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Existence of Positive Solutions to Singular Boundary Value Problems Involving φ-Laplacian

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Abstract: This paper is concerned with the existence of positive solutions to singular Dirichlet boundary value problems involving φ-Laplacian. For non-negative nonlinearity $f = f(t,s)$ satisfying $f(t,0) \not\equiv 0$, the existence of an unbounded solution component is shown. By investigating the shape of the component depending on the behavior of f at ∞, the existence, nonexistence and multiplicity of positive solutions are studied.

Keywords: φ-Laplacian; multiplicity of positive solutions; annular domain

1. Introduction

We are concerned with the existence, nonexistence and multiplicity of positive solutions to the following problem

$$\begin{cases} (d(t)\varphi(c(t)u'))' + \lambda h(t) f(t,u) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{R}_+ := [0,\infty)$ is a parameter, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $c, d \in C([0,1],(0,\infty))$, $h \in C((0,1),(0,\infty))$ and $f \in C([0,1]\times\mathbb{R}_+,\mathbb{R}_+)$.

Problem (1) arises naturally in studying radial solutions to the following equation

$$\begin{cases} \operatorname{div}(w(|x|)A(|\nabla v|)\nabla v) + \lambda k(|x|) g(|x|,v) = 0 & \text{in } \Omega, \\ v = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega = \{x \in \mathbb{R}^N : R_0 < |x| < R_1\}$ with $N \geq 2$ and $0 < R_0 < R_1 < \infty$, $w \in C([R_0,R_1],(0,\infty))$, $k \in C((R_0,R_1),\mathbb{R}_+)$ and $g \in C(\mathbb{R}_+,\mathbb{R}_+)$. Indeed, applying change of variables, $v(|x|) = u(t)$ and $|x| = (R_1-R_0)t + R_0$, we can transform (2) into (1) with $\varphi(t) = A(|t|)t$, $d(t) = w((R_1-R_0)t+R_0)((R_1-R_0)t+R_0)^{N-1}$, $c(t) = \frac{1}{R_1-R_0}$, $h(t) = (R_1-R_0)((R_1-R_0)t+R_0)^{N-1}k((R_1-R_0)t+R_0)$ and $f(t,u) = g((R_1-R_0)t+R_0,u)$ (see, e.g., [1]).

Throughout this paper, unless otherwise stated, we assume that φ satisfies the following hypothesis:

(A) there exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \leq \varphi(xy) \leq \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+.$$

Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing homeomorphism. We denote by $\mathcal{H}_\xi$ the set

$$\{g \in C((0,1), \mathbb{R}_+) : \int_0^{\frac{1}{2}} \xi^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds + \int_{\frac{1}{2}}^1 \xi^{-1}\left(\int_{\frac{1}{2}}^s g(\tau)d\tau\right) ds < \infty\}.$$

Remark 1. *Assume that (A) holds. Then it follows that*

$$\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{3}$$

Indeed, (A) implies $\varphi^{-1}(\varphi(x)\psi_1(y)) \le xy$ for all $x, y \in \mathbb{R}_+$. Replacing x and y with $\varphi^{-1}(x)$ and $\psi_1^{-1}(y)$, respectively, one has $\varphi^{-1}(xy) \le \varphi^{-1}(x)\psi_1^{-1}(y)$ for all $x, y \in \mathbb{R}_+$. Similarly, it can be proved that $\varphi^{-1}(x)\psi_2^{-1}(y) \le \varphi^{-1}(xy)$.

Moreover, $\mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$. Indeed, by (3),

$$\varphi^{-1}(1)\int_0^{\frac{1}{2}} \psi_2^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds \le \int_0^{\frac{1}{2}} \varphi^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds \le \varphi^{-1}(1)\int_0^{\frac{1}{2}} \psi_1^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right) ds,$$

which implies $\mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi \subseteq \mathcal{H}_{\psi_2}$. Clearly, $L^1(0,1) \subseteq \mathcal{H}_{\psi_1}$.

For $f(t,s) = f(s)$, we make the following notations: $f_0 := \lim\limits_{s\to 0^+} \dfrac{f(s)}{\varphi(s)}$ and $f_\infty := \lim\limits_{s\to\infty} \dfrac{f(s)}{\varphi(s)}$.

For $\varphi(s) = |s|^{p-2}s$ with $p > 1$, the existence of positive solutions to problem (1) has been extensively studied in the literature for the past several decades (see References [2–18] and references therein). For example, when $p = 2$, h is at most $O(1/t^{2-\delta})$ as $t \to 0^+$ for some $\delta > 0$ and $f(t,s) = e^s$, in Reference [3], it was shown that there exists $\lambda_0 > 0$ such that (1) has a positive solution for $\lambda \in (0, \lambda_0)$ and it has no positive solution for $\lambda > \lambda_0$. The same result was obtained in Reference [4] under the assumption that h satisfies $\int_0^1 t^a(1-t)^b h(s)ds < \infty$ for some $0 < a, b < 1$ and $f(t,s) = f(s)$ is a nondecreasing function satisfying, for some $c > 0$, $f(s) \ge cs$ for all $s \ge 0$. When $p \in (1, \infty)$, in Reference [5], under the assumption that $h \in L^1(0,1)$ and $f_0 = f_\infty = \infty$, it was shown that there exist $\lambda^* \ge \lambda_* > 0$ such that (1) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, one positive solution for $\lambda \in [\lambda_*, \lambda^*]$ and no positive solution for $\lambda > \lambda_*$. In Reference [6], when $h \in \mathcal{H}_\varphi$ and $f(t,s) = f(s)$ satisfies $f(0) > 0$ and $f_\infty = \infty$, it was shown that $\lambda_* = \lambda^*$.

In Reference [19], for an increasing homeomorphism φ satisfying

$(A)'$ there exist an increasing homeomorphism $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \le \varphi(x)\chi(y) \text{ for all } x, y \in \mathbb{R}_+,$$

the same result as Reference [5] was obtained when $c \equiv d \equiv 1$, $h \in \mathcal{H}_{\psi_1}$ and $f_0 = f_\infty = \infty$. Moreover, if $f(0) > 0$ is assumed, it was shown that $\lambda_* = \lambda^*$. Thus the result of Reference [19] extends the previous results of References [3–6] for p-Laplacian problem to singularly weighted φ-Laplacian one.

It looks like the assumption $(A)'$ is more general than the assumption (A), but it is not true. We point out that the assumption (A) is equivalent to the assumption $(A)'$. Indeed, let ψ_1 be an increasing homeomorphism satisfying the first inequality in the assumption (A). Define $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi_2(0) = 0$ and $\psi_2(y) = 1/(\psi_1(y^{-1}))$ for $y > 0$. Then ψ_2 is an increasing homeomorphism on $\mathbb{R}_+$. For $x, y > 0$, $0 < \varphi(xy)\psi_1(y^{-1}) \le \varphi(x)$, and consequently $\varphi(xy) \le \varphi(x)/\psi_1(y^{-1}) = \varphi(x)\psi_2(y)$. Since the homeomorphism ψ_2 satisfying the second inequality in the assumption (A) can be easily defined from ψ_1, the assumption $(A)'$ is no longer useful.

For more general φ which does not satisfy (A), in Reference [20], when $c \equiv d \equiv 1$ and $0 \le h \in L^1(0,1)$ with $h \not\equiv 0$, it was shown that (1) has a positive solution u_λ for all $\lambda > 0$ satisfying $\lim_{\lambda\to 0^+} u_\lambda = 0$ in

$C^1[0,1]$ under some assumptions on f which induces the sublinear nonlinearity if $\varphi(s) = |s|^{p-1}s$ with $p > 1$. For other interesting results, we refer the reader to References [21–23] and the references therein.

The concavity of solutions plays a crucial role in defining operators on a cone and using fixed point theorems (see, e.g., References [2,6,19] and the references therein). It is well known that solutions to problem (1) with $c \equiv d \equiv 1$ are concave functions on $[0,1]$. However, if $c \not\equiv 1$ and $d \not\equiv 1$, it is not obvious that the solutions to problem (1) are concave functions on $[0,1]$. In Reference [1], under the assumption that d is nondecreasing on $[0,1]$, a lemma ([1], Lemma 2.4) was proved from which a suitable positive cone was defined and various results for positive solutions to problem (1) were proved.

However, the proof of the lemma ([1], Lemma 2.4) is not clear. In the proof of it, the fact $c(t)u'(t)$ is non-increasing on $(0,1)$ is used. However it may not be true, since the fact $d(t)\varphi(c(t)u'(t))$ is nonincreasing on $(0,1)$ does not imply that $\varphi(c(t)u'(t))$ is nonincreasing on $(0,1)$, even though $d(t)$ is non-decreasing on $[0,1]$ (see Remark 2 (1)). Consequently, $c(t)u'(t)$ may not be nonincreasing on $(0,1)$.

In this paper, we show the existence of an unbounded solution component and prove the existence and nonexistence of positive solutions to problem (1) under suitable assumptions on nonlinearity $f(t,s)$. Among other main results, we extend a result of Reference [6] for p-Laplacian problem to general φ-Laplacian one (see Theorem 4 below). For that purpose, we prove a similar result to that of Reference [1] (Lemma 2.4) under the weaker hypotheses to functions g and d (see Lemma 2 below). Also, the result (Theorem 4) extends that of Reference [19] in some way, since we assume that $c \not\equiv 1$, $d \not\equiv 1$ and $h \in \mathcal{H}_\varphi$ in it.

The rest of this paper is organized as follows. In Section 2, a solution operator related to problem (1) is introduced and some preliminaries are given. In Section 3, the main results (Theorems 2–4) are proved and a few examples to illustrate the assumptions in the main results are given.

2. Preliminaries

First we give some notations which will be used in this paper.

The usual maximum norm in a Banach space $C[0,1]$ is denoted by $\|u\|_\infty := \max_{t\in[0,1]} |u(t)|$ for $u \in C[0,1]$, and let $c_0 := \min_{t\in[0,1]} c(t) > 0$, $d_0 := \min_{t\in[0,1]} d(t) > 0$ and $\rho_1 := \dfrac{c_0}{\|c\|_\infty} \dfrac{\psi_2^{-1}\left(\frac{1}{\|d\|_\infty}\right)}{\psi_1^{-1}\left(\frac{1}{d_0}\right)} \in (0,1]$.

Define $\mathcal{K}$ to be a cone in $C[0,1]$ by

$$\mathcal{K} := \{u \in C([0,1],\mathbb{R}_+) : u(t) \geq \tfrac{\rho_1}{4}\|u\|_\infty \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}]\}.$$

Now we introduce a solution operator related to problem (1). Let $g \in \mathcal{H}_\varphi \setminus \{0\}$ be fixed, and define a function $v_g : (0,1) \to \mathbb{R}$ by $v_g(t) = v_g^1(t) - v_g^2(t)$ for $t \in (0,1)$. Here v_g^1 and v_g^2 are functions defined by

$$v_g^1(t) = \int_0^t \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^t g(\tau)d\tau\right)ds \text{ and } v_g^2(t) = \int_t^1 \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_t^s g(\tau)d\tau\right)ds.$$

We claim that v_g^1 is a non-decreasing continuous function on $(0,1)$ satisfying $\lim_{t\to 0+} v_g^1(t) = 0$ and $\lim_{t\to 1^-} v_g^1(t) \in (0,\infty]$. Indeed, from the nonnegativity of $g(\not\equiv 0)$, it follows that v_g^1 is non-decreasing on $(0,1)$ and $\lim_{t\to 1^-} v_g^1(t) \in (0,\infty]$. By (3),

$$0 \leq v_g^1(t) \leq \frac{1}{c_0}\psi_1^{-1}\left(\frac{1}{d_0}\right)\int_0^t \varphi^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau\right)ds \text{ for } t \in (0,\frac{1}{2}).$$

Consequently, since $g \in \mathcal{H}_\varphi$, $\lim_{t\to 0+} v_g^1(t) = 0$. Finally, we prove the continuity of v_g^1 on $(0,1)$. Let $x_0 \in (0,1)$ be fixed and let ϵ be chosen so that $[x_0-\epsilon, x_0+\epsilon] \subseteq (0,1)$. Assume that $\{x_n\}$ is a sequence in $[x_0-\epsilon, x_0+\epsilon]$ satisfying $\lim_{n\to\infty} x_n = x_0$. Let

$$G_n(s) = K_{[0,x_n]}(s)\frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^{x_n} g(\tau)d\tau\right) \text{ and } G_0(s) = K_{[0,x_0]}(s)\frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^{x_0} g(\tau)d\tau\right)$$

for $s \in (0,1)$. Here K_I is the characteristic function of I, that is, $K_I(x) = 1$ for $x \in I$ and $K_I(x) = 0$ for $x \in (0,1) \setminus I$. Then $\lim_{n\to\infty} G_n(s) = G_0(s)$ for each $s \in (0,1)$ and for all n,

$$\begin{aligned} 0 \le G_n(s) &\le K_{[0,x_0+\epsilon]}(s)\frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^{x_0+\epsilon} g(\tau)d\tau\right) ds \\ &\le K_{[0,x_0+\epsilon]}(s)\frac{1}{c_0}\psi_1^{-1}\left(\frac{1}{d_0}\right)\varphi^{-1}\left(\int_s^{x_0+\epsilon} g(\tau)d\tau\right) ds \in L^1(0,1). \end{aligned}$$

By Lebesgue's dominated convergence theorem, ν_g^1 is continuous at $x = x_0$. Thus, the claim is proved.

Similarly, it can be shown that ν_g^2 is a non-increasing continuous function on $(0,1)$ satisfying $\lim_{t\to 0^+} \nu_g^2(t) \in (0,\infty]$ and $\lim_{t\to 1-} \nu_g^2(t) = 0$. Then there exists an interval $[\sigma_g^1, \sigma_g^2] \subsetneq (0,1)$ satisfying $\nu_g(\sigma) = 0$ for all $\sigma \in [\sigma_g^1, \sigma_g^2]$.

Define a function $T : \mathcal{H}_\varphi \to C[0,1]$ by $T(0) = 0$ and, for $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$T(g)(t) = \begin{cases} \int_0^t \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^\sigma g(\tau)d\tau\right) ds, & \text{if } 0 \le t \le \sigma, \\ \int_t^1 \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_\sigma^s g(\tau)d\tau\right) ds, & \text{if } \sigma \le t \le 1, \end{cases} \tag{4}$$

where $\sigma = \sigma(g)$ is a zero of ν_g in $(0,1)$, that is,

$$\int_0^\sigma \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^\sigma g(\tau)d\tau\right) ds = \int_\sigma^1 \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_\sigma^s g(\tau)d\tau\right) ds. \tag{5}$$

We notice that, although $\sigma = \sigma(g)$ is not necessarily unique, the operator T is well defined. Indeed, if σ_1 and σ_2 are zeroes of ν_g in $(0,1)$, then $g(t) = 0$ for $t \in [\sigma_1, \sigma_2]$, in view of the monotonicity of ν_1 and ν_2. Consequently, $T(g)$ is independent of the choice of $\sigma \in [\sigma_1, \sigma_2]$ (see, e.g., Reference [1]).

For $g \in \mathcal{H}_\varphi$, consider the following problem

$$\begin{cases} (d(t)\varphi(c(t)u'))' + g(t) = 0, \ \ t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{6}$$

For $g = 0$, (6) has a unique zero solution due to the boundary conditions.

Lemma 1. *Assume that (A) holds, and let u be a solution to problem (6) with $g \in \mathcal{H}_\varphi \setminus \{0\}$. Then there exists a subinterval $[\sigma_1, \sigma_2]$ of $(0,1)$ such that $u'(t) > 0,\ t \in (0,\sigma_1)$, $u'(t) = 0$ for $t \in [\sigma_1, \sigma_2]$ and $u'(t) < 0,\ t \in (\sigma_2, 1)$. Moreover, $T(g)$ is a unique solution to problem (6) and $T(g) > 0$ on $(0,1)$.*

Proof. Since $g \ne 0$, 0 is not a solution to problem (6). From the fact $(d(t)\varphi(c(t)u'(t)))' = -g(t) \le 0$ for $t \in (0,1)$, it follows that $d(t)\varphi(c(t)u'(t))$ is continuous and non-increasing in $(0,1)$. By the monotonicity of φ, since $u(0) = u(1) = 0$ and $c, d > 0$ on $[0,1]$,

$$\lim_{t\to 0+} d(t)\varphi(c(t)u'(t)) \in (0,\infty] \text{ and } \lim_{t\to 1-} d(t)\varphi(c(t)u'(t)) \in [-\infty, 0).$$

Consequently, there exists a subinterval $[\sigma_1, \sigma_2]$ of $(0,1)$ such that $(d\varphi(cu'))(t) > 0,\ t \in (0,\sigma_1)$, $(d\varphi(cu'))(t) = 0$ for $t \in [\sigma_1, \sigma_2]$ and $(d\varphi(cu'))(t) < 0,\ t \in (\sigma_2, 1)$. Then, by the hypotheses on c, d and φ, $u'(t) > 0,\ t \in (0,\sigma_1)$, $u'(t) = 0$ for $t \in [\sigma_1, \sigma_2]$ and $u'(t) < 0,\ t \in (\sigma_2, 1)$. Clearly, $T(g)$ is a solution to problem (6) and $T(g) > 0$ on $(0,1)$. By directly integrating (6), it can be shown that $T(g)$ is a unique solution to problem (6). $\square$

Remark 2.

(1) *It is easy to show that if $g_1 > 0$, $g_2 > 0$, $g_1 g_2$ is non-increasing and g_1 is non-decreasing on (a,b), then g_2 is non-increasing on (a,b). However, if g_2 is a sign-changing function on (a,b), it is not true that g_2 is non-decreasing on (a,b). For example, $g_1(x) = x^3$ and $g_2(x) = (x-a)(x-b)$ with $0 < a < b < 1$. Let x_1 and x_2 be a local maximum point and a local minimum point of $g_1 g_2$, respectively. Note that $0 < x_1 < a < \frac{a+b}{2} < x_2 < b$, since $g_2'(\frac{a+b}{2}) = 0$ and $(g_1 g_2)'(\frac{a+b}{2}) < 0$. Then g_1 is a positive increasing function on $(0,1)$ and $g_1 g_2$ is decreasing on (x_1, x_2). However, g_2 is decreasing on $(x_1, \frac{a+b}{2})$ and is increasing on $(\frac{a+b}{2}, x_2)$.*

(2) *We notice that if we assume that c and d are non-decreasing on $[0,1]$, by Remark 2 (1), it is easy to check that, for any solution u to problem (6), u' is non-increasing on $(0, \sigma_2]$, which implies that u is a concave function on $[0, \sigma_2]$. However, in general, u may not be a concave function on $[0,1]$.*

Without the monotonicity of d, we prove a result which is analogous to Reference [1] (Lemma 2.4).

Lemma 2. *Assume that (A) hold and let $g \in \mathcal{H}_\varphi$ be given. Then*

$$T(g)(t) \geq \min\{t, 1-t\}\rho_1 \|T(g)\|_\infty \text{ for } t \in [0,1].$$

Proof. For $g = 0$, 0 is a unique solution to problem (6) and there is nothing to prove.

Let $g \in \mathcal{H}_\varphi \setminus \{0\}$ and $\sigma \in (0,1)$ be a constant satisfying (5), i.e., $\|T(g)\|_\infty = T(g)(\sigma)$. By (3), for $t \in (0, \sigma]$,

$$\begin{aligned} T(g)(t) &= \int_0^t \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^\sigma g(\tau)d\tau\right) ds \geq \frac{1}{\|c\|_\infty} \int_0^t \varphi^{-1}\left(\frac{1}{\|d\|_\infty} \int_s^\sigma g(\tau)d\tau\right) ds \\ &\geq \frac{1}{\|c\|_\infty} \psi_2^{-1}\left(\frac{1}{\|d\|_\infty}\right) \int_0^t \varphi^{-1}\left(\int_s^\sigma g(\tau)d\tau\right) ds. \end{aligned}$$

Similarly, $\|T(g)\|_\infty \leq \frac{1}{c_0} \psi_1^{-1}\left(\frac{1}{d_0}\right) \int_0^\sigma \varphi^{-1}\left(\int_s^\sigma g(\tau)d\tau\right) ds$. Recall that $c_0 = \min_{t \in [0,1]} c(t) > 0$ and $d_0 = \min_{t \in [0,1]} d(t) > 0$. Let $w_1(t) := \int_0^t \varphi^{-1}\left(\int_s^\sigma g(\tau)d\tau\right) ds$ for $t \in [0, \sigma]$. Since $g \geq 0$ on $(0,1)$, w_1' is non-increasing on $(0, \sigma]$, so that w_1 is a concave function on $(0, \sigma]$. Consequently $w_1(t) \geq t w_1(\sigma)$ for $t \in (0, \sigma]$, and $T(g)(t) \geq \rho_1 t \|T(g)\|_\infty$ for $t \in [0, \sigma]$. Similarly, $T(g)(t) \geq \rho_1 (1-t) \|T(g)\|_\infty$ for $t \in [\sigma, 1]$, and thus the proof is complete. $\square$

By Lemmas 1 and 2, for each $g \in \mathcal{H}_\varphi$, $T(g) \in \mathcal{K}$, and $(T(g))'(\sigma) = 0$ if and only if $T(g)(\sigma) = \|T(g)\|_\infty$.

From now on, we assume $h \in \mathcal{H}_\varphi$. Define a function $F : \mathbb{R}_+ \times \mathcal{K} \to C(0,1)$ by $F(\lambda, u)(t) = \lambda h(t) f(t, u(t))$ for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$ and $t \in (0,1)$. Clearly, $F(\lambda, u) \in \mathcal{H}_\varphi$ for any $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$.

Define an operator $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ by $H(\lambda, u) = T(F(\lambda, u))$ for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$, i.e., for $(\lambda, u) \in \mathbb{R}_+ \times \mathcal{K}$,

$$H(\lambda, u)(t) = \begin{cases} \int_0^t \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^\sigma F(\lambda, u)(\tau)d\tau\right) ds, & \text{if } 0 \leq t \leq \sigma, \\ \int_t^1 \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_\sigma^s F(\lambda, u)(\tau)d\tau\right) ds, & \text{if } \sigma \leq t \leq 1, \end{cases} \tag{7}$$

where $\sigma = \sigma(\lambda, u)$ is a constant satisfying

$$\int_0^\sigma \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^\sigma F(\lambda, u)(\tau)d\tau\right) ds = \int_\sigma^1 \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_\sigma^s F(\lambda, u)(\tau)d\tau\right) ds. \tag{8}$$

To prove the complete continuity of H, the following lemma is needed.

Lemma 3. *Assume that* (A) *and* $h \in \mathcal{H}_\varphi$ *hold. Let* $M > 0$ *be given and let* $\{(\lambda_n, u_n)\}$ *be a bounded sequence in* $\mathbb{R}_+ \times \mathcal{K}$ *with* $\lambda_n + \|u_n\|_\infty \le M$. *If* $\sigma_n \to 0$ *or* 1 *as* $n \to \infty$, *then* $\|H(\lambda_n, u_n)\|_\infty \to 0$ *and* $F(\lambda_n, u_n)(t) \to 0$ *as* $n \to \infty$ *for each* $t \in (0,1)$. *Here,* $\sigma_n = \sigma(\lambda_n, u_n)$ *is a constant satisfying* (8) *with* $\lambda = \lambda_n$ *and* $u = u_n$.

Proof. We only prove the case $\sigma_n \to 0$ as $n \to \infty$, since the other case can be dealt in a similar manner. Since there exists $N > 0$ such that $\lambda f(t, u(t)) \le N$ for all $(t, \lambda, u) \in [0,1] \times [0, M] \times [0, M]$, by (3),

$$\begin{aligned} \|H(\lambda_n, u_n)\|_\infty = H(\lambda_n, u_n)(\sigma_n) &= \int_0^{\sigma_n} \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^{\sigma_n} \lambda_n h(\tau) f(\tau, u_n(\tau)) d\tau\right) ds \\ &\le \frac{1}{c_0} \psi_1^{-1}(\frac{N}{d_0}) \int_0^{\sigma_n} \varphi^{-1}\left(\int_s^{\sigma_n} h(\tau) d\tau\right) ds. \end{aligned}$$

Consequently, it follows from $h \in \mathcal{H}_\varphi$ that $\|H(\lambda_n, u_n)\|_\infty \to 0$ as $n \to \infty$.

Since σ_n is a constant satisfying (8) with $\lambda = \lambda_n$ and $u = u_n$,

$$\lim_{n \to \infty} \int_{\sigma_n}^1 \frac{1}{c(s)} \varphi^{-1}(\frac{1}{d(s)} \int_{\sigma_n}^s F(\lambda_n, u_n)(\tau) d\tau) ds = 0,$$

which implies that, for all $t \in (0,1)$, $F(\lambda_n, u_n)(t) \to 0$ as $n \to \infty$. $\square$

With Lemma 3, by the argument similar to those in the proof of [2] (Lemma 3), it can be proved that $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ is completely continuous (see also Reference [24], Lemma 3.3). So we omit the proof of it.

Lemma 4. *Assume that* (A) *and* $h \in \mathcal{H}_\varphi$ *hold. Then the operator* $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Finally, we present a well-known theorem for the existence of an unbounded solution component by Leray and Schauder [25]:

Theorem 1. (see, e.g., Reference [26], Corollary 14.12) *Let* X *be a Banach space with* $X \ne \{0\}$ *and let* $\mathcal{K}$ *be a cone in* X. *Consider*

$$x = H(\lambda, x), \tag{9}$$

where $\lambda \in \mathbb{R}_+$ *and* $x \in \mathcal{K}$. *If* $H : \mathbb{R}_+ \times \mathcal{K} \to \mathcal{K}$ *is completely continuous and* $H(0, x) = 0$ *for all* $x \in \mathcal{K}$, *then there exists an unbounded solution component* $\mathcal{C}$ *of* (9) *in* $\mathbb{R}_+ \times \mathcal{K}$ *emanating from* $(0,0)$.

3. Main Results

First, we make a list of assumptions on $f(t,s)$ which will be used in this section.

$(F0)$ $f(t_0, 0) > 0$ for some $t_0 \in (0,1)$.

$(F0)'$ for any $M > 0$, there exists a non-empty interval $(\alpha_M, \beta_M) \subsetneq (0,1)$ such that

$$f(t,s) > 0 \text{ for } (t,s) \in [\alpha_M, \beta_M] \times [0, M].$$

$(F1)$ $\displaystyle\lim_{s \to \infty} \max_{t \in [0,1]} \frac{f(t,s)}{\varphi(s)} = 0.$

$(F1)'$ $\displaystyle\lim_{s \to \infty} \max_{t \in [0,1]} \frac{f(t,s)}{\psi_1(s)} = 0.$ Here ψ_1 is the homeomorphism in the assumption (A).

$(F2)$ there exist $\hat{C} > 0$ and a non-empty interval $(\alpha, \beta) \subseteq (0,1)$ such that

$$f(t,s) \geq \hat{C}\varphi(s) \text{ for } (t,s) \in [\alpha,\beta] \times \mathbb{R}_+.$$

$(F3)$ $f(t,s) > 0$ for all $(t,s) \in [0,1] \times \mathbb{R}_+$ and $\displaystyle\lim_{s\to\infty} \min_{t\in[0,1]} \frac{f(t,s)}{\varphi(s)} = \infty.$

Remark 3.

(1) *It is easy to see that* (1) *has a solution if and only if* $H(\lambda,\cdot)$ *has a fixed point in* $\mathcal{K}$. *Since* $H(0,u) = 0$ *for all* $u \in \mathcal{K}$, 0 *is a unique solution to problem* (1) *with* $\lambda = 0$.

(2) *Assume that* $f(t,0) = 0$ *for all* $t \in [0,1]$. *Then* 0 *is a solution to problem* (1) *for any* $\lambda \in \mathbb{R}_+$.

(3) *Assume that* $(F0)$ *holds. Then* 0 *is not a solution to problem* (1) *with* $\lambda > 0$. *Let* u *be a solution to problem* (1) *with* $\lambda > 0$. *Then, by Lemma* 1, u *is a positive solution, i.e.,* $u(t) > 0$ *for all* $t \in (0,1)$.

By Lemma 4, Theorem 1 and Remark 3, one has the following proposition.

Proposition 1. *Assume that* $(A), (F0)$ *and* $h \in \mathcal{H}_\varphi$ *hold. Then there exists an unbounded solution component* $\mathcal{C}$ *emanating from* $(0,0)$ *in* $\mathbb{R}_+ \times \mathcal{K}$ *such that* (i) $\mathcal{C} \cap (\{0\} \times \mathcal{K}) = \{(0,0)\}$ *and* (ii) *for any* $(\lambda,u) \in \mathcal{C} \setminus \{(0,0)\}$, u *is a positive solution to problem* (1) *with* $\lambda > 0$.

Now we give a lemma which provides useful information about the solution component $\mathcal{C}$ defined in Proposition 1.

Lemma 5. *Assume that* $(A), (F0), (F1)$ *and* $h \in \mathcal{H}_{\psi_1}$ *hold. Let* $J = [0,l]$ *be a compact interval with* $l > 0$. *Then there exists* $M_J > 0$ *such that* $\|u\|_\infty \leq M_J$ *for any positive solutions* u *to problem* (1) *with* $\lambda \in J$.

Proof. Let $m = (4l)^{-1}\psi_1(h_*^{-1}) > 0$. Here

$$h_* = \max\left\{ \frac{1}{c_0}\int_0^{\frac{1}{2}} \psi_1^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau\right)ds, \frac{1}{c_0}\int_{\frac{1}{2}}^1 \psi_1^{-1}\left(\frac{1}{d_0}\int_{\frac{1}{2}}^s h(\tau)d\tau\right)ds \right\} > 0.$$

By $(F1)$, there exists $s_m > 0$ such that $f(t,s) \leq m\varphi(s)$ for $(t,s) \in [0,1] \times [s_m,\infty)$. Set $C_m = \max\{f(t,s) : (t,s) \in [0,1] \times [0,s_m]\} > 0$. Assume to the contrary that there exists a sequence $\{(\lambda_n,u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, for sufficiently large $N > 0$, $C_m \leq m\varphi(\|u_N\|_\infty)$.

Let σ_N be a constant satisfying $\|u_N\|_\infty = u_N(\sigma_N)$. Assume $\sigma_N \leq \frac{1}{2}$, since the case $\sigma_N > \frac{1}{2}$ can be dealt in a similar manner. Then, by (3),

$$\begin{aligned}
\|u_N\|_\infty &= \int_0^{\sigma_N} \frac{1}{c(s)}\varphi^{-1}\left(\frac{1}{d(s)}\int_s^{\sigma_N} \lambda_N h(\tau) f(\tau,u_N(\tau))d\tau\right)ds \\
&\leq \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau l(C_m + m\varphi(\|u_N\|_\infty))\right)ds \\
&\leq \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau 2lm\varphi(\|u_N\|_\infty)\right)ds \\
&\leq h_*\varphi^{-1}(2lm\varphi(\|u_N\|_\infty)) \leq h_*\psi_1^{-1}(2lm)\|u_N\|_\infty,
\end{aligned}$$

which implies $m \geq (2l)^{-1}\psi_1((h_*)^{-1})$. This contradicts the choice of m. □

By similar arguments used to prove Lemma 5, one can prove the following result which shows the same property for the solution component $\mathcal{C}$. For the convenience of readers, we give the proof of it.

Lemma 6. *Assume that $(A), (F0), (F1)'$ and $h \in \mathcal{H}_\varphi$ hold. Let $J = [0, l]$ be a compact interval with $l > 0$. Then there exists $M_J > 0$ such that $\|u\|_\infty \le M_J$ for any positive solutions u to problem (1) with $\lambda \in J$.*

Proof. Let $m' = (4l)^{-1}\psi_1(h_{**}^{-1}) > 0$. Here

$$h_{**} = \max\left\{ \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau\right) ds, \frac{1}{c_0}\int_{\frac{1}{2}}^{1} \varphi^{-1}\left(\frac{1}{d_0}\int_{\frac{1}{2}}^{s} h(\tau)d\tau\right) ds \right\}.$$

By $(F1)'$, there exists $s_{m'} > 0$ such that $f(t,s) \le m'\psi_1(s)$ for $(t,s) \in [0,1] \times [s_{m'}, \infty)$. Set $C_{m'} = \max\{f(t,s) : (t,s) \in [0,1] \times [0, s_{m'}]\} > 0$. Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in J$ and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, for sufficiently large $N > 0$, $C_{m'} \le m'\psi_1(\|u_N\|_\infty)$.

Let σ_N be a constant satisfying $\|u_N\|_\infty = u_N(\sigma_N)$. Assume $\sigma_N \le \frac{1}{2}$, since the case $\sigma_N > \frac{1}{2}$ can be dealt in a similar manner. Then

$$\begin{aligned}
\|u_N\|_\infty &\le \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau l(C_{m'} + m'\psi_1(\|u_N\|_\infty))\right) ds \\
&\le \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau 2lm'\psi_1(\|u_N\|_\infty)\right) ds \\
&\le \frac{1}{c_0}\int_0^{\frac{1}{2}} \varphi^{-1}\left(\frac{1}{d_0}\int_s^{\frac{1}{2}} h(\tau)d\tau \psi_1(\|u_N\|_\infty)\right) ds\psi_1^{-1}(2lm') \\
&\le h_{**}\psi_1^{-1}(2lm')\|u_N\|_\infty,
\end{aligned}$$

which contradicts the choice of m'. □

We remark that the assumptions in Lemma 5 are different from ones in Lemma 6. Indeed, let $\varphi(s) = s + s^2$ and $\psi_1(s) = \min\{s, s^2\}$ for $s \in \mathbb{R}_+$. Then the first inequality in the assumption $(A2)$ is satisfied. Clearly, $(F1)'$ implies $(F1)$, since $\varphi(1)\psi_1(s) \le \varphi(s)$ for all $s \in \mathbb{R}_+$. For $f(t,s) = s$, $\lim_{s\to\infty} \frac{s}{\varphi(s)} = 0$, but $\lim_{s\to\infty} \frac{s}{\psi_1(s)} = 1$. Consequently, $(F1)$ does not imply $(F1)'$. Since $\mathcal{H}_{\psi_1} \subseteq \mathcal{H}_\varphi$, we give an example of h satisfying $h \in \mathcal{H}_\varphi \setminus \mathcal{H}_{\psi_1}$. Let $h(t) = t^{-2}$ for $t > 0$. Note that $\psi_1^{-1}(s) = \max\{\sqrt{s}, s\}$ and $\varphi^{-1}(s) = \frac{-1+\sqrt{1+4s}}{2}$ for $s \in \mathbb{R}_+$. Then $h \in \mathcal{H}_\varphi$, but $h \notin \mathcal{H}_{\psi_1}$, since

$$\varphi^{-1}\left(\int_s^{\frac{1}{2}} \tau^{-2}d\tau\right) = \varphi^{-1}\left(s^{-1} - 2\right) = \frac{-1 + \sqrt{1 + 4(s^{-1} - 2)}}{2} \in L^1\left(0, \frac{1}{2}\right)$$

and

$$\psi_1^{-1}\left(\int_s^{\frac{1}{2}} \tau^{-2}d\tau\right) = \psi_1^{-1}\left(s^{-1} - 2\right) = s^{-1} - 2 \text{ for } s \in (0, \tfrac{1}{3}).$$

Now we give the first main result in this paper.

Theorem 2. *Assume that $(A), (F0)$ and either $(F1)$ and $h \in \mathcal{H}_{\psi_1}$ or $(F1)'$ and $h \in \mathcal{H}_\varphi$ hold. Then for any $\lambda \in (0, \infty)$, there exists a positive solution u_λ to problem (1) such that $(\lambda, u_\lambda) \in \mathcal{C}$ and $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0^+$. Moreover, if $(F0)'$ is assumed instead of $(F0)$, then $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Here, $\mathcal{C}$ is the solution component defined in Proposition 1.*

Proof. Let $\lambda^* = \sup\{\lambda : (\lambda, u_\lambda) \in \mathcal{C}\}$. Since $\mathcal{C}$ is unbounded in $\mathbb{R}_+ \times \mathcal{K}$, by Lemma 5 or Lemma 6, $\lambda^* = \infty$, so that for any $\lambda \in (0, \infty)$, there exists a positive solution u_λ to problem (1) such that $(\lambda, u_\lambda) \in \mathcal{C}$ and $\|u_\lambda\|_\infty \to 0$ as $\lambda \to 0^+$.

Next, we show that if $(F0)'$ is assumed instead of $(F0)$, then $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to \infty$. Assume to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ in $\mathcal{C}$ such that $\lambda_n \to \infty$ as $n \to \infty$, but there exists $M > 0$ such that $\|u_n\|_\infty \le M$ for all n. Then, by $(F0)'$, there exists $\delta_M > 0$ such that $f(t, u_n(t)) \ge \delta_M$ for all n and all $t \in [\alpha_M, \beta_M]$. For each n, let σ_n be a constant satisfying $u_n(\sigma_n) = \|u_n\|_\infty$ and let $\gamma_M = \frac{\alpha_M + \beta_M}{2}$. Suppose that $\sigma_n \ge \gamma_M$ (the case $\sigma_n < \gamma_M$ is similar). Then

$$\begin{aligned} \|u_n\|_\infty \ge u_n(\alpha_M) &= \int_0^{\alpha_M} \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^{\sigma_n} \lambda_n h(\tau) f(\tau, u_n(\tau)) d\tau\right) ds \\ &\ge \int_0^{\alpha_M} \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_{\alpha_M}^{\gamma_M} h(\tau) d\tau \lambda_n \delta_M\right) ds \\ &\ge \gamma_M^* \varphi^{-1}(h_M^* \lambda_n) \to \infty \text{ as } n \to \infty, \end{aligned}$$

which contradicts the fact that $\|u_n\|_\infty \le M$ for all n. Here,

$$\gamma_M^* := \frac{1}{\|c\|_\infty} \min\{\alpha_M, 1 - \beta_M\} > 0 \text{ and } h_M^* = \frac{\delta_M}{\|d\|_\infty} \min\left\{\int_{\alpha_M}^{\gamma_M} h(\tau) d\tau, \int_{\gamma_M}^{\beta_M} h(\tau) d\tau\right\} > 0.$$

Thus, the proof is complete. □

Next we give a lemma about the λ-direction block for positive solutions to problem (1).

Lemma 7. *Assume that $(A), (F2)$ and $h \in \mathcal{H}_\varphi$ hold. Then there exists $\bar{\lambda} > 0$ such that (1) has no positive solution for $\lambda > \bar{\lambda}$.*

Proof. Let u be a positive solution to problem (1) with $\lambda > 0$ and $u(\sigma) = \|u\|_\infty$. By $(F2)$, $f(t,s) > \hat{C}\varphi(s)$ for $(t,s) \in [\alpha, \beta] \times \mathbb{R}_+$. Let $\gamma = \frac{\alpha+\beta}{2}$. We only consider the case $\sigma \ge \gamma$, since the case $\sigma < \gamma$ can be dealt in a similar manner. By Lemma 1, $u(t) \ge u(\alpha)$ for $t \in [\alpha, \gamma]$, and consequently, $f(t, u(t)) \ge \hat{C}\varphi(u(\alpha))$ for $t \in [\alpha, \gamma]$. Then, by (3),

$$\begin{aligned} u(\alpha) &= \int_0^\alpha \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^\sigma \lambda h(\tau) f(\tau, u(\tau)) d\tau\right) ds \\ &\ge \frac{1}{\|c\|_\infty} \int_0^\alpha \varphi^{-1}\left(\int_\alpha^\gamma h(\tau) d\tau \|d\|_\infty^{-1} \lambda \hat{C} \varphi(u(\alpha))\right) ds \\ &\ge h^* \varphi^{-1}(\|d\|_\infty^{-1} \lambda \hat{C} \varphi(u(\alpha))) \ge h^* \psi_2^{-1}(\|d\|_\infty^{-1} \lambda \hat{C}) u(\alpha). \end{aligned}$$

Here $h^* = \dfrac{1}{\|c\|_\infty} \min\left\{\displaystyle\int_0^\alpha \psi_2^{-1}\left(\int_\alpha^\gamma h(\tau) d\tau\right) ds, \int_\beta^1 \psi_2^{-1}\left(\int_\gamma^\beta h(\tau) d\tau\right) ds\right\} > 0$. Consequently,

$$\lambda \le \frac{\|d\|_\infty}{\hat{C}} \psi_2\left(\frac{1}{h^*}\right) =: \bar{\lambda},$$

which completes the proof. □

Now we give the second main result in this paper.

Theorem 3. *Assume that $(A), (F2)$ and $h \in \mathcal{H}_\varphi$ hold. Then there exists $\lambda^* > 0$ such that (1) has at least one positive solution for $\lambda \in (0, \lambda^*)$ and no positive solution for $\lambda > \lambda^*$.*

Proof. By Proposition 1, there exists at least one positive solution to problem (1) for all small $\lambda > 0$. Let λ_1 be a positive number such that (1) has a positive solution u_1 for $\lambda = \lambda_1$. To complete the proof of Theorem 3, by Lemma 7, it suffices to show that (1) has a positive solution for all $\lambda \in (0, \lambda_1)$.

Let $\lambda \in (0, \lambda_1)$ be fixed, and consider the following modified problem

$$\begin{cases} (d(t)\varphi(c(t)u'))' + \lambda h(t)\bar{f}(t,u) = 0, \ \ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{10}$$

where $\bar{f}(t,u) = f(t, \gamma(t,u))$ and $\gamma : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\gamma(t,u) = \begin{cases} u_1(t), & \text{if } \ u \geq u_1(t), \\ u, & \text{if } \ 0 < u < u_1(t), \\ 0, & \text{if } \ u \leq 0. \end{cases}$$

Define $T_\lambda : C[0,1] \to C[0,1]$ by $T_\lambda(u) = T(\hat{F}(u))$ for $u \in C[0,1]$, where $\hat{F}(u)(t) = \lambda h(t)\bar{f}(t, u(t))$ for $u \in C[0,1]$ and $t \in (0,1)$. Then it is easy to see that u is a solution to problem (10) if and only if $u = T_\lambda u$, and T_λ is completely continuous on $C[0,1]$. From the definition of γ and the continuity of f, it follows that there exists $N_1 > 0$ such that $||T_\lambda u||_\infty < N_1$ for all $u \in C[0,1]$. Then, by Schauder fixed point theorem, there exists $u_\lambda \in C[0,1]$ such that $T_\lambda u_\lambda = u_\lambda$. Consequently, by Lemma 1, u_λ is a positive solution to problem (10). We claim that $u_\lambda(t) \leq u_1(t)$ for all $t \in [0,1]$. If the claim is not true, since $u_\lambda(0) = u_\lambda(1) = u_1(0) = u_1(0) = 0$, there exists an interval $(t_1, t_2) \subseteq (0,1)$ such that $u_\lambda(t) > u_1(t)$ for all $t \in (t_1, t_2)$, $u_\lambda(t_1) = u_1(t_1)$ and $u_\lambda(t_2) = u_1(t_2)$. Then there exists $\hat{t} \in (t_1, t_2)$ such that

$$(u_\lambda - u_1)(\hat{t}) = \max\{(u_\lambda - u_1)(t) : t \in [t_1, t_2]\} > 0, \tag{11}$$

i.e., $u'_\lambda(\hat{t}) = u'_1(\hat{t})$. By the definition of γ and the fact $\lambda < \lambda_1$,

$$-(d(t)\varphi(c(t)u'_\lambda(t)))' \leq -(d(t)\varphi(c(t)u'_1(t)))' \text{ for } t \in (t_1, t_2). \tag{12}$$

For $t \in (t_1, \hat{t}]$, integrating (12) from t to $\hat{t}$, we have $u'_\lambda(t) \leq u'_1(t)$. Integrating it from t_1 and $\hat{t}$ again, $u_\lambda(\hat{t}) \leq u_1(\hat{t})$, which contradicts (11). Consequently, the claim is proved and u_λ is a positive solution to problem (1) by the definition of γ. Thus, the proof is complete. □

Next we give a lemma about *a priori* estimates for solutions to problem (1).

Lemma 8. *Assume that* $(A), (F3)$ *and* $h \in \mathcal{H}_\varphi$ *hold. Let* $I = [l_1, \infty)$ *with* $l_1 > 0$ *be given. Then there exists* $M_I > 0$ *such that* $\|u\|_\infty \leq M_I$ *for any positive solutions* u *to problem* (1) *with* $\lambda \in I$.

Proof. Suppose to the contrary that there exists a sequence $\{(\lambda_n, u_n)\}$ such that u_n is a positive solution to problem (1) with $\lambda = \lambda_n \in I$ and $||u_n||_\infty \to \infty$ as $n \to \infty$.

Take $C^* = 4\|d\|_\infty (h_0 l_1)^{-1}\psi_2(4\|c\|_\infty) + 1$, where $h_0 = \min\{h(t) : t \in [\frac{1}{4}, \frac{3}{4}]\} > 0$. By $(F3)$, there exists $K > 0$ such that $f(t,s) \geq C^*\varphi(s)$ for $(t,s) \in [0,1] \times (K, \infty)$. Since $u_n \in \mathcal{K}$ for all n,

$$u_n(t) \geq \frac{\rho_1}{4}\|u_n\|_\infty \text{ for } t \in [\tfrac{1}{4}, \tfrac{3}{4}].$$

For sufficiently large $N > 0$,

$$F(\lambda_N, u_N)(t) = \lambda_N h(t) f(t, u_N(t)) \geq l_1 C^* h(t)\varphi(u_N(t)) \text{ for all } t \in [\tfrac{1}{4}, \tfrac{3}{4}].$$

Let σ_N be a constant satisfying $u_N(\sigma_N) = \|u_N\|_\infty$. We only consider the case $\sigma_N \geq \frac{1}{2}$, since the case $\sigma_N < \frac{1}{2}$ can be proved similarly. Since $u_N(t) \geq u_N(\frac{1}{4})$ for $t \in [\frac{1}{4}, \sigma_N]$,

$$
\begin{aligned}
u_N\left(\frac{1}{4}\right) &= \int_0^{\frac{1}{4}} \frac{1}{c(s)} \varphi^{-1}\left(\frac{1}{d(s)} \int_s^{\sigma} F(\lambda_N, u_N)(\tau) d\tau\right) ds \\
&\geq \frac{1}{\|c\|_\infty} \int_0^{\frac{1}{4}} \varphi^{-1}\left(\int_{\frac{1}{4}}^{\frac{1}{2}} h(\tau) d\tau \|d\|_\infty^{-1} l_1 C^* \varphi(u_N\left(\frac{1}{4}\right))\right) ds \\
&\geq \frac{1}{4\|c\|_\infty} \psi_2^{-1}\left(\frac{h_0 l_1 C^*}{4\|d\|_\infty}\right) u_N\left(\frac{1}{4}\right),
\end{aligned}
$$

which contradicts the choice of C^*, and thus the proof is complete. □

Remark 4. *Assume that $(F3)$ holds. Since $\lim_{s\to\infty} \min_{t\in[0,1]} \dfrac{f(t,s)}{\varphi(s)} = \infty$, there exists $M > 0$ such that $f(t,s) > \varphi(s)$ for all $(t,s) \in [0,1] \times [M, \infty)$. By the positivity of $f(t,s)$, there exists $\hat{C} \in (0,1)$ such that*

$$
f(t,s) \geq \hat{C}\varphi(s) \text{ for all } (t,s) \in [0,1] \times [0, M].
$$

Consequently, $f(t,s) \geq \hat{C}\varphi(s)$ for all $(t,s) \in [0,1] \times \mathbb{R}_+$. Thus $(F3)$ implies $(F2)$.

Now we give the third main result in this paper.

Theorem 4. *Assume that $(A), (F3)$ and $h \in \mathcal{H}_\varphi$ hold. Then there exists $\lambda_* > 0$ such that (1) has two positive solutions for $\lambda \in (0, \lambda_*)$, at least one positive solution for $\lambda = \lambda_*$ and no positive solution for $\lambda > \lambda_*$. Moreover, for $\lambda \in (0, \lambda_*)$, two positive solutions u_λ^1 and u_λ^2 can be chosen so that $\|u_\lambda^1\|_\infty \to 0$ and $\|u_\lambda^2\|_\infty \to \infty$ as $\lambda \to 0^+$.*

Proof. Let $\lambda_* := \sup\{\hat{\lambda} > 0 : (1) \text{ has two positive solutions for all } \lambda \in (0, \hat{\lambda})\}$. Then, by Proposition 1, Lemmas 7 and 8, $\lambda_* \in (0, \infty)$ is well-defined. Indeed, let $\{(\lambda_n, u_n)\}$ be a sequence in the unbounded solution component $\mathcal{C}$ defined in Proposition 1 satisfying $\lambda_n + \|u_n\|_\infty \to \infty$ as $n \to \infty$. By Lemma 7, $\lambda_n \leq \bar{\lambda}$, and $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Then, by Lemma 8, $\lambda_n \to 0$ as $n \to \infty$. Thus the shape of the continuum of $\mathcal{C}$ is determined. Consequently, (1) has two positive solutions u_λ^1, u_λ^2 for all small $\lambda > 0$ such that $\|u_\lambda^1\|_\infty \to 0$ and $\|u_\lambda^2\|_\infty \to \infty$ as $\lambda \to 0^+$, and it has no positive solution for all large $\lambda > 0$. Thus $\lambda_* \in (0, \infty)$ is well-defined.

By the choice of λ_*, (1) has at least two positive solutions for $\lambda \in (0, \lambda_*)$, and, by the complete continuity of H and Lemma 8, it has at least one positive solution for $\lambda = \lambda_*$. By the same argument as in the proof of Reference [6] (Theorem 1.1), (1) has no positive solution for $\lambda > \lambda_*$, and thus the proof is complete. □

Finally, we give a few examples which illustrates the assumptions in the main results.

Example 1. *Let φ be an odd function satisfying $\varphi(x) = x + x^2$ for $x \in \mathbb{R}_+$. It is easy to check that (A) is satisfied for $\psi_1(y) = \min\{y, y^2\}$ and $\psi_2(y) = \max\{y, y^2\}$. Let $h : (0,1) \to (0, \infty)$ be a function defined by $h(t) = t^{-a}(1-t)^{-b}$ for $t \in (0,1)$. It is easy to see that $h \in \mathcal{H}_{\psi_1} \setminus L^1(0,1)$ for any $a, b \in [1,2)$ and $h \in \mathcal{H}_\varphi \setminus L^1(0,1)$ for any $a, b \in [1,3)$.*

Finally, we give some examples of $f = f(t,s)$ satisfying the assumptions in the main results.

(1) *Let $f(t,s) = \max\{0, s(1-s)\}$ for $(t,s) \in [0,1] \times \mathbb{R}_+$. Clearly, $(F0)$ and $(F1)'$ are satisfied.*
(2) *Let $f(t,s)$ be any nonnegative continuous function satisfying*

$$f(t,s) = 1 \text{ for } (t,s) \in \left[\frac{s+1}{s+2}, \frac{s+2}{s+3}\right] \times \mathbb{R}_+$$

and

$$f(t,s) \leq [\varphi(s)]^{\frac{1}{2}} + 1 \text{ (resp., } f(t,s) \leq [\varphi(s)]^{\frac{1}{3}} + 1) \text{ for } (t,s) \in [0,1] \times \mathbb{R}_+.$$

Then $(F0)'$ *is satisfied for* $(\alpha_M, \beta_M) = \left(\frac{s+1}{s+2}, \frac{s+2}{s+3}\right)$ *and* $(F1)$ *(resp.,* $(F1)'$*) is satisfied.*

(3) *Let* $f(t,s)$ *be any nonnegative continuous function satisfying*

$$f(t,s) = (1+t)\varphi(s) + 1 \text{ for } (t,s) \in \left[\frac{1}{4}, \frac{3}{4}\right] \times \mathbb{R}_+ \text{ and } f(t,s) \leq 2\varphi(s) + 1 \text{ for } (t,s) \in [0,1] \times \mathbb{R}_+.$$

Then $(F2)$ *is satisfied for* $\hat{C} = \frac{5}{4}$ *and* $(\alpha, \beta) = (\frac{1}{4}, \frac{3}{4})$*, but* $(F3)$ *does not hold, since* $\lim_{s\to\infty} \min_{t\in[0,1]} \dfrac{f(t,s)}{\varphi(s)} \leq 2.$

(4) *Let* $f(t,s) = e^s$ *or* $f(t,s) = 1 + (\sin t + 2 + s)\varphi(s)$ *for* $(t,s) \in [0,1] \times \mathbb{R}_+$*. Then* $(F3)$ *is satisfied.*

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