

Article

Some New Oscillation Criteria for Second Order Neutral Differential Equations with Delayed Arguments

Omar Bazighifan ^{1,†}  and Clemente Cesarano ^{2,*,†} ¹ Department of Mathematics, Faculty of Science, Hadhramout University, Hadhramout 50512, Yemen² Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy

* Correspondence: c.cesarano@uninettunouniversity.net

† These authors contributed equally to this work.

Received: 10 June 2019; Accepted: 10 July 2019; Published: 11 July 2019



Abstract: In this paper, we study the oscillation of second-order neutral differential equations with delayed arguments. Some new oscillatory criteria are obtained by a Riccati transformation. To illustrate the importance of the results, one example is also given.

Keywords: oscillatory solutions; nonoscillatory solutions; second-order; neutral differential equations

1. Introduction

The main focus of this study was the oscillation criteria of the solution of second-order delay differential equations of the form

$$\left[a(\ell) (z'(\ell))^\beta \right]' + p(\ell) f(z(\tau(\ell))) = 0, \quad \ell \geq \ell_0, \quad (1)$$

where $z(\ell) = x(\ell) + q(\ell)x(\sigma(\ell))$ and β is a quotient of odd positive integers. Throughout the paper, we always assume that:

(H₁) $a \in C^1([\ell_0, \infty), \mathbb{R}^+)$, $a'(\ell) \geq 0$, $a(\ell) > 0$, $p, q \in C([\ell_0, \infty), [0, \infty))$, $0 \leq p(t) < 1$, $\sigma, \tau \in C([\ell_0, \infty), \mathbb{R})$, $\tau(\ell) \leq \ell$, $\sigma(\ell) \leq \ell$, $\lim_{\ell \rightarrow \infty} \tau(\ell) = \infty$ and $\lim_{\ell \rightarrow \infty} \sigma(\ell) = \infty$,

(H₂) $f \in C(\mathbb{R}, \mathbb{R})$, $f(u)/u^\beta \geq k > 0$, for $u \neq 0$.

By a solution of (1), we mean a function $z \in C([\ell_0, \infty), \mathbb{R})$, $\ell_z \geq \ell_0$, which has the property $a(\ell) [z'(\ell)]^\beta \in C([\ell_0, \infty), \mathbb{R})$, and satisfies (1) on $[\ell_z, \infty)$. We consider only those solutions z of (1) which satisfy $\sup\{|z(\ell)| : \ell \geq \ell_z\} > 0$, for all $\ell > \ell_z$. We assume that (1) possesses such a solution. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[\ell_z, \infty)$, otherwise it is called non-oscillatory. (1) is said to be oscillatory if all its solutions are oscillatory. Likewise, the equation itself is called oscillatory if all of its solutions are oscillatory.

Differential equations are of great importance in the branches of mathematics and other sciences. In 1918, researchers were interested in studying differential equations. Since then, there has been much research on the subject of the oscillation of differential and functional differential equations—see [1–10].

The differential equation in which the highest-order derivative of the unknown function appears both with and without delay is called a neutral delay differential equation. In past years, researchers have been interested in the oscillation of neutral differential equations—see [11–14].

Many authors have discussed the oscillations of second-order differential equations, and have also proposed several ways to achieve oscillation for these equations. For treatments on this subject,

we refer the reader to the texts, [15–21]. Here are some of the results that served as motivation for this work.

Cesarano and Bazighifan [22] discussed the equation

$$[a(\ell) w'(\ell)]' + q(\ell) f(w(\tau(\ell))) = 0 \tag{2}$$

and used the classical Riccati transformation technique.

Moaaz and Bazighifan [23] considered the oscillatory properties of second-order delay differential equations

$$[a(\ell) (w'(\ell))^\beta]' + p(\ell) f(w(\tau(\ell))) = 0,$$

under the condition

$$\int_{\ell_0}^{\infty} \frac{1}{a^{\frac{1}{\beta}}(\ell)} d\ell < \infty$$

and he proved it was oscillatory if

$$kq(\ell) \frac{\tau^2(\ell)}{\ell^2} > M.$$

Grace et al. [24] studied the differential equations

$$[a(\ell) (z'(\ell))^\beta]' + p(\ell) z^\beta(\tau(\ell)) = 0, \quad \ell \geq \ell_0,$$

under the conditions

$$\int_{\ell_0}^{\infty} \frac{1}{a^{\frac{1}{\beta}}(\ell)} d\ell = \infty.$$

Trench [25] used the comparison technique for the following

$$[a(\ell) w'(\ell)]' + q(\ell) w(\tau(\ell)) = 0,$$

which they compared with the first-order differential equation, and on the condition

$$\int_{\ell_0}^{\infty} \frac{1}{a(\ell)} dz = \infty.$$

In this paper we used the Riccati transformation technique, which differs from those reported in [26] to establish some conditions for the oscillation of (1) under the condition

$$\int_{\ell_0}^{\infty} \frac{1}{a^{\frac{1}{\beta}}(s)} ds < \infty. \tag{3}$$

An example is presented to illustrate our main results.

We begin with the following lemma.

Lemma 1 (See [1], Lemma 2.1). *Let $\beta \geq 1$ be a ratio of two numbers, $G, H, U, V \in \mathbb{R}$. Then,*

$$G^{\frac{\beta+1}{\beta}} - (G - H)^{\frac{\beta+1}{\beta}} \leq \frac{H^{\frac{1}{\beta}}}{\beta} [(1 + \beta)G - H], \quad GH \geq 0,$$

and

$$Uy - Vy^{\frac{\beta+1}{\beta}} \leq \frac{\beta^\beta}{(\beta + 1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}, \quad V > 0.$$

2. Main Results

In this section, we state the oscillation criteria for (1). To facilitate this, we refer to the following:

$$B(\ell) := \int_{\ell}^{\infty} \frac{1}{a^{\frac{1}{\beta}}(s)} ds.$$

$$A(\ell) = kq(\ell) (1 - p(\tau(\ell)))^{\beta}.$$

$$D(\tau(\ell)) = B(\ell + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) B^{\beta}(\tau(s)) A(s) ds.$$

$$\check{D}(\ell) = \exp\left(-\beta \int_{\tau(\ell)}^{\ell} \frac{a(s)^{\frac{-1}{\beta}}}{D(s)} ds\right).$$

$$\Phi(\ell) = \delta(\ell) \left[A(\ell) - \frac{1 - \beta}{a^{\frac{1}{\beta}}(\ell) \eta^{\beta+1}(\ell)} \right].$$

$$\theta(\ell) = \frac{\delta'_+(\ell)}{\delta(\ell)} + \frac{(\beta + 1)}{a^{\frac{1}{\beta}}(\ell) \eta(\ell)}.$$

$$\rho'_+(\ell) := \max\{0, \rho'(\ell)\} \text{ and } \delta'_+(\ell) := \max\{0, \delta'(\ell)\}$$

Theorem 1. Assume that (3) holds. If there exist positive functions $\rho, \delta \in C^1([\ell_0, \infty), (0, \infty))$ such that

$$\int_{\ell_0}^{\infty} \left[\rho(s) A(s) \check{D}(s) - \frac{a(s) (\rho'_+(s))^{\beta+1}}{(\beta + 1)^{\beta+1} \rho^{\beta}(s)} \right] ds = \infty \tag{4}$$

and

$$\int_{\ell_0}^{\infty} \left(\Phi(s) - \frac{\delta(s) a(s) (\theta(s))^{\beta+1}}{(\beta + 1)^{\beta+1}} \right) ds = \infty, \tag{5}$$

then every solution of (1) is oscillatory.

Proof. Let x be a non-oscillatory solution of Equation (1), defined in the interval $[\ell_0, \infty)$. Without loss of generality, we may assume that $x(\ell) > 0$. It follows from (1) that there are two possible cases, for $\ell \geq \ell_1$, where $\ell_1 \geq \ell_0$ is sufficiently large:

$$(C_1) \quad z'(\ell) > 0, \left(a(\ell) (z'(\ell))^{\alpha} \right) < 0,$$

$$(C_2) \quad z'(\ell) < 0, \left(a(\ell) (z'(\ell))^{\alpha} \right) < 0.$$

Assume that Case (C₁) holds. Since $\tau(\ell) \leq t$ and $z'(\ell) > 0$, we get

$$\begin{aligned} x(\ell) &= z(\ell) - p(\ell)x(\tau(\ell)) \\ &\geq (1 - p(\ell))z(\ell), \end{aligned} \tag{6}$$

which, together with (1), implies that

$$\begin{aligned} \left[a(\ell) (z'(\ell))^{\beta} \right]' &\leq -kq(\ell) f(x(\tau(\ell))) \\ &\leq -kq(\ell) x^{\beta}(\tau(\ell)) \\ &\leq -kq(\ell) (1 - p(\tau(\ell)))^{\beta} z^{\beta}(\tau(\ell)) \\ &\leq -A(\ell) z^{\beta}(\tau(\ell)). \end{aligned} \tag{7}$$

On the other hand, we see

$$\begin{aligned} z(\ell) &= z(\ell_1) + \int_{\ell_1}^{\ell} \frac{1}{a(s)^{\frac{1}{\beta}}} a(s)^{\frac{1}{\beta}} z'(s) ds \\ &\geq B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell). \end{aligned} \tag{8}$$

Thus, we easily prove that

$$\left(z(\ell) - B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)' = -B(\ell) \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)'.$$

Applying the chain rule, it is easy to see that

$$B(\ell) \left(a(\ell) (z'(\ell))^{\beta} \right)' = \beta B(\ell) \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)^{\beta-1} \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)'.$$

By virtue of (7), the latter equality yields

$$-B(\ell) \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)' = \frac{1}{\beta} B(\ell) \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)^{1-\beta} A(\ell) z^{\beta}(\tau(\ell)).$$

Thus, we obtain

$$\left(z(\ell) - B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)' \geq \frac{1}{\beta} B(\ell) \left(a(\ell)^{\frac{1}{\beta}} z'(\ell) \right)^{1-\beta} A(\ell) z^{\beta}(\tau(\ell)).$$

Integrating from ℓ_1 to ℓ , we get

$$z(\ell) \geq B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell) + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) \left(a(s)^{\frac{1}{\beta}} z'(s) \right)^{1-\beta} A(s) z^{\beta}(\tau(s)) ds.$$

From (8), we get

$$\begin{aligned} z(\ell) &\geq B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell) \\ &\quad + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) \left(a(s)^{\frac{1}{\beta}} z'(s) \right)^{1-\beta} A(s) B^{\beta}(\tau(s)) a(\tau(s)) (z'(\tau(s)))^{\beta} ds, \end{aligned}$$

and this

$$\begin{aligned} z(\ell) &\geq B(\ell) a(\ell)^{\frac{1}{\beta}} z'(\ell) \\ &\quad + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) \left(a(s)^{\frac{1}{\beta}} z'(s) \right)^{1-\beta} A(s) B^{\beta}(s) a(s) (z'(s))^{\beta} ds, \\ &\geq a(\ell)^{\frac{1}{\beta}} z'(\ell) \left(B(\ell) + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) B^{\beta}(\tau(s)) A(s) ds \right). \end{aligned}$$

Thus, we conclude that

$$z(\tau(\ell)) \geq a(\tau(\ell))^{\frac{1}{\beta}} z'(\tau(\ell)) D(\tau(\ell)). \tag{9}$$

Define the function $\omega(\ell)$ by

$$\omega(\ell) := \rho(\ell) \frac{a(\ell) (z')^{\beta}(\ell)}{z^{\beta}(\ell)}, \tag{10}$$

then $\omega(\ell) > 0$ for $\ell \geq \ell_1$ and

$$\begin{aligned} \omega'(\ell) &= \rho'(\ell) \frac{a(\ell)(z')^\beta(\ell)}{z^\beta(\ell)} + \rho(\ell) \frac{(a(z')^\beta)'(\ell)}{z^\beta(\ell)} - \beta\rho(\ell) \frac{z^{\beta-1}(\ell)z'(\ell)a(\ell)(z')^\beta(\ell)}{z^{2\beta}(\ell)}, \quad (11) \\ &= \rho'(\ell) \frac{a(\ell)(z')^\beta(\ell)}{z^\beta(\ell)} + \rho(\ell) \frac{(a(z')^\beta)'(\ell)}{z^\beta(\ell)} - \beta\rho(\ell)a(\ell)\left(\frac{z'(\ell)}{z(\ell)}\right)^{\beta+1}. \end{aligned}$$

From (9), we obtain

$$z(\ell) \geq a(\ell)^{\frac{1}{\beta}} z'(\ell) D(\ell) \quad (12)$$

and this

$$\frac{z'(\ell)}{z(\ell)} \leq \frac{1}{a(\ell)^{\frac{1}{\beta}} D(\ell)}.$$

Integrating the latter inequality from $\tau(\ell)$ to ℓ , we get

$$\frac{z'(\tau(\ell))}{z(\ell)} \geq \exp\left(-\int_{\tau(\ell)}^{\ell} \frac{a(s)^{\frac{-1}{\beta}}}{D(s)} ds\right). \quad (13)$$

Combining (7) and (13), it follows that

$$\begin{aligned} \frac{[a(\ell)(z'''(\ell))^\beta]'(\ell)}{z^\beta(\ell)} &\leq -A(\ell) \left(\frac{z(\tau(\ell))}{z(\ell)}\right)^\beta \quad (14) \\ &\leq -A(\ell) \exp\left(-\beta \int_{\tau(\ell)}^{\ell} \frac{a(s)^{\frac{-1}{\beta}}}{D(s)} ds\right) \\ &\leq -A(\ell) \check{D}(\ell). \end{aligned}$$

By (10) and (11), we obtain that

$$\omega'(\ell) \leq \frac{\rho'_+(\ell)}{\rho(\ell)} \omega(\ell) - \rho(\ell) A(\ell) \check{D}(\ell) - \frac{\beta}{(\rho(\ell)a(\ell))^{\frac{1}{\beta}}} \omega^{\frac{\beta+1}{\beta}}(\ell).$$

Now, we let

$$G := \frac{\rho'_+(\ell)}{\rho(\ell)}, \quad H := \frac{\beta}{(\rho(\ell)a(\ell))^{\frac{1}{\beta}}}, \quad y := \omega(\ell).$$

Applying the Lemma 1, we find

$$\frac{\rho'_+(\ell)}{\rho(\ell)} \omega(\ell) - \frac{\beta}{(\rho(\ell)a(\ell))^{\frac{1}{\beta}}} \omega(\ell)^{\frac{\beta+1}{\beta}} \leq \frac{a(\ell)(\rho'_+(\ell))^{\beta+1}}{(\beta+1)^{\beta+1} \rho^\beta(\ell)}.$$

Hence, we obtain

$$\omega'(\ell) \leq -\rho(\ell) A(\ell) \check{D}(\ell) + \frac{a(\ell)(\rho'_+(\ell))^{\beta+1}}{(\beta+1)^{\beta+1} \rho^\beta(\ell)}. \quad (15)$$

Integrating from ℓ_1 to ℓ , we get

$$\int_{\ell_1}^{\ell} \left[\rho(s) A(s) \check{D}(s) - \frac{a(s)(\rho'_+(s))^{\beta+1}}{(\beta+1)^{\beta+1} \rho^\beta(s)} \right] ds \leq \omega(\ell_1), \quad (16)$$

which contradicts (4).

Assume that Case (C₂) holds. It follows from $(a(\ell)(z'(\ell))^\beta) \leq 0$, that we obtain

$$z'(s) \leq \left(\frac{a(\ell)}{a(s)}\right)^{1/\beta} z'(\ell).$$

Integrating from ℓ to ℓ_1 , we get

$$z(\ell_1) \leq z(\ell) + a^{1/\beta}(\ell) z'(\ell) \int_{\ell}^{\ell_1} a^{-1/\beta}(s) ds. \tag{17}$$

Letting $\ell_1 \rightarrow \infty$, we obtain

$$z(\ell) \geq -B(\ell)a(\ell)^{1/\beta} z'(\ell),$$

which implies that

$$\left(\frac{z(\ell)}{B(s)}\right)' \geq 0.$$

Define the function $\psi(t)$ by

$$\psi(t) := \delta(\ell) \left[\frac{a(\ell)(z'(\ell))^\beta}{z^\beta(\ell)} + \frac{1}{\eta^\beta(\ell)} \right], \tag{18}$$

then $\psi(t) > 0$ for $t \geq t_1$ and

$$\begin{aligned} \psi'(\ell) &= \delta'(\ell) \frac{a(\ell)(z'(\ell))^\beta}{z^\beta(\ell)} + \delta(\ell) \frac{(a(\ell)(z'(\ell))^\beta)'}{z^\beta(\ell)} \\ &\quad - \beta\delta(\ell)a(\ell) \frac{(z')^{\beta+1}(\ell)}{z^{\beta+1}(\ell)} + \frac{\alpha\delta(\ell)}{a^{1/\beta}(\ell)\eta^{\beta+1}(\ell)}. \end{aligned}$$

Using (18), we obtain

$$\begin{aligned} \psi'(\ell) &= \frac{\delta'_+(\ell)}{\delta(\ell)} \psi(\ell) + \delta(\ell) \frac{(a(\ell)(z'(\ell))^\beta)'}{z^\beta(\ell)} \\ &\quad - \beta\delta(\ell)a(\ell) \left[\frac{\psi(\ell)}{\delta(\ell)a(\ell)} - \frac{1}{a(\ell)\eta^\beta(\ell)} \right]^{\frac{\beta+1}{\beta}} + \frac{\beta\delta(\ell)}{a^{1/\beta}(\ell)\eta^{\beta+1}(\ell)}. \end{aligned} \tag{19}$$

Using Lemma 1 with $G = \frac{\psi(\ell)}{\delta(\ell)a(\ell)}$, $H = \frac{1}{a(\ell)\eta^\beta(\ell)}$, we get

$$\begin{aligned} \left[\frac{\psi(\ell)}{\delta(\ell)a(\ell)} - \frac{1}{a(\ell)\eta^\beta(\ell)} \right]^{\frac{\beta+1}{\beta}} &\geq \left(\frac{\psi(\ell)}{\delta(\ell)a(\ell)} \right)^{\frac{\beta+1}{\beta}} \\ &\quad - \frac{1}{\beta a^{1/\beta}(\ell)\eta(\ell)} \left((\beta+1) \frac{\psi(\ell)}{\delta(\ell)a(\ell)} - \frac{1}{a(\ell)\eta^\beta(\ell)} \right). \end{aligned} \tag{20}$$

From (7), (19), and (20), we obtain

$$\begin{aligned} \psi'(\ell) \leq & \frac{\delta'_+(\ell)}{\delta(\ell)} \psi(\ell) - \delta(\ell) A(\ell) - \beta \delta(\ell) a(\ell) \left(\frac{\psi(\ell)}{\delta(\ell) a(\ell)} \right)^{\frac{\beta+1}{\beta}} \\ & - \beta \delta(\ell) a(\ell) \left[\frac{-1}{\beta a^{\frac{1}{\beta}}(\ell) \eta(\ell)} \left((\beta+1) \frac{\psi(\ell)}{\delta(\ell) a(\ell)} - \frac{1}{a(\ell) \eta^\beta(\ell)} \right) \right]. \end{aligned}$$

This implies that

$$\begin{aligned} \psi'(\ell) \leq & \left(\frac{\delta'_+(\ell)}{\delta(\ell)} + \frac{(\beta+1)}{a^{\frac{1}{\beta}}(\ell) \eta(\ell)} \right) \psi(\ell) - \frac{\beta}{(\delta(\ell) a(\ell))^{\frac{1}{\beta}}} \psi^{\frac{\beta+1}{\beta}}(\ell) \\ & - \delta(\ell) \left[A(\ell) - \frac{1-\beta}{a^{\frac{1}{\beta}}(\ell) \eta^{\beta+1}(\ell)} \right]. \end{aligned} \tag{21}$$

Thus, by (19) yield

$$\psi'(\ell) \leq -\Phi(\ell) + \theta(\ell) \psi(\ell) - \frac{\beta}{(\delta(\ell) a(\ell))^{\frac{1}{\beta}}} \psi^{\frac{\beta+1}{\beta}}(\ell). \tag{22}$$

Applying the Lemma 1 with $U = \theta(\ell)$, $V = \frac{\beta}{(\delta(\ell) a(\ell))^{\frac{1}{\beta}}}$ and $y = \psi(\ell)$, we get

$$\psi'(\ell) \leq -\Phi(\ell) + \frac{\delta(\ell) a(\ell) (\theta(\ell))^{\beta+1}}{(\beta+1)^{\beta+1}}. \tag{23}$$

Integrating from ℓ_1 to ℓ , we get

$$\int_{\ell_1}^{\ell} \left(\Phi(s) - \frac{\delta(s) a(s) (\theta(s))^{\beta+1}}{(\beta+1)^{\beta+1}} \right) ds \leq \psi(\ell_1) - \psi(\ell) \leq \psi(\ell_1),$$

which contradicts (5). The proof is complete. \square

Example 1. As an illustrative example, we consider the following equation:

$$\left(\ell^2 \left(x(\ell) + \frac{1}{3} x\left(\frac{\ell}{2}\right) \right)' \right)' + x\left(\frac{\ell}{3}\right) = 0, \ell \geq 1. \tag{24}$$

Let

$$\beta = 1, a(\ell) = \ell^2, p(\ell) = \frac{1}{3}, q(\ell) = 1, \sigma(\ell) = \frac{\ell}{2}, \tau(\ell) = \frac{\ell}{3}.$$

If we now set $\delta(\ell) = \rho(\ell) = 1$ and $k = 1$, It is easy to see that all conditions of Theorem 1 are satisfied.

$$B(\ell) := \int_{\ell_0}^{\infty} \frac{1}{a^{1/\beta}(s)} ds = \frac{1}{\ell} < \infty.$$

$$A(\ell) = kq(\ell) (1 - p(\tau(\ell)))^\beta = \left(1 - \frac{1}{3} \right) = \frac{2}{3}.$$

$$D(\tau(\ell)) = B(\ell) + \frac{1}{\beta} \int_{\ell_1}^{\ell} B(s) B^\beta(\tau(s)) A(s) ds = \frac{1}{\ell} \int_{\ell_1}^{\ell} \frac{2}{3} \frac{1}{s\tau(s)}$$

$$\check{D}(\ell) = \exp\left(-\beta \int_{\tau(\ell)}^{\ell} \frac{a(s)^{\frac{-1}{\beta}}}{D(s)} ds\right)$$

$$\int_{\ell_0}^{\infty} \left[\rho(s) A(s) \check{D}(s) - \frac{a(s) (\rho'_+(s))^{\beta+1}}{(\beta+1)^{\beta+1} \rho^{\beta}(s)} \right] ds = \int_{\ell_0}^{\infty} \frac{2}{3} ds = \infty$$

and

$$\int_{\ell_0}^{\infty} \left(\Phi(s) - \frac{\delta(s) a(s) (\theta(s))^{\beta+1}}{(\beta+1)^{\beta+1}} \right) ds = \infty.$$

Hence, by Theorem 1, every solution of Equation (24) is oscillatory.

3. Conclusions

This article was interested in the oscillation criteria of the solution of second-order delay differential equations of (1). It has also been illustrated through an example that the results obtained are an improvement on the previous results. Our technique lies in using the generalized Riccati substitution, which differs from those reported in [26]. We offered some new sufficient conditions, which ensure that any solution of Equation (1) oscillates under the condition (3). Equation (1) is a neutral delay differential equation when $\tau(\ell) \leq \ell$, $\sigma(\ell) \leq \ell$. Furthermore, we could study $\tau(\ell) \geq \ell$, and be able to get the oscillation criteria of Equation (1) if $z(\ell) = x(\ell) - q(\ell)x(\sigma(\ell))$ in our future work.

Author Contributions: The authors claim to have contributed equally and significantly in this paper. All authors read and approved the final manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The authors thank the reviewers for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: There are no competing interests between the authors.

References

1. Agarwal, R.; Zhang, C.; Li, T. Some remarks on oscillation of second order neutral differential equations. *Appl. Math. Comput.* **2016**, *274*, 178–181. [[CrossRef](#)]
2. Liang, H.; Li, Q.; Zhang, Z. New oscillatory criteria for higher-order nonlinear neutral delay differential equation. *Non. Anal.* **2008**, *69*, 1719–1731. [[CrossRef](#)]
3. Koplatadze, R. Criteria for the oscillation of Solution of differential inequalities and second order equations with retarded argument. *Trudy Instituta Prikladnoj Matematiki Imeni IN Vekua* **1986**, *17*, 104–121.
4. Kiguradze, I.T.; Chanturiya, T.A. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*; Kluwer Academic Publisher: Dordrecht, The Netherlands, 1993.
5. Baculikova, B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. *Appl. Math. Lett.* **2018**, *11*, 68–75. [[CrossRef](#)]
6. Bazighifan, O.; Elabbasy, E.M.; Moaaz, O. Oscillation of higher-order differential equations with distributed delay. *J. Inequal. Appl.* **2019**, *2019*, 55. [[CrossRef](#)]
7. Bazighifan, O. On Oscillatory and Asymptotic Behavior of Fourth Order Nonlinear Neutral Delay Differential Equations. *Int. J. Modern Math. Sci.* **2019**, *17*, 21–30.
8. Assante, D.; Cesarano, C.; Foranaro, C.; Vazquez, L. Higher Order and Fractional Diffusive Equations. *J. Eng. Sci. Technol. Rev.* **2015**, *8*, 202–204. [[CrossRef](#)]
9. Cesarano, C.; Foranaro, C.; Vazquez, L. Operational results in bi-orthogonal Hermite functions. *Acta Math. Univ. Comen.* **2016**, *85*, 43–68.
10. Dattoli, G.; Ricci, P.; Cesarano, C. Beyond the monomiality: The monumbrality principle. *J. Comput. Anal. Appl.* **2004**, *6*, 77–83.
11. Gyori, I.; Ladas, G. *Oscillation Theory of Delay Differential Equations with Applications*; Clarendon Press: Oxford, UK, 1991.

12. Fite, W. Concerning the zeros of the solutions of certain differential equations. *Trans. Am. Math. Soc.* **1918**, *19*, 341–352. [[CrossRef](#)]
13. Ladde, G.; Lakshmikantham, V.; Zhang, B. *Oscillation Theory of Differential Equations with Deviating Arguments*; Marcel Dekker: New York, NY, USA, 1987.
14. Hale, J.K. Partial neutral functional differential equations. *Rev. Roum. Math. Pures Appl.* **1994**, *33*, 339–344.
15. Litsyn, E.; Stavroulakis, I.P. On the oscillation of solutions of higher order Emden–Fowler state dependent advanced differential equations. *Nonlinear Anal.* **2001**, *47*, 3877–3883. [[CrossRef](#)]
16. Cesarano, C.; Pinelas, S.; Al-Showaikh, F.; Bazighifan, O. Asymptotic Properties of Solutions of Fourth-Order Delay Differential Equations. *Symmetry* **2019**, *11*, 628. [[CrossRef](#)]
17. Cesarano, C.; Bazighifan, O. Oscillation of fourth-order functional differential equations with distributed delay. *Axioms* **2019**, *8*, 61. [[CrossRef](#)]
18. Moaaz, O.; Elabbasy, E.M.; Bazighifan, O. On the asymptotic behavior of fourth-order functional differential equations. *Adv. Differ. Equ.* **2017**, *2017*, 261. [[CrossRef](#)]
19. Marin, M.; Ochsner, A. The effect of a dipolar structure on the Holder stability in Green-Naghdi thermoelasticity. *Cont. Mech. Thermodyn* **2017**, *29*, 1365–1374. [[CrossRef](#)]
20. Wei, J.J. Oscillation of second order delay differential equation. *Ann. Differ. Equ.* **1988**, *4*, 473–478.
21. Philos, C. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delay. *Arch. Math.* **1981**, *36*, 168–178. [[CrossRef](#)]
22. Cesarano, C.; Bazighifan, O. Qualitative behavior of solutions of second order differential equations. *Symmetry* **2019**, *11*, 777. [[CrossRef](#)]
23. Moaaz, O.; Bazighifan, O. Oscillation criteria for second-order quasi-linear neutral functional differential equation. *Discrete Contin. Dyn. Syst. Ser. S* **2019**, *2019*, 181023.
24. Grace, S.R.; Graef, J.; Tunç, E. Oscillatory behavior of second order damped neutral differential equations with distributed deviating arguments. *Miskolc Math. Notes* **2017**, *18*, 759–769. [[CrossRef](#)]
25. Trench, W. Canonical forms and principal systems for general disconjugate equations. *Trans. Am. Math. Soc.* **1973**, *189*, 319–327. [[CrossRef](#)]
26. Chatzarakis, G.E.; Dzurina, J.; Jadlovská, I. New oscillation criteria for second-order half-linear advanced differential equations. *Appl. Math. Comput.* **2019**, *347*, 404–416. [[CrossRef](#)]



© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).