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Generalized Mittag–Leffler Stability of Hilfer Fractional Order Nonlinear Dynamic System

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Abstract: This article studies the generalized Mittag–Leffler stability of Hilfer fractional nonautonomous system by using the Lyapunov direct method. A new Hilfer type fractional comparison principle is also proved. The novelty of this article is the fractional Lyapunov direct method combined with the Hilfer type fractional comparison principle. Finally, our main results are explained by some examples.

Keywords: generalized Mittag–Leffler stability; Hilfer fractional nonautonomous system; Hilfer type fractional comparison principle; fractional Lyapunov direct method

1. Introduction

Fractional calculus, as one of the more powerful tools to deal with complex phenomena, is getting more and more attention. Moreover, it has been applied in various areas such as control theory, cosmology, economic, physics, etc. For details, readers refer to the works in [1–8]. Recently, researchers have taken an increased interest in the development of the Hilfer fractional derivative that is defined in Definition 1. As stated in [9–11], Hilfer fractional derivative contains classical fractional derivatives. For example, the Hilfer fractional derivative is consistent with the Riemann–Liouville or Caputo fractional derivative for $\beta = 0$ or $\beta = 1$, respectively. More specifics about the Hilfer fractional derivative can be found in [9–17].

Lately, fractional calculus has become more common in control problems. Different fractional order controllers are significant in almost every field of the control subject. Stability is one of the important properties of the control problem. Therefore several researchers have investigated the stability of fractional order systems. Up to now, it has made great strides [18–32]. The Lyapunov direct method (LDM) is one of the more important methods to analyze stability of fractional order systems.

For nonlinear systems, the solutions of nonlinear differential equations are often difficult to express. LDM [33–39] offers an excellent method to analyze the property of the solution without solving this differential equation. Since LDM can be used in any order system, it shows that this method has great superiority. This method directly infers the stability of the system through a Lyapunov function for the system. LDM is a sufficient condition for judging system stability. In other words, even if the Lyapunov function candidate is not found, the system may also be stable.

In 2009, Li et al. [20] investigated the Mittag–Leffler stability of fractional order nonlinear dynamic systems:

$$_{t_0}D_t^{\alpha} x(t) = f(t, x),$$

with initial condition $x(t_0)$, where D denotes either the Caputo or Riemann–Liouville fractional operator, $\alpha \in (0,1)$, $f : [t_0, \infty] \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty] \times \Omega$, and $\Omega \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

In 2014, Aguila–Camacho et al. [25] considered the stability of fractional order nonlinear time-varying systems:

$$ {}_{t_0}^{c}D_t^{\alpha} x(t) = f(x, t), $$

where $\alpha \in (0,1)$, ${}_{t_0}^{c}D_t^{\alpha}$ denotes the Caputo fractional derivative and t represents the time.

In 2017, Yang et al. [28] investigated the Mittag–Leffler stability of nonlinear fractional-order systems with impulses

$$ \begin{cases} D_{0,t}^{\alpha} u(t) = Au(t) + g(t, u(t)), & t \neq t_k, \\ \Delta u(t_k) = u(t_k^+) - u(t_k^-) = J_k(u(t_k)), & t = t_k, \quad k \in \mathbb{Z}_+, \end{cases} $$

where $D_{0,t}^{\alpha}$ denotes the Caputo fractional derivative of order α, $(0 < \alpha < 1)$, $u(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is a constant matrix, $g(t, u(t)) \in \mathbb{R}^n$ is the nonlinear term with $g(t, 0) = 0$, $J_k(\cdot)$ standing for the jump operator of impulsive, and the impulsive moments satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ with $lim_{k \to +\infty} t_k = \infty$.

To the best of our knowledge, while some research has been carried on the stability of the Riemann–Liouville or Caputo fractional order systems, no single study exists which has investigated the stability of the Hilfer fractional order system by using LDM. In this context, the dual index of the Hilfer fractional derivative is complex but fascinating. To overcome the difficulty of proving the stability of the given system, we developed a new Hilfer type fractional comparison principle, which plays a vital role in this article. This paper has three key aims. Firstly, the generalized Mittag–Leffler (G-M-L) stability is proposed. Then a new Hilfer type fractional comparison principle is proved. Finally, the analysis of fractional LDM is studied.

2. Preliminaries

In this section, we give some definitions and related lemmas.

Definition 1 ([9]). *The Hilfer fractional derivative of order α and type β for a function g is defined as*

$$ ({}_0D_t^{\alpha,\beta} g)(t) = ({}_0I_t^{\beta(1-\alpha)} \frac{d}{dt} {}_0I_t^{(1-\beta)(1-\alpha)} g)(t), \ \ t > 0, 0 < \alpha < 1, 0 \le \beta \le 1, $$

where ${}_0I_t^{(\cdot)}$ denotes Riemann-Liouville fractional integral.

Lemma 1 ([40]). *Let $0 < \alpha < 1$, then*

$$ {}_0I_t^{\alpha} {}_0D_t^{\alpha} g(x) = g(x) - \frac{{}_0I_t^{1-\alpha} g(0)}{\Gamma(\alpha)} x^{\alpha - 1}. $$

Lemma 2 ([14]). *Let $0 < \alpha < 1$, $0 \le \beta \le 1$, and $\gamma = \alpha + \beta - \alpha\beta$, then*

$$ {}_0I_t^{\gamma} {}_0D_t^{\gamma} g(x) = {}_0I_t^{\alpha} {}_0D_t^{\alpha,\beta} g(x). $$

Remark 1 ([9]). *The Laplace transform of Hilfer fractional derivative is*

$$ \mathcal{L}[{}_0D_t^{\alpha,\beta} f(x)](s) = s^{\alpha} \mathcal{L}[f(x)](s) - s^{\beta(1-\alpha)} ({}_0I_t^{(1-\beta)(1-\alpha)} f)(0^+), \ \ 0 < \alpha < 1. $$

Definition 2 ([4]). *The one-parameter and two-parameter Mittag–Leffler functions are defined by respectively*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0$, $\beta > 0$, and $z \in \mathbb{C}$.

Consider the stability of the Hilfer fractional nonautonomous system

$$ {}_{t_0}D_t^{\alpha,\beta} x(t) = g(t, x), \tag{1}$$

with fractional integral type initial condition ${}_{t_0}I_t^{(1-\beta)(1-\alpha)} x(t_0) = C$, where $C \geq 0$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, $g : [t_0, \infty] \times U \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0, \infty] \times U$, and $0 \in U \subset \mathbb{R}^n$.

Definition 3. *The equilibrium point of ${}_{t_0}D_t^{\alpha,\beta} x(t) = g(t, x)$ is a constant x_0, iff $g(t, x_0) = 0$.*

Definition 4. *(G-M-L Stability) The solution of (1) is called G-M-L stable if*

$$\|x(t)\| \leq \{k[{}_{t_0}I_t^{1-\gamma} x(t_0)](t - t_0)^{-\gamma} E_{\alpha,1-\gamma}(-\mu(t - t_0)^\alpha)\}^c, \tag{2}$$

where $\alpha \in (0, 1)$, $\beta \in [0, 1]$, t_0 is initial time, $\mu \geq 0$, $-\alpha < \gamma < 1 - \alpha$, $c > 0$, $k(0) = 0$, and $k(x)$ is locally Lipschitz on $x \in \mathbb{R}^n$ with Lipschitz constant k_0.

Remark 2. *G-M-L Stability implies asymptotic stability.*

Definition 5 ([41]). *ω is called a K-class function, if $\omega(0) = 0$, and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is strictly increasing.*

Now, we develop a new Hilfer type fractional comparison principle, which plays a vital role in the proof of our main theorems.

Lemma 3. *(Hilfer Type Fractional Comparison Principle) Let ${}_0D_t^{\alpha,\beta} m(t) \geq {}_0D_t^{\alpha,\beta} n(t)$ and $m(0) = n(0)$, where $\alpha \in (0, 1)$, $\beta \in [0, 1]$. Then $m(t) \geq n(t)$.*

Proof. According to ${}_0D_t^{\alpha,\beta} m(t) \geq {}_0D_t^{\alpha,\beta} n(t)$, there exists a non-negative function $f(t)$ such that

$$ {}_0D_t^{\alpha,\beta} m(t) = f(t) + {}_0D_t^{\alpha,\beta} n(t). \tag{3}$$

Applying Remark 1 to (3), we obtain

$$s^\alpha M(s) - s^{\beta(1-\alpha)}[{}_0I^{(1-\beta)(1-\alpha)} m(0^+)] = F(s) + s^\alpha N(s) - s^{\beta(1-\alpha)}[{}_0I^{(1-\beta)(1-\alpha)} n(0^+)]. \tag{4}$$

From $m(0) = n(0)$, we get

$$M(s) = s^{-\alpha} F(s) + N(s). \tag{5}$$

Taking the inverse Laplace transform of (5), we have

$$m(t) \ = \ {}_0D_t^{-\alpha} f(t) + n(t). \tag{6}$$

Since $f(t) \geq 0$, we have ${}_0D_t^{-\alpha} f(t) \geq 0$, which implies that

$$m(t) \geq n(t).$$

□

3. Main Theory

In this section, let us firstly give a simple introduction to the LDM. If one can seek out a Lyapunov function for the given system, then the system is stable. Note that LDM is a sufficient condition for judging system stability. In other words, when the Lyapunov function is not found, the system may also be stable, so we cannot conclude that the system is unstable. In this article, we get G-M-L stability of the Hilfer fractional nonautonomous system by using the LDM. What is more, we apply a new Hilfer type fractional comparison principle and class-K functions to investigate the fractional LDM, which is a completely new attempt of stability analysis of the Hilfer fractional dynamic system.

Theorem 1. *Let an equilibrium point of system (1) be $x = 0$ and $U \subset \mathbb{R}^n$ be a domain containing the origin. Let $W(t, x(t)) : [0, \infty) \times U \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x satisfying*

$$a_1 \|x\|^m \leq W(t, x(t)) \leq a_2 \|x\|^{mc}, \tag{7}$$

$${}_0D_t^{\alpha,\beta} W(t, x(t)) \leq -a_3 \|x\|^{mc}, \tag{8}$$

where $t \geq 0$, $x \in U$, $\alpha \in (0, 1)$, $\beta \in [0, 1]$, a_1, a_2, a_3, m and c are arbitrary positive constants. Then the equilibrium point $x = 0$ is G-M-L stable.

Proof. According to (7) and (8), we have

$${}_0D_t^{\alpha,\beta} W(t, x(t)) \leq -\frac{a_3}{a_2} W(t, x(t)). \tag{9}$$

There is a function $Y(t) \geq 0$ such that

$${}_0D_t^{\alpha,\beta} W(t, x(t)) + Y(t) = -\frac{a_3}{a_2} W(t, x(t)). \tag{10}$$

From Lemmas 1 and 2, we have

$$W(t) = \frac{C}{\Gamma(\gamma)} t^{\gamma-1} - \frac{a_3}{a_2} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} W(s) ds - {}_0I_t^{\alpha} Y(t), \ \ \gamma = \alpha + \beta - \alpha\beta, \tag{11}$$

where $W(t) = W(t, x(t))$, ${}_0I_t^{(1-\alpha)(1-\beta)} W(0) = C \geq 0$.

Then we apply the method of successive approximations to solve Equation (11), that is,

$$W(t) = Ct^{\gamma-1} E_{\alpha,\gamma}(-\frac{a_3}{a_2} t^{\alpha}) - Y(t) * [t^{\alpha-1} E_{\alpha,\alpha}(-\frac{a_3}{a_2} t^{\alpha})]. \tag{12}$$

If $x(0) = 0$, then ${}_0I_t^{(1-\alpha)(1-\beta)} W(0) = 0$. Thus the solution to (1) is $x = 0$.

If $x(0) \neq 0$, ${}_0I_t^{(1-\alpha)(1-\beta)}W(0) > 0$. Because $t^{\alpha-1}$ and $E_{\alpha,\alpha}(-\frac{a_3}{a_2}t^{\alpha})$ [21] are non-negative functions for $0 < \alpha < 1$, we obtain

$$W(t) \leq Ct^{\gamma-1}E_{\alpha,\gamma}(-\frac{a_3}{a_2}t^{\alpha}). \tag{13}$$

Taking (13) in (7), we can get

$$\|x(t)\| \leq [\frac{C}{a_1}t^{\gamma-1}E_{\alpha,\gamma}(-\frac{a_3}{a_2}t^{\alpha})]^{\frac{1}{m}}, \tag{14}$$

where $\frac{C}{a_1} > 0$ for $x(0) \neq 0$.

Let $h = \frac{C}{a_1} = \frac{{}_0I_t^{1-\gamma}W(0)}{a_1} = \frac{{}_0I_t^{1-\gamma}W(0,x(0))}{a_1} \geq 0$, then we get

$$\|x(t)\| \leq [ht^{\gamma-1}E_{\alpha,\gamma}(-\frac{a_3}{a_2}t^{\alpha})]^{\frac{1}{m}}, \tag{15}$$

where $h = 0$ holds iff $x(0) = 0$. Since $W(t,x)$ is locally Lipschitz in x and $V(0,x(0)) = 0$ iff $x(0) = 0$, we get that $h = \frac{{}_0I_t^{1-\gamma}W(0,x(0))}{a_1}$ is Lipschitz with respect to $x(0)$ and $h(0) = 0$ as well, which shows the G-M-L stability of (1). □

Theorem 2. *Let an equilibrium point of Hilfer nonautonomous fractional order system (1) be $x = 0$. Suppose that there exists a Lyapunov function $V(t,x(t))$ and class-K functions ω_i $(i = 1,2,3)$ such that*

$$\omega_1(\|x\|) \leq V(t,x(t)) \leq \omega_2(\|x\|) \tag{16}$$

and

$${}_0D_t^{\alpha,\beta}V(t,x(t)) \leq -\omega_3(\|x\|), \tag{17}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$. Then the system (1) is asymptotically stable.

Proof. According to (16) and (17), we have

$${}_0D_t^{\alpha,\beta}V(t,x(t)) \leq -\omega_3(\omega_2^{-1}(V(t,x(t)))). \tag{18}$$

Because $V(t,x(t))$ is bounded by the unique non-negative solution of the scalar differential equation

$${}_0D_t^{\alpha,\beta}f(t) = -\omega_3(\omega_2^{-1}(f(t))), \quad f(0) = V(0,x(0)). \tag{19}$$

It follows from Definition 3 that $f(t) = 0$ for $t \geq 0$ if $f(0) = 0$, since $\omega_3\omega_2^{-1}$ is a class-K function.
If not, $f(t) \geq 0$ on $t \in [0,+\infty)$, and in view of (19), we have ${}_0D_t^{\alpha,\beta}f(t) \leq 0$.
From Lemma 3, we have

$$f(t) \leq f(0), \quad t \in (0,+\infty). \tag{20}$$

Then we can get the asymptotic stability of (19) by contradiction.
Case 1: Assume that there is a constant $t_1 \geq 0$ such that

$${}_0D_{t_1}^{\alpha,\beta}f(t) = -\omega_3(\omega_2^{-1}(f(t_1))) = 0, \tag{21}$$

which means that

$${}_0D_t^{\alpha,\beta}f(t) = {}_{t_1}D_t^{\alpha,\beta}f(t) = -\omega_3(\omega_2^{-1}(f(t))) \tag{22}$$

for any $t \geq t_1$. According to Definition 3, the equilibrium point of ${}_{t_1}D_t^{\alpha,\beta} f(t) = -\omega_3(\omega_2^{-1}(f(t)))$ is $x = 0$. Then $f(t) = 0$ for $t \geq t_1$ if $f(t_1) = 0$.

Case 2: Suppose there is $\epsilon > 0$ satisfying $f(t) \geq \epsilon$ for $t \geq 0$. From (20), we have

$$0 < \epsilon \leq f(t) \leq f(0), \quad t \geq 0. \tag{23}$$

Taking (23) in (19), we obtain

$$\begin{aligned} -\omega_3(\omega_2^{-1}(f(t))) &\leq -\omega_3(\omega_2^{-1}(\epsilon)) \\ &= -\frac{\omega_3(\omega_2^{-1}(\epsilon))}{f(0)} f(0) \\ &\leq -hf(t), \end{aligned} \tag{24}$$

where $0 < h = \frac{\omega_3(\omega_2^{-1}(\epsilon))}{f(0)}$, we have

$${}_0D_t^{\alpha,\beta} f(t) = -\omega_3(\omega_2^{-1}(f(t))) \leq -hf(t). \tag{25}$$

By using Theorem 1, we obtain

$$f(t) \leq Ct^{\gamma-1} E_{\alpha,\gamma}(-ht^{\alpha}), \tag{26}$$

which contradicts the assumption $f(t) \geq \epsilon$.

On the basis of the discussions in **Case 1** and **Case 2**, we get $lim_{t\to\infty} f(t) = 0$. Then from (16) and V(t,x(t)) is bounded by $f(t)$, we obtain $lim_{t\to\infty} x(t) = 0$. □

Remark 3. *When $\beta = 0$ or $\beta = 1$, the stability of ${}_{t_0}D_t^{\alpha,\beta} x(t) = g(t,x)$ has been proved by Li, Chen and Podlubny [21]. Our main Theorems generalize and improve Theorems 5.1 and 6.2 of literature [21].*

4. Examples

Example 1. *For the system*

$${}_0D_t^{\alpha,\beta} y(t) = g(y), \tag{27}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$, $y(0) = y_0$, the equilibrium point of (27) is $y = 0$, $\|y\|_2 \leq \tilde{h}\|g(y)\|_2$, where $\tilde{h} > 0$, $\|\cdot\|_2$ denotes the 2-norm, and $g(y)\frac{dg(y)}{dy}\dot{y} \leq 0$. Hence the equilibrium point $y = 0$ is stable.

Proof. Let the Lyapunov candidate be $V(y) = g^2(y)$, because $g(y)\frac{dg(y)}{dy}\dot{y} \leq 0$, we have

$$\frac{dV}{dt} = \frac{dV}{dy}\dot{y}(t) = 2g(y)\frac{dg(y)}{dy}\dot{y} \leq 0. \tag{28}$$

Since $\|y\|_2 \leq \tilde{h}\|g(y)\|_2$ and the equilibrium point is $y = 0$, we have $\|y\|_2^2 \leq \tilde{h}^2\|g(y)\|_2^2 \leq \tilde{h}^2 V(y_0)$. Consequently, $y = 0$ is stable. □

Example 2. *For the Hilfer fractional order system*

$${}_0D_t^{\alpha,\beta} x(t) = -x^3(t), \tag{29}$$

where $\alpha \in (0,1)$, $\beta \in [0,1]$ and $x(0) \geq 0$ is the initial condition. The equilibrium point $x = 0$ is asymptotically stable.

Proof. Let the Lyapunov candidate be $V(x) = x^4$, we obtain $\dot{V}(X(t)) = 4x^3(t)\dot{x}(t)$, where $\dot{x}$ denotes the derivative of x with respect to t.

By Lemmas 1 and 2, we have

$$ {}_0I_t^{\alpha}{}_0D_t^{\alpha,\beta}x(t) = {}_0I_t^{\gamma}{}_0D_t^{\gamma}x(t) = x(t) - \frac{{}_0I_t^{1-\gamma}x(0)}{\Gamma(\gamma)}t^{\gamma-1} = -{}_0I_t^{\alpha}x^3(t), \ \ \gamma = \alpha + \beta - \alpha\beta, $$

where ${}_0I_t^{1-\gamma}x(0) = C$, then

$$ x(t) = \frac{C}{\Gamma(\gamma)}t^{\gamma-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x^3(s)ds, $$

and

$$ \dot{x}(t) = \frac{C(\gamma-1)}{\Gamma(\gamma)}t^{\gamma-2} - \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t (t-s)^{\alpha-1}x^3(s)ds. $$

It follows from $0 < \gamma < 1$ and the proof of Example 14 [20] that we have $x(0)\dot{x}(t) < 0$ and $V(x(t)) = x^4(t)$ is a decreasing function.

Suppose there exists a positive constant ξ satisfying $x(0)x(t) \geq \xi$ for all $t \geq 0$, we have

$$ \begin{aligned} {}_0D_t^{\alpha,\beta}V &= {}_0I_t^{\beta(1-\alpha)}\frac{d}{dt}{}_0I_t^{(1-\beta)(1-\alpha)}V = {}_0I_t^{\beta(1-\alpha)}\frac{d}{dt}{}_0D_t^{-(1-\beta)(1-\alpha)-1}4x^3\dot{x} \\ &\leq \frac{4\xi^3}{x^4(0)}{}_0I_t^{\beta(1-\alpha)}\frac{d}{dt}{}_0D_t^{-(1-\beta)(1-\alpha)-1}x(0)\dot{x} \\ &\leq \frac{4\xi^3}{x^4(0)}{}_0I_t^{\beta(1-\alpha)}\frac{d}{dt}{}_0D_t^{-(1-\beta)(1-\alpha)-1}\Big[\frac{x(0)C(\gamma-1)}{\Gamma(\gamma)}t^{\gamma-2} - \frac{x^4(0)t^{\alpha-1}}{\Gamma(\alpha)}\Big] \\ &= -4\xi^3 = \frac{-4\xi^3}{x^4(0)}x^4(0) \leq -a_3V, \end{aligned} $$

where $a_3 = \frac{4\xi^3}{x^4(0)} > 0$. It follows from Theorem 1 that $lim_{t\to\infty}V(x(t)) = lim_{t\to\infty}x^4(t) = 0$ which contradicts the assumption $x(0)x(t) > \xi$. Therefore, the equilibrium point $x = 0$ is asymptotically stable. □

Remark 4. *When $\beta = 1$, the Hilfer fractional derivative becomes the Caputo fractional derivative. In this case, Example 2 is an extension of Example 14 [20].*

5. Conclusions

In this paper, we studied the generalized Mittag–Leffler stability of Hilfer fractional nonautonomous system by using the Lyapunov direct method. The definition of the generalized Mittag–Leffler stability and a new Hilfer type fractional comparison principle were proposed, which enriches the knowledge of the system theory. Since the Hilfer fractional derivative includes many classical fractional derivatives, our conclusions can also be widely applied to many fractional order systems.

At present, research on the Caputo–Fabrizio fractional differential equations, which is a new research engine in the field of fractional calculus, is becoming more and more active. For its new development, see [42–46]. As an extension of our conclusion, we present an open question, namely how to develop the stability of the Caputo–Fabrizio fractional nonautonomous system by using the Lyapunov direct method. The biggest difficulty for this is to perfectly establish a new Caputo–Fabrizio type fractional comparison principle.

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