

Article

Hermite-Hadamard Type Inequalities for Interval (h_1, h_2) -Convex Functions

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Abstract: We introduce the concept of interval (h_1, h_2) -convex functions. Under the new concept, we establish some new interval Hermite-Hadamard type inequalities, which generalize those in the literature. Also, we give some interesting examples.

Keywords: Hermite-Hadamard inequality; interval-valued functions; (h_1, h_2) -convex

1. Introduction

Interval analysis was introduced in numerical analysis by Moore in the celebrated book [1]. Over the past 50 years, it has attracted considerable interest and has been applied in various fields, such as interval differential equations [2], aeroelasticity [3], aerodynamic load analysis [4], and so on. For more profound results and applications, see [5–9].

It is known that inequalities play an important role in almost all branches of mathematics as well as in other areas of science. Among the many types of inequalities, those carrying the names of Jensen, Hermite-Hadamard, Hardy, Ostrowski, Minkowski and Opial et al. have a deep significance and have made a great impact in substantial fields of research. Recently, some of these inequalities have been extended to interval-valued functions by Chalco-Cano et al.; see, e.g., [10–16]. Surprisingly enough, interval Hermite-Hadamard type inequalities has perhaps not received enough attention [17]. For convenience, we recall the classical Hermite-Hadamard inequality. Let f be convex, then

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(t)dt \leq \frac{f(u)+f(v)}{2}.$$

This inequality has been developed for different classes of convexity [18–26]. Especially, since the h -convex concept was proposed by Varosanec in 2007 [27], a number of authors have already studied more refined Hermite-Hadamard inequalities involving h -convex functions [28–33].

In 2018, Awan et al. introduced (h_1, h_2) -convex functions and proved the following inequality [34]:

Theorem 1. Let $f : [u, v] \rightarrow \mathbb{R}$. If f is (h_1, h_2) -convex, and $h_1(\frac{1}{2})h_2(\frac{1}{2}) \neq 0$. Then

$$\frac{1}{2h_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v f(t)dt \leq [f(u)+f(v)] \int_0^1 h_1(x)h_2(1-x)dx.$$

Motivated by Awan et al., our main objective is to generalize the results above by constructing interval Hermite-Hadamard type inequalities for (h_1, h_2) -convex functions. Also, we present some examples to illustrate our theorems. Our results generalize some known inequalities presented

in [17,32,34,35]. Furthermore, the present results can be considered as tools for further research in interval convex analysis, interval nonlinear programming, inequalities for fuzzy-interval-valued functions, among others.

We give preliminaries in Section 2. In Section 3, we introduce interval (h_1, h_2) -convex concept, and obtain some interval Hermite-Hadamard type inequalities. Moreover, some interesting examples are given. In Section 4, we give conclusions and future work.

2. Preliminaries

For the basic notations and definitions on interval analysis, see [17]. The family of all intervals and positive intervals of \mathbb{R} are denoted by $\mathbb{R}_{\mathcal{I}}$ and $\mathbb{R}_{\mathcal{I}}^+$, respectively. For interval $[\underline{u}, \bar{u}]$ and $[\underline{v}, \bar{v}]$, the Hausdorff distance is defined by

$$d([\underline{u}, \bar{u}], [\underline{v}, \bar{v}]) = \max \{ |\underline{u} - \underline{v}|, |\bar{u} - \bar{v}| \}.$$

Then, $(\mathbb{R}_{\mathcal{I}}, d)$ is complete.

A set of numbers $\{t_{i-1}, \xi_i, t_i\}_{i=1}^m$ is said to be a tagged partition P of $[u, v]$ if

$$u = t_0 < t_1 < \dots < t_m = v$$

and if $t_{i-1} \leq \xi_i \leq t_i$ for all $i = 1, 2, \dots, m$. Moreover, if we let $\Delta t_i = t_i - t_{i-1}$, then the partition is called δ -fine if $\Delta t_i < \delta$ for each i . We denote by $\mathcal{P}(\delta, [u, v])$ the family of all δ -fine partitions of $[u, v]$. Given $P \in \mathcal{P}(\delta, [u, v])$, we define a integral sum of $f : [u, v] \rightarrow \mathbb{R}_{\mathcal{I}}$ as follows:

$$S(f, P, \delta, [u, v]) = \sum_{i=1}^m f(\xi_i)(t_i - t_{i-1}).$$

Definition 1. Let $f : [u, v] \rightarrow \mathbb{R}_{\mathcal{I}}$. f is called IR-integrable on $[u, v]$ with IR-integral $A = (IR) \int_u^v f(t)dt$, if there exists an $A \in \mathbb{R}_{\mathcal{I}}$ such that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(S(f, P, \delta, [u, v]), A) < \epsilon$$

for each $P \in \mathcal{P}(\delta, [u, v])$. Let $\mathcal{IR}_{([u, v])}$ denote the set of all IR-integrable functions on $[u, v]$.

Definition 2. Let $h_1, h_2 : [0, 1] \subseteq J \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$ (Awan et al. [34]). $f : J \rightarrow \mathbb{R}^+$ is called (h_1, h_2) -convex, or that $f \in SX((h_1, h_2), J, \mathbb{R})$, if for any $s, t \in J$ and $x \in (0, 1)$ one has

$$f(xs + (1 - x)t) \leq h_1(x)h_2(1 - x)f(s) + h_1(1 - x)h_2(x)f(t). \tag{1}$$

Remark 1. If $h_2 \equiv 1$, then Definition 2 reduces to h -convex in [27].

If $h_1 = h_2 \equiv 1$, then Definition 2 reduces to P -function in [18].

If $h_1(t) = t^s, h_2 \equiv 1$, then Definition 2 reduces to s -convex in [36].

We end this section of preliminaries by introducing the new concept of interval (h_1, h_2) -convexity. This idea is inspired by Costa [12]. Note that for interval $[\underline{u}, \bar{u}]$ and $[\underline{v}, \bar{v}]$, the inclusion " \subseteq " is defined by

$$[\underline{u}, \bar{u}] \subseteq [\underline{v}, \bar{v}] \iff \underline{v} \leq \underline{u}, \bar{u} \leq \bar{v}.$$

Definition 3. Let $h_1, h_2 : [0, 1] \subseteq J \rightarrow \mathbb{R}^+$ such that $h_1, h_2 \not\equiv 0$. $f : J \rightarrow \mathbb{R}_{\mathcal{I}}^+$ is called interval (h_1, h_2) -convex, if for all $s, t \in J$ and $x \in (0, 1)$ one has

$$h_1(x)h_2(1 - x)f(s) + h_1(1 - x)h_2(x)f(t) \subseteq f(xs + (1 - x)t). \tag{2}$$

The set of all interval (h_1, h_2) -convex function is denoted by $SX((h_1, h_2), J, \mathbb{R}_T^+)$.

3. Interval Hermite-Hadamard Type Inequality

In what follows, let $H(x, y) = h_1(x)h_2(y)$ for $x, y \in [0, 1]$.

Theorem 2. Let $f : [u, v] \rightarrow \mathbb{R}_T^+, h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ and $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. If $f \in SX((h_1, h_2), [u, v], \mathbb{R}_T^+)$ and $f \in \mathcal{IR}_{([u, v])}$, then

$$\frac{1}{2H(\frac{1}{2}, \frac{1}{2})} f\left(\frac{u+v}{2}\right) \supseteq \frac{1}{v-u} \int_u^v f(t)dt \supseteq [f(u) + f(v)] \int_0^1 H(x, 1-x)dx. \tag{3}$$

Proof. By hypothesis, we have

$$H\left(\frac{1}{2}, \frac{1}{2}\right) f(xu + (1-x)v) + H\left(\frac{1}{2}, \frac{1}{2}\right) f((1-x)u + xv) \subseteq f\left(\frac{u+v}{2}\right).$$

Then

$$\begin{aligned} \int_0^1 \underline{f}(xu + (1-x)v)dx + \int_0^1 \underline{f}((1-x)u + xv)dx &\geq \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \int_0^1 \underline{f}\left(\frac{u+v}{2}\right)dx, \\ \int_0^1 \bar{f}(xu + (1-x)v)dx + \int_0^1 \bar{f}((1-x)u + xv)dx &\leq \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \int_0^1 \bar{f}\left(\frac{u+v}{2}\right)dx. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{2}{v-u} \int_u^v \underline{f}(t)dt &\geq \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \int_0^1 \underline{f}\left(\frac{u+v}{2}\right)dx = \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \underline{f}\left(\frac{u+v}{2}\right), \\ \frac{2}{v-u} \int_u^v \bar{f}(t)dt &\leq \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \int_0^1 \bar{f}\left(\frac{u+v}{2}\right)dx = \frac{1}{H(\frac{1}{2}, \frac{1}{2})} \bar{f}\left(\frac{u+v}{2}\right). \end{aligned}$$

This implies

$$\frac{1}{H(\frac{1}{2}, \frac{1}{2})} \left[\underline{f}\left(\frac{u+v}{2}\right), \bar{f}\left(\frac{u+v}{2}\right) \right] \supseteq \frac{2}{v-u} \left[\int_u^v \underline{f}(t)dt, \int_u^v \bar{f}(t)dt \right].$$

Thus,

$$\frac{1}{2H(\frac{1}{2}, \frac{1}{2})} f\left(\frac{u+v}{2}\right) \supseteq \frac{1}{v-u} \int_u^v f(t)dt.$$

In the same way as above, we have

$$\frac{1}{v-u} \int_u^v f(t)dt \supseteq [f(u) + f(v)] \int_0^1 H(x, 1-x)dx,$$

and the result follows. \square

Remark 2. If $H(x, y) \equiv h_1(x)$, then Theorem 2 reduces to ([17], Theorem 4.1).

If $h_1(x) = x^s, h_2 \equiv 1$, then Theorem 2 reduces to ([37], Theorem 4).

If $h_1 = h_2 \equiv 1$, then inequality (3) in Theorem 2 reduces to inequality for P-function.

If $\underline{f} = \bar{f}$, then Theorem 2 reduces to ([34], Theorem 1). Furthermore, If $h_2 \equiv 1$, then we get ([32], Theorem 6).

Example 1. Suppose that $h_1(x) = x$, $h_2(x) \equiv 1$ for $x \in [0, 1]$, $[u, v] = [-1, 1]$, and $f : [u, v] \rightarrow \mathbb{R}_T^+$ be defined by $f(t) = [t^2, 4 - e^t]$. Then

$$\begin{aligned} \frac{1}{2H\left(\frac{1}{2}, \frac{1}{2}\right)} f\left(\frac{u+v}{2}\right) &= f(0) = [0, 3], \\ \frac{1}{v-u} \int_u^v f(t) dt &= \frac{1}{2} \left[\int_{-1}^1 t^2 dt, \int_{-1}^1 (6 - e^t) dt \right] = \left[\frac{1}{3}, 4 - \frac{e - e^{-1}}{2} \right], \\ [f(u) + f(v)] \int_0^1 H(x, 1-x) dx &= \left[1, 4 - \frac{e + e^{-1}}{2} \right]. \end{aligned}$$

Then, we obtain that

$$[0, 3] \supseteq \left[\frac{1}{3}, 4 - \frac{e - e^{-1}}{2} \right] \supseteq \left[\frac{1}{3}, 4 - \frac{e + e^{-1}}{2} \right].$$

Consequently, Theorem 2 is verified.

The next result generalizes Theorem 3.1 of [35] and Theorem 4.3 of [17].

Theorem 3. Let $f : [u, v] \rightarrow \mathbb{R}_T^+$, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ and $H\left(\frac{1}{2}, \frac{1}{2}\right) \neq 0$. If $f \in SX((h_1, h_2), [u, v], \mathbb{R}_T^+)$ and $f \in \mathcal{IR}_{([u, v])}$, then

$$\begin{aligned} \frac{1}{4H^2\left(\frac{1}{2}, \frac{1}{2}\right)} f\left(\frac{u+v}{2}\right) &\supseteq \Delta_1 \supseteq \frac{1}{v-u} \int_u^v f(t) dt \\ &\supseteq \Delta_2 \supseteq [f(u) + f(v)] \left[\frac{1}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) \right] \int_0^1 H(x, 1-x) dx, \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \frac{1}{4H\left(\frac{1}{2}, \frac{1}{2}\right)} \left[f\left(\frac{3u+v}{4}\right) + f\left(\frac{u+3v}{4}\right) \right], \\ \Delta_2 &= \left[\frac{f(u) + f(v)}{2} + f\left(\frac{u+v}{2}\right) \right] \int_0^1 H(x, 1-x) dx. \end{aligned}$$

Proof. For $[u, \frac{u+v}{2}]$, one has

$$\begin{aligned} H\left(\frac{1}{2}, \frac{1}{2}\right) f\left(xu + (1-x)\frac{u+v}{2}\right) &+ H\left(\frac{1}{2}, \frac{1}{2}\right) f\left((1-x)u + x\frac{u+v}{2}\right) \\ &\subseteq f\left(\frac{xu + (1-x)\frac{u+v}{2}}{2} + \frac{(1-x)u + x\frac{u+v}{2}}{2}\right) = f\left(\frac{3u+v}{4}\right). \end{aligned}$$

Consequently, we get

$$\frac{1}{4H\left(\frac{1}{2}, \frac{1}{2}\right)} f\left(\frac{3u+v}{4}\right) \supseteq \frac{1}{v-u} \int_u^{\frac{u+v}{2}} f(t) dt.$$

In the same way as above, for $[\frac{u+v}{2}, v]$, we have

$$\frac{1}{4H\left(\frac{1}{2}, \frac{1}{2}\right)} f\left(\frac{u+3v}{4}\right) \supseteq \frac{1}{v-u} \int_{\frac{u+v}{2}}^v f(t) dt.$$

Hence,

$$\Delta_1 = \frac{1}{4H(\frac{1}{2}, \frac{1}{2})} \left[f\left(\frac{3u+v}{4}\right) + f\left(\frac{u+3v}{4}\right) \right] \supseteq \frac{1}{v-u} \int_u^v f(t)dt.$$

Thanks to Theorem 2, one has

$$\begin{aligned} & \frac{1}{4\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2} f\left(\frac{u+v}{2}\right) \\ &= \frac{1}{4\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2} f\left(\frac{1}{2} \cdot \frac{3u+v}{4} + \frac{1}{2} \cdot \frac{u+3v}{4}\right) \\ &\supseteq \frac{1}{4\left[H\left(\frac{1}{2}, \frac{1}{2}\right)\right]^2} \left[H\left(\frac{1}{2}, \frac{1}{2}\right) f\left(\frac{3u+v}{4}\right) + H\left(\frac{1}{2}, \frac{1}{2}\right) f\left(\frac{u+3v}{4}\right) \right] \\ &\supseteq \Delta_1 \\ &\supseteq \frac{1}{v-u} \int_u^v f(t)dt \\ &\supseteq \frac{1}{2} \left[f(u) + f(v) + 2f\left(\frac{u+v}{2}\right) \right] \int_0^1 H(x, 1-x)dx \\ &= \Delta_2 \\ &\supseteq \left[\frac{f(u) + f(v)}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) (f(u) + f(v)) \right] \int_0^1 H(x, 1-x)dx \\ &\supseteq [f(u) + f(v)] \left[\frac{1}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) \right] \int_0^1 H(x, 1-x)dx, \end{aligned}$$

and the result follows. □

Example 2. Furthermore, by Example 1, we have

$$\begin{aligned} \Delta_1 &= \frac{1}{2} \left[f\left(-\frac{1}{2}\right), f\left(\frac{1}{2}\right) \right] = \left[\frac{1}{4}, 4 - \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2} \right], \\ \Delta_2 &= \frac{1}{2} \left(\left[1, 4 - \frac{e + e^{-1}}{2} \right] + [0, 3] \right) = \left[\frac{1}{2}, \frac{7}{2} - \frac{e + e^{-1}}{4} \right], \\ & [f(u) + f(v)] \left[\frac{1}{2} + H\left(\frac{1}{2}, \frac{1}{2}\right) \right] \int_0^1 H(x, 1-x)dx = \left[1, 4 - \frac{e + e^{-1}}{2} \right]. \end{aligned}$$

Then, we obtain that

$$[0, 3] \supseteq \left[\frac{1}{4}, 4 - \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2} \right] \supseteq \left[\frac{1}{3}, 4 - \frac{e - e^{-1}}{2} \right] \supseteq \left[\frac{1}{2}, \frac{7}{2} - \frac{e + e^{-1}}{4} \right] \supseteq \left[1, 4 - \frac{e + e^{-1}}{2} \right].$$

Consequently, Theorem 3 is verified.

Similarly, we get the following result, which generalizes Theorem 3 of [34] and Theorem 4.5 of [17].

Theorem 4. Let $f, g : [u, v] \rightarrow \mathbb{R}_I^\pm$, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$ and $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. If $f, g \in SX((h_1, h_2), [u, v], \mathbb{R}_I^\pm)$ and $fg \in \mathcal{IR}_{([u, v])}$, then

$$\frac{1}{v-u} \int_u^v f(t)g(t)dt \supseteq M(u, v) \int_0^1 H^2(x, 1-x)dx + N(u, v) \int_0^1 H(x, x)H(1-x, 1-x)dx,$$

where

$$M(u, v) = f(u)g(u) + f(v)g(v), N(u, v) = f(u)g(v) + f(v)g(u).$$

Example 3. Suppose that $h_1(x) = x, h_2(x) \equiv 1, [u, v] = [0, 1]$ and

$$f(t) = [t^2, 4 - e^t], g(t) = [t, 3 - t^2].$$

Then

$$\begin{aligned} \frac{1}{v-u} \int_u^v f(t)g(t)dt &= \int_0^1 [t^3, (4 - e^t)(3 - t^2)]dt = \left[\frac{1}{4}, \frac{35}{3} - 2e\right], \\ M(u, v) \int_0^1 H^2(x, 1-x)dx &= M(0, 1) \int_0^1 x^2dx = \left[\frac{1}{3}, \frac{17}{3} - \frac{2}{3}e\right], \\ N(u, v) \int_0^1 H(x, x)H(1-x, 1-x)dx &= N(0, 1) \int_0^1 x^2dx = \left[0, 3 - \frac{e}{2}\right]. \end{aligned}$$

It follows that

$$\left[\frac{1}{4}, \frac{35}{3} - 2e\right] \supseteq \left[\frac{1}{3}, \frac{17}{3} - \frac{2}{3}e\right] + \left[0, 3 - \frac{e}{2}\right] = \left[\frac{1}{3}, \frac{26}{3} - \frac{7}{6}e\right].$$

Consequently, Theorem 4 is verified.

The next result generalizes Theorem 2 of [34] and Theorem 4.6 of [17].

Theorem 5. Let $f, g : [u, v] \rightarrow \mathbb{R}_T^+$, $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^+$, and $H(\frac{1}{2}, \frac{1}{2}) \neq 0$. If $f, g \in SX((h_1, h_2), [u, v], \mathbb{R}_T^+)$ and $fg \in \mathcal{IR}_{([u, v])}$, then

$$\begin{aligned} \frac{1}{2H^2(\frac{1}{2}, \frac{1}{2})} f\left(\frac{u+v}{2}\right)g\left(\frac{u+v}{2}\right) &\supseteq \frac{1}{v-u} \int_u^v f(t)g(t)dt + N(u, v) \int_0^1 H^2(x, 1-x)dx \\ &+ M(u, v) \int_0^1 H(x, x)H(1-x, 1-x)dx. \end{aligned}$$

Proof. By hypothesis, one has

$$\begin{aligned} &f\left(\frac{u+v}{2}\right)g\left(\frac{u+v}{2}\right) \\ &\supseteq H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[\underline{f}(xu + (1-x)v)\underline{g}(xu + (1-x)v), \bar{f}(xu + (1-x)v)\bar{g}(xu + (1-x)v) \right] \\ &+ H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[\underline{f}(xu + (1-x)v)\underline{g}((1-x)u + xv), \bar{f}(xu + (1-x)v)\bar{g}((1-x)u + xv) \right] \\ &+ H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[\underline{f}((1-x)u + xv)\underline{g}(xu + (1-x)v), \bar{f}((1-x)u + xv)\bar{g}(xu + (1-x)v) \right] \\ &+ H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[\underline{f}((1-x)u + xv)\underline{g}((1-x)u + xv), \bar{f}((1-x)u + xv)\bar{g}((1-x)u + xv) \right] \\ &\supseteq H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[f(xu + (1-x)v)g(xu + (1-x)v) + f((1-x)u + xv)g((1-x)u + xv) \right] \\ &+ H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[(H(x, 1-x)f(u) + H(1-x, x)f(v))(H(1-x, x)g(u) + H(x, 1-x)g(v)) \right. \\ &\quad \left. + (H(1-x, x)f(u) + H(x, 1-x)f(v))(H(x, 1-x)g(u) + H(1-x, x)g(v)) \right] \\ &= H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[f(xu + (1-x)v)g(xu + (1-x)v) + f((1-x)u + xv)g((1-x)u + xv) \right] \\ &+ 2H^2\left(\frac{1}{2}, \frac{1}{2}\right) \left[H(x, x)H(1-x, 1-x)M(u, v) + H^2(x, 1-x)N(u, v) \right] \end{aligned}$$

Integrating over $[0, 1]$, and the result follows. \square

Example 4. Furthermore, by Example 3, we get

$$\begin{aligned} \frac{1}{2H^2\left(\frac{1}{2}, \frac{1}{2}\right)} f\left(\frac{u+v}{2}\right) g\left(\frac{u+v}{2}\right) &= 2f\left(\frac{1}{2}\right)g\left(\frac{1}{2}\right) = \left[\frac{1}{4}, 22 - \frac{11}{2}\sqrt{e}\right], \\ N(u, v) \int_0^1 H^2(x, 1-x) dx &= N(0, 1) \int_0^1 x^2 dx = [0, 6 - e], \\ M(u, v) \int_0^1 H(x, x)H(1-x, 1-x) dx &= M(0, 1) \int_0^1 (x - x^2) dx = \left[\frac{1}{6}, \frac{17}{6} - \frac{e}{3}\right]. \end{aligned}$$

It follows that

$$\left[\frac{1}{4}, 22 - \frac{11}{2}\sqrt{e}\right] \supseteq [0, 6 - e] + \left[\frac{1}{6}, \frac{17}{6} - \frac{e}{3}\right] + \left[\frac{1}{4}, \frac{35}{3} - 2e\right] = \left[\frac{5}{12}, \frac{123}{6} - \frac{10}{3}e\right].$$

Consequently, Theorem 5 is verified.

4. Conclusions

We introduced interval (h_1, h_2) -convex and presented some new interval Hermite-Hadamard type inequalities. Our results generalize some known Hermite-Hadamard type inequalities and will be useful in developing the theory of interval differential (or integral) inequalities and interval convex analysis. As a future research direction, we intend to investigate inequalities for fuzzy-interval-valued functions, and some applications in interval nonlinear programming.

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