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Fractional Langevin Equations with Nonlocal Integral Boundary Conditions

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Abstract: In this paper, we investigate a class of nonlinear Langevin equations involving two fractional orders with nonlocal integral and three-point boundary conditions. Using the Banach contraction principle, Krasnoselskii's and the nonlinear alternative Leray Schauder theorems, the existence and uniqueness results of solutions are proven. The paper was appended examples which illustrate the applicability of the results.

Keywords: fractional Langevin equations; fixed point theorem; existence and uniqueness; integral boundary condition

MSC: 26A33; 34A08; 34A12; 34B15

1. Introduction

In recent years, the theory of fractional calculus has been developed rapidly. The existence and uniqueness of solutions for boundary value problems of fractional differential equations have been extensively studied [1–8], and an extensive list of references given in that respect. The significance of fractional differential equations comes from its applications in a several fields such as chemistry, physics, biology, aerodynamics, fitting of experimental data, signal and image processing, economics, control theory, biophysics, blood flow phenomena, etc. The Langevin equation (drafted for first by Langevin in 1908) is found to be an effectual tool to describe the progression of physical phenomena in fluctuating environments [9]. Alongside the intensive development of fractional derivative, the fractional Langevin equations have been introduced by Mainardi and Pironi [10], which was followed by many articles interested with the existence and uniqueness of solutions for fractional Langevin equations [11–25] and the references given therein.

There are several contributions focusing on the boundary value problems of fractional differential equations, mainly on the existence and uniqueness of the solutions with integrals boundary conditions [3,6]. As far as we know, few contributions associated with integrals boundary conditions for fractional Langevin equations have been published in [13,14,19,20].

Inspired by the papers mentioned above, in this paper, we study the existence and uniqueness of solutions for the following boundary value problem of the Langevin equation with two different fractional orders:

$$^{c}D^{\beta}(^{c}D^{\alpha}+\lambda)u(t)=f(t,u(t)),\qquad t\in[0,1] \tag{1}$$

subject to the nonlocal integral boundary condition

$$u(0)=0,\quad u(1)=0,\quad {^{c}D^{\alpha}}u(0)+{^{c}D^{\alpha}}u(1)=\mu\int_0^{\eta}u(s)ds \tag{2}$$

where ${}^cD^\alpha$ and ${}^cD^\beta$ are the Caputo's fractional derivatives of orders $0 < \alpha < 1$ and $1 < \beta \leq 2$, $\lambda, \mu \in \mathbb{R}$, $0 < \eta < 1$ and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function.

In the beginning of the paper, we provide some basic concepts on the fractional integrals and derivatives. Then, we are based on the Banach contraction principle and Krasnoselskii's fixed point theorem to study the existence and uniqueness solution of the three points boundary problems (1) and (2).

2. Preliminaries and Relevant Lemmas

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later. We are indebted to the terminologies used in the books [7,8].

Definition 1. *The Riemann-Lioville fractional integral of order $\alpha > 0$ for a continuous function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$$

provided that the right-hand-side integral exists, where $\Gamma(\alpha)$ denotes the Gamma function is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \qquad \alpha > 0.$$

Definition 2. *Let $n \in \mathbb{N}$ be a positive integer and α be a positive real such that $n - 1 < \alpha \leq n$, then the fractional derivative of a function $f : [0, \infty) \to \mathbb{R}$ in the Caputo sense is defined as*

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds$$

provided that the right-hand-side integral exists and is finite. We notice that the Caputo derivative of a constant is zero.

Lemma 1. *Let α and β be positive and real. If u is a continuous function, then we have*

$$I^\alpha I^\beta u(t) = I^{\alpha+\beta} u(t)$$

Lemma 2. *Let $n \in \mathbb{N}$ and $n - 1 < \alpha \leq n$. If u is a continuous function, then we have*

$$I^\alpha\, {}^cD^\alpha u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1} t^{n-1}$$

Let us now consider the linear fractional Langevin differential equation

$${}^cD^\beta({}^cD^\alpha + \lambda)u(t) = h(t), \qquad t \in [0,1] \tag{3}$$

supplemented with the nonlocal integral boundary condition

$$u(0) = 0, \quad u(1) = 0, \quad {}^cD^\alpha u(0) + {}^c D^\alpha u(1) = \mu \int_0^\eta u(s)ds \tag{4}$$

where $0 < \alpha < 1$ and $1 < \beta \leq 2$, $\lambda, \mu \in \mathbb{R}$, $0 < \eta < 1$ and $h \in C[0,1]$.

Lemma 3. *If $h \in C[0,1]$, then the unique solution of the boundary value problem (3) and (4) is given by*

$$
\begin{aligned}
u(t) = & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds - \frac{t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s) ds \\
& + \frac{\lambda t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s) ds + \frac{\mu t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)} \int_0^\eta u(s) ds \\
& - \frac{t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds
\end{aligned}
$$

Proof. From Lemmas 1, 2 and the relevant lemma in [13], it follows that

$$
\begin{aligned}
{}^cD^\alpha u(t) = & I^\beta h(t) + a_0 + a_1 t - \lambda u(t) \\
= & \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds + a_0 + a_1 t - \lambda u(t)
\end{aligned}
$$

and

$$
\begin{aligned}
u(t) = & I^{\alpha+\beta} h(t) + I^\alpha a_0 + I^\alpha a_1 t - I^\alpha \lambda u(t) + a_2 \\
= & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} h(s) ds + a_0 \frac{t^\alpha}{\Gamma(\alpha+1)} + a_1 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \\
& - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds + a_2
\end{aligned}
\tag{5}
$$

Using the boundary conditions (4) gives $a_2 = 0$,

$$
\begin{aligned}
a_0 = & \frac{\Gamma(\alpha+2)}{1-\alpha} \left(\frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s) ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s) ds \right) \\
& + \frac{1}{1-\alpha} \left(\mu \int_0^\eta u(s) ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s) ds \right)
\end{aligned}
$$

and

$$
\begin{aligned}
a_1 = & \frac{2\Gamma(\alpha+2)}{1-\alpha} \left(\frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s) ds - \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} h(s) ds \right) \\
& - \frac{(1+\alpha)}{1-\alpha} \left(\mu \int_0^\eta u(s) ds - \frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s) ds \right)
\end{aligned}
$$

Substituting the above values of a_0, a_1 and a_3 in (5) to obtain the desired results. □

Lemma 4. *For all $\alpha \in (0,1)$, we have*

1. $\max_{t\in[0,1]} |t^\alpha(1-t)| = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}$
2. $\max_{t\in[0,1]} |t^\alpha(2t-\alpha-1)| = \begin{cases} 1-\alpha & \text{if} \quad \alpha \le \frac{1}{2} \\ \left(\frac{\alpha}{2}\right)^\alpha & \text{if} \quad \alpha > \frac{1}{2} \end{cases}$

Proof. The first statement has been proven by Zhou and Qiao [14]. They also proved that $\max_{t\in[0,1]} |t^\alpha(2t-\alpha-1)| = \max\{\left(\frac{\alpha}{2}\right)^\alpha, 1-\alpha\}$. Now, assume that

$$
g(\alpha) = \left(\frac{\alpha}{2}\right)^\alpha - (1-\alpha).
$$

Differentiation gives

$$g'(\alpha) = \left(\frac{\alpha}{2}\right)^{\alpha} (\ln \alpha - \ln 2 + 1) + 1$$
$$g''(\alpha) = \left(\frac{\alpha}{2}\right)^{\alpha} \left((\ln \alpha - \ln 2 + 1)^2 + \frac{1}{\alpha}\right) > 0$$

which means that $g'(\alpha)$ is increasing on $(0, 1)$. It is easy to see that $\lim_{\alpha \to 0^+} g'(\alpha) = -\infty$ and $g'(1/2) = (1/2)(1 - \ln 4) + 1 \sim 0.806853 > 0$ which implies that there exists $0 < \alpha_0 < 1/2$ such that $g'(\alpha) < 0$ for all $\alpha \in (0, \alpha_0)$ and $g'(\alpha) > 0$ for all $\alpha \in (\alpha_0, 1]$. It follows that $g(\alpha)$ is decreasing on $(0, \alpha_0)$ and increasing on $(\alpha_0, 1)$. Since $\lim_{\alpha \to 0^+} g(\alpha) = 0$, $g(1/2) = 0$ and $g(1) = 1/2$, then we obtain the desired results. □

In the proofs of our main existence results for the problem (1) and (2), we will use the Banach contraction mapping principle, the Krasnoselskii fixed point theorem for the sum of two operators and nonlinear alternative Leray-Schauder theorem presented below:

Lemma 5. *([26]) Let Ω be a closed convex and nonempty subset of Banach space $\mathbb{E}$. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two operators such that:*

1. $\mathcal{F}_1 x + \mathcal{F}_2 y \in \Omega, \qquad x, y \in \Omega$
2. *$\mathcal{F}_1$ is compact and continuous on Ω*
3. *$\mathcal{F}_2$ is a contraction mapping on Ω.*

Then there exists $z \in \Omega$ such that $z = \mathcal{F}_1 z + \mathcal{F}_2 z$.

Lemma 6. *([27,28]) Let $\mathbb{E}$ be a Banach space, C be a closed and convex subset of $\mathbb{E}$, U be an open subset of C and $0 \in U$. Suppose that the operator $\mathcal{T} : \overline{U} \to C$ is a continuous and compact map (that is, $\mathcal{T}(\overline{U})$ is a relatively compact subset of C). Then either*

(i) *$\mathcal{T}$ has a fixed point in $x^* \in \overline{U}$, or*
(ii) *there is $x \in \partial U$ (the boundary of U in C) and $\delta > 1$ such that $\delta x = \mathcal{T}(x)$.*

3. Main Results

Let $\mathbb{E} = C([0, 1], \mathbb{R})$ be the Banach space of all continuous functions from $[0, 1] \longrightarrow \mathbb{R}$ endowed the norm defined by

$$\|u\| = \sup \{|u(t)|, t \in [0, 1]\}.$$

Before stating and proving the main results, we introduce the following hypotheses. Assume that

($\mathcal{H}_1$) The function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a jointly continuous.
($\mathcal{H}_2$) The function f satisfies

$$|f(t, u) - f(t, v)| \leq Ł|u - v|, \qquad \forall t \in [0, 1], u, v \in \mathbb{R}$$

where Ł is the Lipschitz constant.
($\mathcal{H}_3$) There exists a nonnegative function $\psi \in C([0, 1], \mathbb{R}_+)$ such that

$$|f(t, u)| \leq \psi(t), \qquad \forall (t, u) \in ([0, 1], R).$$

($\mathcal{H}_4$) There exist two nonnegative functions $p, q \in L_1([0, 1])$ such that

$$|f(t, u)| \leq p(t)|u| + q(t), \qquad \forall (t, u) \in ([0, 1], R).$$

For computational convenience, we set

$$\Lambda = \Delta_1 + \text{Ł}\Lambda_1 \tag{6}$$

$$\Delta = \frac{\text{Ł}\nu}{\Gamma(\alpha+\beta+1)} + \frac{\lambda\nu}{\Gamma(\alpha+1)} + A\left(\eta|\mu| + \frac{\text{Ł}}{\Gamma(\beta+1)}\right) \tag{7}$$

$$Q = \Delta_1 + \frac{1+\nu}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}p(s)ds + \frac{A}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}p(s)ds \tag{8}$$

$$Q_1 = \frac{1+\nu}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}q(s)ds + \frac{A}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}q(s)ds \tag{9}$$

where

$$\Lambda_1 = \frac{1+\nu}{\Gamma(\alpha+\beta+1)} + \frac{A}{\Gamma(\beta+1)} \tag{10}$$

$$\Delta_1 = \frac{|\lambda|(1+\nu)}{\Gamma(\alpha+1)} + \eta|\mu|A \tag{11}$$

and

$$A = \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}(1-\alpha)\Gamma(\alpha+1)}$$

$$\nu = \begin{cases} 1 \quad \text{if} \quad \alpha \le \frac{1}{2} \\ \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\right)^\alpha \quad \text{if} \quad \alpha > \frac{1}{2} \end{cases}$$

In view of Lemma 3, we transform problem (1) and (2) as

$$u = T(u) = T_1(u) + T_2(u) \tag{12}$$

where the operators $T_i : \mathbb{E} \longrightarrow \mathbb{E}, i = 1,2$ are defined by

$$(T_1u)(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,u(s))ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds \tag{13}$$

and

$$(T_2u)(t) = -\frac{t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-1}f(s,u(s))ds + \frac{\lambda t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}u(s)ds + \frac{\mu t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)}\int_0^\eta u(s)ds - \frac{t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}f(s,u(s))ds \tag{14}$$

Theorem 1. *Assume that the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold. Then the boundary value problem (1) and (2) has a unique solution if $\Lambda < 1$, where Λ is given by (6).*

Proof. Define the closed ball $\mathcal{B}_r = \{u \in \mathbb{E} : \|u\| \le r\}$ with the radius $r \ge M\Lambda_1/(1-\Lambda)$ where Λ_1 is given by (10) and

$$M = \sup_{t\in[0,1]} |f(t,0)|.$$

Then, for $u \in \mathcal{B}_r$, we have

$$\begin{aligned}\|f(t,u(t))\| &= \sup_{t\in[0,1]} |f(t,u(t)) - f(t,0) + f(t,0)| \\ &\leq \sup_{t\in[0,1]} |f(t,u(t)) - f(t,0)| + \sup_{t\in[0,1]} |f(t,0)| \leq Ł\|u\| + M \leq Łr + M.\end{aligned}$$

From this and Lemma 4, we obtain

$$\begin{aligned}\|(Tu)(t)\| = \sup_{t\in[0,1]} \Bigg| & \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} f(s,u(s))ds \\ & - \frac{t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} f(s,u(s))ds \\ & + \frac{\lambda t^\alpha(2t-\alpha-1)}{(1-\alpha)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} u(s)ds + \frac{\mu t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)} \int_0^\eta u(s)ds \\ & - \frac{t^\alpha(1-t)}{(1-\alpha)\Gamma(\alpha+1)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,u(s))ds - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds \Bigg| \\ & \leq \frac{Łr+M}{\Gamma(\alpha+\beta)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha+\beta-1} ds + \frac{\nu(Łr+M)}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} ds \\ & + \frac{|\lambda|\nu r}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} ds + A|\mu| r \int_0^\eta ds + \frac{A(Łr+M)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} ds \\ & + \frac{|\lambda| r}{\Gamma(\alpha)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha-1} ds \\ & = \frac{(Łr+M)(1+\nu)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r(1+\nu)}{\Gamma(\alpha+1)} + A\left(\eta|\mu| r + \frac{Łr+M}{\Gamma(\beta+1)}\right) \\ & \leq r.\end{aligned}$$

which implies that $\|Tu\| \leq r$. Let $u, v \in \mathbb{E}$ for each $t \in [0,1]$, we obtain

$$\begin{aligned}\|(Tu)(t) - (Tv)(t)\| \leq & \frac{1}{\Gamma(\alpha+\beta)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha+\beta-1} |f(s,u(s)) - f(s,v(s))| ds \\ & + \frac{\nu}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} |f(s,u(s)) - f(s,v(s))| ds \\ & + \frac{|\lambda|\nu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |u(s) - v(s)| ds + |\mu| A \int_0^\eta |u(s) - v(s)| ds \\ & + \frac{A}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s,u(s)) - f(s,v(s))| ds \\ & + \frac{\lambda}{\Gamma(\alpha)} \sup_{t\in[0,1]} \int_0^t (t-s)^{\alpha-1} |u(s) - v(s)| ds \\ & \leq \|u-v\| \left\{ \frac{Ł(1+\nu)}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda|(1+\nu)}{\Gamma(\alpha+1)} + A\left(\eta|\mu| + \frac{Ł}{\Gamma(\beta+1)}\right) \right\} \\ & = \Lambda\|u-v\|\end{aligned}$$

By virtue of the hypothesis of $\Lambda < 1$, then the operator T is contraction mapping. Therefore, the boundary value problem (1) and (2) has a unique solution. This ends the proof. □

Theorem 2. *Assume that the assumptions ($\mathcal{H}_1$), ($\mathcal{H}_2$) and ($\mathcal{H}_3$) hold. Then the boundary value problem (1) and (2) has at least one solution if $\Delta < 1$ where Δ is given by (7).*

Proof. Let the two operators T_1 and T_2 be defined as in (13) and (14). Setting $\sup_{t\in[0,1]}|\psi(t)| \le \|\psi\|$. Let the closed ball $\mathcal{B}_r = \{u \in \mathbb{E} : \|u\| \leqslant r\}$ be defined for

$$r \ge \Lambda_1\|\psi\|\,(|1-\Delta_1|)^{-1}$$

where Λ_1 and Δ_1 are given by (10) and (11), respectively. Then, for $u, v \in \mathcal{B}_r$, it follows that

$$\|T_1 u + T_2 v\| \le \Lambda_1\|\psi\| + r\Delta_1 \le r$$

which concludes that $T_1 u + T_2 v \in \mathcal{B}_r$. In view of the assumption ($\mathcal{H}_2$), it can easily be shown that T_2 is a contraction mapping if $\Delta < 1$. The function f is continuous according to the assumption ($\mathcal{H}_1$) which implies that T_1 is continuous. Now, for $u \in \mathcal{B}_r$, it follows that

$$\begin{aligned}
\|(T_1u)(t)\| &= \sup_{t\in[0,1]}\left|\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,u(s))ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds\right| \\
&\le \frac{\|\psi\|}{\Gamma(\alpha+\beta)}\sup_{t\in[0,1]}\int_0^t (t-s)^{\alpha+\beta-1}ds + \frac{|\lambda|\|u\|}{\Gamma(\alpha)}\sup_{t\in[0,1]}\int_0^t (t-s)^{\alpha-1}ds \\
&= \frac{\|\psi\|}{\Gamma(\alpha+\beta+1)}\sup_{t\in[0,1]} t^{\alpha+\beta} + \frac{|\lambda| r}{\Gamma(\alpha+1)}\sup_{t\in[0,1]} t^{\alpha} \\
&\le \frac{\|\psi\|}{\Gamma(\alpha+\beta+1)} + \frac{|\lambda| r}{\Gamma(\alpha+1)}
\end{aligned}$$

which yields that the operator T_1 is uniformly bounded. Let $0 \le t_1 < t_2 \le 1$, then for $u \in \mathcal{B}_r$, we have

$$\begin{aligned}
||(T_1u)(t_2) - (T_1u)(t_1)|| = &\left\|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_2} (t_2-s)^{\alpha+\beta-1}f(s,u(s))ds\right. \\
&- \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1} (t_1-s)^{\alpha+\beta-1}f(s,u(s))ds \\
&\left.- \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1}u(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}u(s)ds\right\| \\
= &\left\|\frac{1}{\Gamma(\alpha+\beta)}\int_0^{t_1} [(t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1}]f(s,u(s))ds\right. \\
&+ \frac{1}{\Gamma(\alpha+\beta)}\int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1}f(s,u(s))ds \\
&+ \frac{\lambda}{\Gamma(\alpha)}\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]u(s)ds \\
&\left.- \frac{\lambda}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}u(s)ds\right\| \\
\le &\frac{\|\psi\|}{\Gamma(\alpha+\beta+1)}\left(t_2^{\alpha+\beta} - t_1^{\alpha+\beta}\right) + \frac{|\lambda| r}{\Gamma(\alpha+1)}\left(2(t_2-t_1)^{\alpha} + t_1^{\alpha} - t_2^{\alpha}\right)
\end{aligned}$$

which is independent of u and approaches zero when letting $t_2 \to t_1$. Thus, T_1 is relatively compact on $\mathcal{B}_r$. Hence, by the Arzela-Ascoli Theorem, the operator T_1 is completely continuous on $\mathcal{B}_r$. Therefore, according to the Krasnoselskii Theorem that was mentioned in Lemma 5, the boundary value problem (1) and (2) has at least one solution on $\mathcal{B}_r$. This completes the proof. □

Theorem 3. *Assume that the assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_4$) hold. Then the boundary value problem (1) and (2) has at least one solution if $Q < 1$ where Q is given by (8).*

Proof. Let $\Omega = \{u \in \mathbb{E} : \|u\| < \ell\}$ be an open subset of the Banach space $\mathbb{E}$ with $\ell = Q_1(1-Q)^{-1}$ where Q_1 is given by (9). As in Theorem 2, it is easy to see that the operator $T : \overline{\Omega} \to \mathbb{E}$ given by (12) is completely continuous, thus the proof is omitted here. Now, assume that $u \in \partial\Omega$ such that $\delta u = T(u)$ for $\delta > 1$. Then, we have

$$\begin{aligned}
\delta\ell = \delta\|u(t)\| = \|(Tu)(t)\| &= \sup_{t\in[0,1]} |(Tu)(t)| \\
&\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1}|f(s,u(s))|ds \\
&+ \frac{\nu}{\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1}|f(s,u(s))|ds \\
&+ \frac{|\lambda|\nu}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}|u(s)|ds + |\mu|A \int_0^\eta |u(s)|ds \\
&+ \frac{A}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}|f(s,u(s))|ds \\
&+ \frac{|\lambda|}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}|u(s)|ds \\
&\leq Q\ell + Q_1.
\end{aligned}$$

This leads to $\delta \leq 1$ which contradicts with the hypothesis $\delta > 1$. Therefore, according to the nonlinear alternative Leray-Schauder type (Lemma 6), the boundary value problem (1) and (2) has at least one solution in $[0,1]$. □

4. An Example

We present an example to better illustrate our main results.

Example 1. *Consider the following boundary value problem for fractional Langevin equations:*

$$\begin{cases} {}^cD^{\frac{3}{2}}({}^cD^{\frac{2}{5}} + \frac{1}{8})u(t) = f(t,u(t)), \qquad 0 < t < 1 \\ u(0) = 0, \qquad u(1) = 0, \qquad {}^cD^{\frac{2}{5}}u(0) + {}^c D^{\frac{2}{5}}u(1) = 2\int_0^{\frac{1}{10}} u(s)ds \end{cases} \tag{15}$$

where $\alpha = 2/5$, $\beta = 3/2$, $\lambda = 1/8$, $\mu = 2, \eta = 1/10$ *and* $f(t,u(t))$ *will be determined according to the assumptions of theorems.*

Using the given data, we find that $\nu = 1$ (since $\alpha = 2/5 < 1/2$) and $A = 0.812907$. Consequently, $\Lambda_1 = 1.70599$ and $\Delta_1 = 0.444346$, where Λ_1 and Δ_1 are given by (10) and (11), respectively.

Case I: Banach fixed point theorem

In order to illustrate Theorem 1, we take:

$$f(t,u(t)) = Ł(1 + t\sin(tu)). \tag{16}$$

It is easy to see that the function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the condition

$$|f(t,u(t)) - f(t,v(t))| = Ł|t\sin(tu) - t\sin(tv)| \leq Łt^2 \int_v^u |\cos ts|ds \leq Ł|u-v|$$

where Ł is the Lipschitz constant. Observe that the condition $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold true. To verify the assumption of Theorem 1 ($\Lambda < 1$) where Λ is given by (6) we have to choose $Ł < 0.325707$. Thus all the assumptions of Theorem 1 are satisfied. So the boundary value problem (1) and (2) has a unique solution on $[0,1]$ if $Ł < 0.325707$.

Case II: Krasnoselskii's theorem

In order to illustrate Theorem 2, take $f(t, u(t))$ as given in (16) and $\psi(t) = 1 + t$. Clearly the three hypotheses of Theorem 2 are satisfied. From the condition $\Delta < 1$ where Δ is given by (7), we have to choose Ł < 0.60111. Thus the conclusion of Theorem 2 applies to the boundary value problem (1) and (2) if Ł < 0.60111.

Case III: Leray-Schauder nonlinear alternative theorem

To illustrate Theorem 3, we take

$$f(t, u(t)) = \frac{(ut^2 + ut^{\frac{5}{2}}) + t}{1 + t^{\frac{1}{2}}}, \tag{17}$$

which implies that $p(t) = t^2$, $q(t) = \frac{t}{1+t^{\frac{1}{2}}}$, and $Q = 0.777662 < 1$. Thus all conditions of Theorem 3 are satisfied and consequently, there exists at least one solution for the boundary value problem (1) and (2) with $f(t, u(t))$ given by (17) on $[0, 1]$.

5. Conclusions

The existence and uniqueness of solutions for nonlocal integral and three-point boundary value problem including the Langevin equation with two fractional orders has been studied. We applied the fractional calculus concepts together with fixed point theorems to prove the existence and uniqueness results. To investigate our problem, we apply the Banach contraction principle, Krasnoselskii's fixed point theorem and the nonlinear alternative Leray-Schauder theorem. Our approach is simple to apply a variety of real-world issues.

It is worth pointing out that in view of the domain of Ł < 0.325707 in Case I and the domain of Ł < 0.60111 in Case II, we find that it was extended in the case of investigation of the existence and this is to be expected.

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