and $v^{1\Omega_3 *}$, and the movement restriction is recommended for the first 22 and 25 days only. However, when we give a less important value to B_j, 10^8 for example, we can observe that a long period of this strategy minimized the function I^{Ω_1} more significantly, and it can be minimized and even earlier than this when we take $A_j = B_j = 10^4$, as seen below in the right-hand side of the figure.

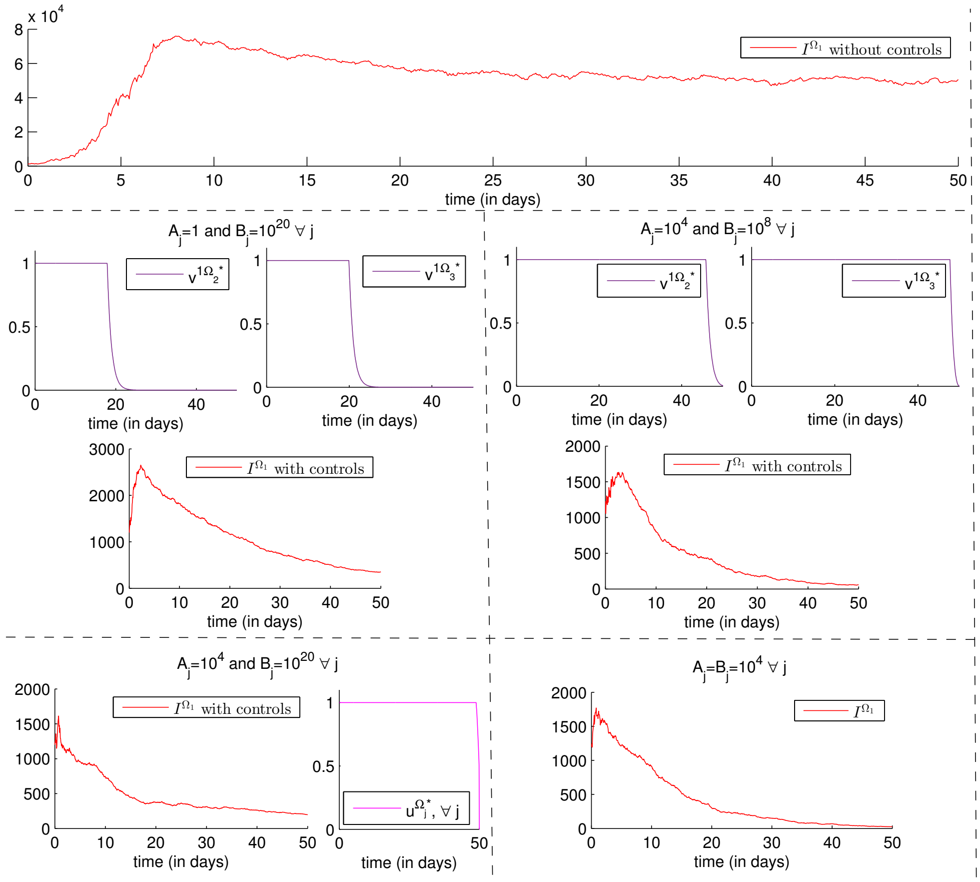

Figure 7. I^{Ω_1} and optimal controls $u^{\Omega_j*}, j = 1, 2, 3$, $v^{1\Omega_2*}$ and $v^{1\Omega_3*}$ for different values of A_j and B_j.

As the last Figure 8, we are interested in doing the same as in the previous figure, but in the case of less important values for initial conditions $I_0^{\Omega_j}, j = 1, 2, 3$ and smaller values for optimal controls $u^{\Omega_j*}, j = 1, 2, 3$, $v^{1\Omega_2*}$, and $v^{1\Omega_3*}$. From this figure, we observe that when there is no control, the function I^{Ω_1} does not increase to larger values as observed in the first figures, and then, we understand that we will not need to vaccinate everyone and close all borders for all times of the control strategy period. As predicted, we observe that when we add the value 10^{20} to both denominators A_j and B_j, implying an important reduction of all optimal controls, the number of infected people in Ω_1 decreased to very small values after 20 days only, and this can be done more effectively when vaccination is prolonged, as we can observe in the right-hand side of the figure, when we add 10^{10} to A_j.

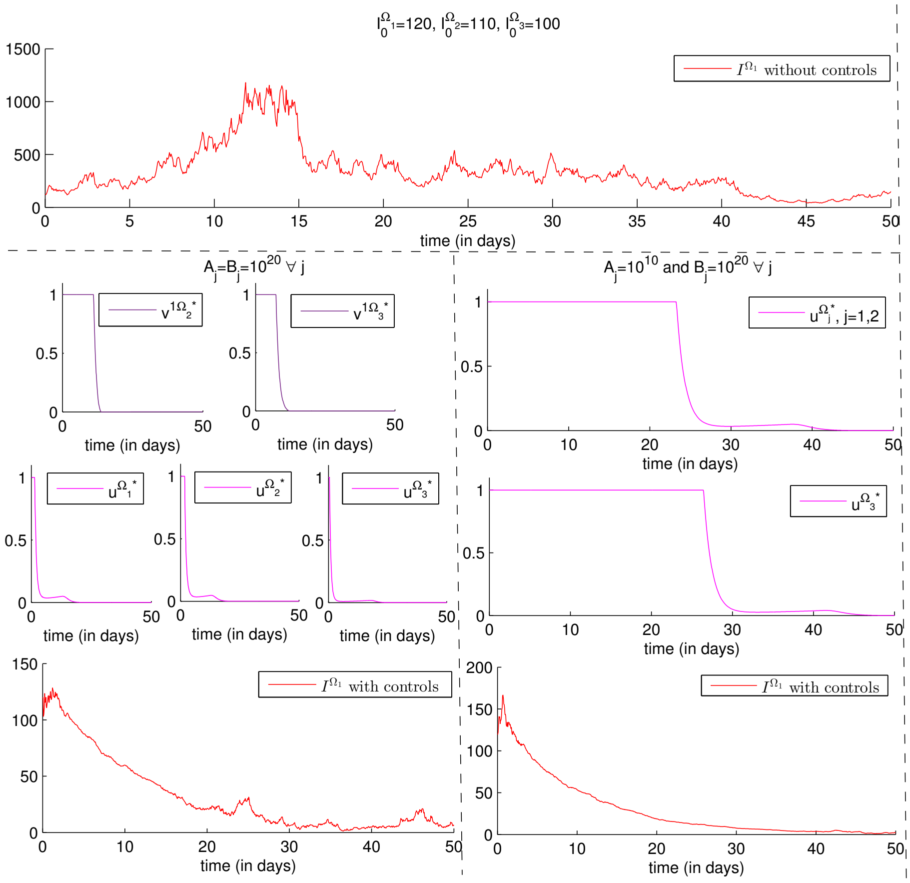

Figure 8. I^{Ω_1} and optimal controls $u^{\Omega_j*}, j = 1,2,3$, $v^{1\Omega_2*}$, and $v^{1\Omega_3*}$ for different values of A_j and B_j and less important initial condition of infection.

6. Conclusions

In this paper, we have proposed a stochastic model for the study of infection dynamics when an epidemic emerges in regions that are connected by any kind of anthropological movement, considering perturbations that are due to media, infodemics, and escapes. We have also suggested two optimal control approaches related to vaccination and movement restriction policies based on a stochastic version of Pontryagin's maximum principle.

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Nomenclature

All symbols used in this paper are presented in the following nomenclature.

t	time
T	time horizon
Ω	a large domain or region
x^{Ω}	$\mathbb{R}^n$ random state vector associated with Ω
u^{Ω}	$\mathbb{R}^m$ random control vector associated with Ω
$(\Theta, \mathcal{F}, \mathbb{P})$	probability space
W^{Ω}	vector-valued Wiener process associated with Ω all over $(\Theta, \mathcal{F}, \mathbb{P})$
f	$\mathbb{R}^n$ vector-valued nonlinear function
g	matrix-valued nonlinear diffusion
Ω_j	a region, an area, or subdomain of Ω $(1 \leq j \leq p)$
S^{Ω_j}	number of susceptibles in Ω_j
I^{Ω_j}	number of infectives in Ω_j
R^{Ω_j}	number of removed in Ω_j
ρ^{jk}	stochastic proportion of adequate contacts in Ω_j between a susceptible from Ω_j and an infective from Ω_k
βjk	deterministic proportion of adequate contacts in Ω_j between a susceptible from Ω_j and an infective from Ω_k
σ_j	intensities of fluctuations caused by media
d_j	birth and death rate
γ_j	recovery rate
N^{Ω_j}	population size corresponding to Ω_j
u^{Ω_j}	vaccination control introduced in Ω_j
θ^{Ω_j}	perturbation of control function u^{Ω_j} associated with Ω_j
$u^{\Omega_j}_{min}$	minimal bound of $u^{\Omega_j}(t)$
$u^{\Omega_j}_{max}$	maximal bound of $u^{\Omega_j}(t)$
J	objective function
f_0	current gain function
α^I_j	weight parameter associated with the number of infectives in Ω_j
α^R_j	weight parameter associated with the number of removed in Ω_j
A_j	vaccination control severity weight in Ω_j
B_j	movement restriction control severity weight in Ω_j
U_j	vaccination control set
μ^{Ω}	adjoint state variable associated with x^{Ω}
ν^{Ω}	adjoint matrix diffusion associated with g
I_H	set of indices of regions at a high-risk of infection
$V^{I_H}_j$	movement restriction control set
$v^{j\Omega_k}$	movement restriction control introduced in Ω_j to prevent infection from Ω_k
ϑ^{jk}	perturbation of control function $v^{j\Omega_k}$ associated to Ω_j
ς_{jk}	intensities of fluctuations caused by escapes
$v^{\Omega_j}_{min}$	minimal bound of $v^{j\Omega_k}(t)$
$v^{\Omega_j}_{max}$	maximal bound of $v^{j\Omega_k}(t)$

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