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Conformal and Geodesic Mappings onto Some Special Spaces

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Abstract: In this paper, we consider conformal mappings of Riemannian spaces onto Ricci-2-symmetric Riemannian spaces and geodesic mappings of spaces with affine connections onto Ricci-2-symmetric spaces. The main equations for the mappings are obtained as a closed system of Cauchy-type differential equations in covariant derivatives. We find the number of essential parameters which the solution of the system depends on. A similar approach was applied for the case of conformal mappings of Riemannian spaces onto Ricci-m-symmetric Riemannian spaces, as well as geodesic mappings of spaces with affine connections onto Ricci-m-symmetric spaces.

Keywords: space with affine connection; riemannian space; ricci-m-symmetric space; conformal mapping; geodesic mapping

MSC: 53B05; 53B50; 35M10

1. Introduction

Historically, conformal mappings have been considered in many monographs, surveys, and papers. The theory of conformal mappings has very important applications in general relativity (e.g., [1–5]).

The question of whether a Riemannian space admits a conformal mapping onto some Einstein space was addressed by Brinkmann [6] and reduced to the question of whether solutions of some nonlinear system of Cauchy-type PDEs with respect to unknown functions exist. This subject was set out in a monograph written by Petrov [3].

In the papers [7,8], the main equations of the above said mappings were reduced to a linear system of differential equations in covariant derivatives. The mobility degree, with respect to conformal mappings onto Einstein spaces, was also found.

In [8], the authors found an estimation of the first lacuna in a distribution of degree of Riemannian space mobility groups, with respect to conformal mappings, onto Einstein spaces. It was proved in [8] that, with respect to the conformal mappings, the maximal degree of mobility was admitted by conformal flat spaces, and only by them. The paper presents a criterion in tensor form for Riemannian spaces, different to conformally Euclidean ones, for which the maximal degree of mobility $r = n - 1$, where n is the dimension of the spaces $(n > 2)$. Hence, the estimation of the first lacuna in a distribution of degree of mobility was obtained, and the spaces with the maximal degree of mobility, different to conformally Euclidean ones, were distinguished.

In the above said explorations, it was supposed that all geometric objects under consideration belonged to a sufficiently high class of smoothness.

The paper [7] presents the minimal conditions on the differentiability of objects under consideration to be satisfied by conformal mappings of Riemannian spaces onto Einstein spaces. The main equations for the mappings are obtained as a closed system of Cauchy-type differential equations in covariant derivatives, taking into account the minimal requirements on the differentiability of metrics of spaces which are conformally equivalent.

The paper [9] is devoted to conformal mappings of Riemannian spaces onto Ricci-symmetric spaces. The main equations for the mappings were reduced to a closed system of Cauchy-type differential equations in covariant derivatives. The authors also found the number of essential parameters on which the solution of the system depends on. It is worth noting that the system is nonlinear.

In a series of papers [10–12], Kaigorodov studied two-symmetric spaces and their generalizations. He discovered several examples of two-symmetric spaces. Obviously, every two-symmetric space is Ricci-two-symmetric. On the other hand, any (pseudo-) Riemannian space admits nontrivial conformal mappings. Hence, it is not hard to get a new (pseudo-) Riemannian space which conforms to the initial Ricci-two-symmetric space. Here, we consider the converse problem: given a (pseudo-) Riemannian space, how we could find conformal mappings onto Ricci-two-symmetric spaces.

The theory goes back to the paper [13] of Levi-Civita, in which the problem on the search for Riemannian spaces with common geodesics was stated and solved in a special coordinate system. Regarding this, we note a remarkable fact—that this problem is related to the study of equations of dynamics of mechanical systems.

The theory of geodesic mappings was developed by Thomas, Weyl, Shirokov, Solodovnikov, Sinyukov, Mikeš, and others [2,3,14–16].

The best-known equations are the Levi-Civita equations obtained by Levi-Civita himself for the case of Riemannian spaces. Later, Weyl obtained the same equations for geodesic mappings between spaces with affine connections.

Sinyukov [16] has proved that the main equations for geodesic mappings of (pseudo-)Riemannian spaces are equivalent to some linear system of Cauchy-type differential equations in covariant derivatives.

The paper [17] extends the results to the case of geodesic mappings of equiaffine spaces with affine connections onto (pseudo-) Riemannian spaces. Geodesic mappings of generalized symmetric and recurrent (pseudo-) Riemannian spaces were studied by Mikeš [18].

In the paper [19], the authors proved that the main equations of geodesic mappings of spaces with affine connections onto Ricci-symmetric spaces were equivalent to some system of Cauchy-type differential equations in covariant derivatives. In this paper, the main equations for conformal mappings of Riemannian spaces onto Ricci-2-symmetric Riemannian spaces and geodesic mappings of spaces with affine connections onto Ricci-2-symmetric spaces are obtained as closed-system Cauchy-type differential equations in covariant derivatives. We find the number of essential parameters which the solution of the system depends on, and the obtained results are extended for the case of conformal mappings of Riemannian spaces onto Ricci-m-symmetric Riemannian spaces and geodesic mappings of spaces with affine connections onto Ricci-m-symmetric spaces. We suppose, throughout the paper, that all geometric objects under consideration are continuous and sufficiently smooth.

2. Basic Concepts of Conformal Mappings

Let us consider the conformal mapping f of a Riemannian space V_n with the metric tensor g onto a Riemannian space $\overline{V}_n$ with the metric tensor $\overline{g}$. Note that both spaces V_n and $\overline{V}_n$ are based on the same smooth manifold.

Let us suppose that the Riemannian spaces V_n and $\overline{V}_n$ are referred to a common coordinate system $x^1, x^2, \dots, x^n$ relative to a mapping—see [14], p. 181.

A diffeomorphism $f: \quad V_n \longrightarrow \overline{V}_n$ is called *conformal mapping* if, in a common coordinate system, $x^1, x^2, \ldots, x^n$ is relative to the mapping of their metric tensors g and $\overline{g}$ are proportional, and the components of the tensors are in the relation

$$\overline{g}_{ij}(x) = e^{2\psi(x)} \cdot g_{ij}(x), \tag{1}$$

where $\psi(x)$ is a function of the x's—see, for example, [16], p. 68.

From (1), it follows that conformal mappings preserve angles between tangent vectors of any pair of curves. Conformal mappings are completely characterized by that property.

From (1), it also follows that relations between the Christoffel symbols formed with respect to the two metric tensors are given by

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j(x) + \delta^h_j \psi_i(x) - \psi^h(x) \cdot g_{ij}(x), \tag{2}$$

where $\psi_i = \dfrac{\partial \psi}{\partial x^i}$ are partial derivatives of ψ with respect to x^i, $\psi^h(x) = g^{h\alpha}\psi_\alpha$, g^{ij} are components of the inverse matrix to g_{ij}, and δ^h_i is the Kronecker delta.

A conformal mapping is called *homothetic* if the function $\psi(x)$ is a constant—that is, $\overline{g}_{ij}(x) = c \cdot g_{ij}(x)$ (cf. e.g., [14], p. 198). The condition is equivalent to $\psi_i(x) = 0$; hence, the mapping is also an affine one.

Let us recall that in the Riemannian space V_n with a metric tensor $g_{ij}(x)$, the Riemann tensor, Ricci tensor, and scalar curvature are defined by the metric tensor as follows:

$$R^h_{ijk} = \frac{\partial \Gamma^h_{ik}}{\partial x^j} - \frac{\partial \Gamma^h_{ij}}{\partial x^k} + \Gamma^\alpha_{ik}\Gamma^h_{\alpha j} - \Gamma^\alpha_{ij}\Gamma^h_{\alpha k}, \qquad R_{ij} = R^\alpha_{ij\alpha}, \qquad R = R_{\alpha\beta} g^{\alpha\beta}.$$

It is known [2,3,14,16] that under conformal mappings, a relationship between the Riemann tensors is presented by the formulas:

$$\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik} + g_{ij}\psi^h_k - g_{ik}\psi^h_j + (\delta^h_k g_{ij} - \delta^h_j g_{ik}) \triangle_1 \psi, \tag{3}$$

where $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$, $\psi^h_k = g^{h\alpha}\psi_{\alpha k}$, $\triangle_1 \psi = g^{\alpha\beta}\psi_\alpha \psi_\beta$, and the symbol "," denotes the covariant derivative with respect to the metric tensor of a space V_n.

Contracting the equations (3) for h and k in the reduction, we get

$$\psi_{i,j} = \frac{\mu}{n-2} g_{ij} + \psi_i \psi_j - \frac{1}{n-2}\left(\overline{R}_{ij} - R_{ij}\right), \tag{4}$$

where μ is a certain invariant.

3. Conformal Mappings of Riemannian Spaces onto Ricci-2-Symmetric Riemannian Spaces

A space $\overline{A}_n$ with affine connection (Riemannian space $\overline{V}_n$) is called Ricci-m-symmetric if its Ricci tensor $\overline{R}_{ij}$ satisfies the condition

$$\overline{R}_{ij|k_1 k_2 \ldots k_m} = 0, \tag{5}$$

where the symbol "$|$" denotes a covariant derivative with respect to the connection of the space $\overline{A}_n$ (cf. e.g., [10–12,14], p. 338). In particular, for the case of Ricci-2-symmetric spaces, (5) is written as follows:

$$\overline{R}_{ij|km} = 0. \tag{6}$$

Let us consider conformal mappings of a Riemannian space V_n with the metric tensor g onto some Ricci-2-symmetric Riemannian space $\overline{V}_n$ with the metric tensor $\overline{g}$. If the spaces V_n and $\overline{V}_n$ are referred to the common coordinate system $x = (x^1, x^2, \ldots, x^n)$, then we get

$$\overline{R}_{ij|k} = \overline{R}_{ij,k} - P^{\alpha}_{ki}\overline{R}_{\alpha j} - P^{\alpha}_{kj}\overline{R}_{i\alpha}. \tag{7}$$

Taking account of (2), it follows from (7) that

$$\overline{R}_{ij|k} = \overline{R}_{ij,k} - \psi_i\overline{R}_{kj} - \psi_j\overline{R}_{ik} - 2\psi_k\overline{R}_{ij} + \psi^{\alpha}g_{ik}\overline{R}_{\alpha j} + \psi^{\alpha}g_{jk}\overline{R}_{i\alpha}. \tag{8}$$

Differentiating (8) with respect to x^m in the space V_n and taking into account $\psi^{\alpha}_{,m} = g^{\alpha\beta}\psi_{\beta,m}$, we obtain

$$\begin{aligned}\left(\overline{R}_{ij|k}\right)_{,m} = {}& \overline{R}_{ij,km} - \psi_{i,m}\overline{R}_{kj} - \psi_i\overline{R}_{kj,m} - \psi_{j,m}\overline{R}_{ik} - \psi_j\overline{R}_{ik,m} - 2\psi_{k,m}\overline{R}_{ij} - 2\psi_k\overline{R}_{ij,m}+\\ &+ g^{\alpha\beta}g_{ik}\overline{R}_{\alpha j}\psi_{\beta,m} + \psi^{\alpha}g_{ik}\overline{R}_{\alpha j,m} + g^{\alpha\beta}g_{jk}\overline{R}_{i\alpha}\psi_{\beta,m} + \psi^{\alpha}g_{jk}\overline{R}_{i\alpha,m}\,.\end{aligned} \tag{9}$$

According to the definition of a covariant derivative, we get

$$\left(\overline{R}_{ij|k}\right)_{,m} = \overline{R}_{ij|km} + P^{\alpha}_{mi}\overline{R}_{\alpha j|k} + P^{\alpha}_{mj}\overline{R}_{i\alpha|k} + P^{\alpha}_{mk}\overline{R}_{ij|\alpha}\,. \tag{10}$$

Taking account of (9) and (10), we have

$$\begin{aligned}\overline{R}_{ij|km} = {}& \overline{R}_{ij,km} - \psi_{i,m}\overline{R}_{kj} - \psi_i\overline{R}_{kj,m} - \psi_{j,m}\overline{R}_{ik} - \psi_j\overline{R}_{ik,m} - 2\psi_{k,m}\overline{R}_{ij} - 2\psi_k\overline{R}_{ij,m} + g^{\alpha\beta}g_{ik}\overline{R}_{\alpha j}\psi_{\beta,m}+\\ &+ \psi^{\alpha}g_{ik}\overline{R}_{\alpha j,m} + g^{\alpha\beta}g_{jk}\overline{R}_{i\alpha}\psi_{\beta,m} + \psi^{\alpha}g_{jk}\overline{R}_{i\alpha,m} - P^{\alpha}_{mi}\overline{R}_{\alpha j|k} - P^{\alpha}_{mj}\overline{R}_{i\alpha|k} - P^{\alpha}_{mk}\overline{R}_{ij|\alpha}\,.\end{aligned} \tag{11}$$

We introduce the tensor $\overline{R}_{ijk}$ defined by

$$\overline{R}_{ij,k} = \overline{R}_{ijk}\,. \tag{12}$$

Since the space $\overline{V}_n$ is Ricci-2-symmetric (i.e., (6) holds), it follows from (11) that

$$\begin{aligned}\overline{R}_{ijk,m} = {}& \theta_{im}\overline{R}_{kj} + \psi_i\overline{R}_{kjm} + \theta_{jm}\overline{R}_{ik} + \psi_j\overline{R}_{ikm} + 2\theta_{km}\overline{R}_{ij} + 2\psi_k\overline{R}_{ijm} - g^{\alpha\beta}\theta_{\beta m}(g_{ik}\overline{R}_{\alpha j} + g_{jk}\overline{R}_{i\alpha})-\\ &- \psi^{\alpha}(g_{ik}\overline{R}_{\alpha jm} + g_{jk}\overline{R}_{i\alpha m}) + \theta^{\alpha}_{mi}\theta_{\alpha jk} + \theta^{\alpha}_{mj}\theta_{i\alpha k} + \theta^{\alpha}_{mk}\theta_{ij\alpha}\,,\end{aligned} \tag{13}$$

where

$$\begin{aligned}&\theta_{ij} = \frac{\mu}{n-2}\,g_{ij} + \psi_i\psi_j - \frac{1}{n-2}\left(\overline{R}_{ij} - R_{ij}\right),\\ &\theta_{ijk} = \overline{R}_{ijk} - \psi_i\overline{R}_{kj} - \psi_j\overline{R}_{ik} - 2\psi_k\overline{R}_{ij} + \psi^{\alpha}g_{ik}\overline{R}_{\alpha j} + \psi^{\alpha}g_{jk}\overline{R}_{i\alpha},\\ &\theta^h_{ij} = \delta^h_i\psi_j + \delta^h_j\psi_i - \psi^h g_{ij}.\end{aligned}$$

Let us differentiate (4) with respect to x^k in the space V_n and alternate the obtained result in j and k. In view of the Ricci identity and the fact that the Ricci tensor is symmetric, we get

$$\begin{aligned}(n-2)\psi_{\alpha}R^{\alpha}_{ijk} = {}& -g_{ij}\mu_{,k} + g_{ik}\mu_{,j} - g^{\alpha\beta}\psi_{\alpha}\left(g_{ik}\overline{R}_{\beta j} - g_{ij}\overline{R}_{\beta k}\right)+\\ &+ R_{ik,j} - R_{ij,k} + R_{ij}\psi_k - R_{ik}\psi_j + \mu(g_{ij}\psi_k - g_{ik}\psi_j).\end{aligned} \tag{14}$$

Let us multiply (14) by g^{ij} and contract for l and j. According to the Voss-Weyl formula $R_{ij,k}g^{jk} = \frac{1}{2}R_{,i}$, we obtain

$$(n-1)\mu_{,k} = g^{\alpha\beta}\left[(n-2)\psi_{\gamma}R^{\gamma}_{\beta k\alpha} - (n-1)\psi_{\beta}\overline{R}_{\alpha k} - \psi_{\beta}R_{\alpha k}\right] + [R + (n-1)\mu]\psi_k - \frac{1}{2}R_{,k}\,. \tag{15}$$

Also, we have the notation

$$\psi_i = \psi_{,i}\,. \tag{16}$$

Obviously, in the space $\overline{V}_n$, the Equations (4), (12), (13), (15), and (16) form a closed mixed system of Cauchy-type PDEs with respect to functions $\psi(x)$, $\psi_i(x)$, $\mu(x)$, $\overline{R}_{ij}(x)$, and $\overline{R}_{ijk}(x)$, and the functions $\overline{R}_{ij}(x)$ must satisfy the algebraic conditions $\overline{R}_{ij}(x) = \overline{R}_{ij}(x)$. Hence, we have proved:

Theorem 1. *In order for a Riemannian space V_n to admit a conformal mapping onto a Ricci-2-symmetric space $\overline{V}_n$, it is necessary and sufficient that the mixed Cauchy-type system of differential equations in covariant derivatives* (4), (12), (13), (15), *and* (16) *has a solution with respect to functions $\psi(x)$, $\psi_i(x)$, $\mu(x)$, $\overline{R}_{ijk}(x)$, and $\overline{R}_{ij}(x)$ $(= \overline{R}_{ij}(x))$.*

It is obvious that the general solution of the mixed Cauchy-type system depends on no more than $\frac{1}{2}n \cdot (n+1)^2 + n + 2$ essential parameters.

It is easy to see that if we differentiate covariantly with respect to x^ρ in the space V_n, and taking into account the definition of a covariant derivative writing expression for the tensor $(\overline{R}_{ij|km})_{,\rho}$, then we have obtained the expression for the covariant derivative of $\overline{R}_{ij|km\rho}$ through $\overline{R}_{ij,km\rho}$.

Hence, in the case when the space $\overline{V}_n$ is Ricci-3-symmetric, the main equations for the mapping can be written in the form of a closed, Cauchy-type system of equations in covariant derivatives.

Obviously, continuing this way, it is readily shown that the main equations for conformal mappings of Riemannian spaces onto Ricci-m-symmetric spaces can also be presented as a closed system of Cauchy-type equations in covariant derivatives.

4. Basic Concepts of Geodesic Mappings

A curve is called *geodesic* if the tangent vector field along the curve is parallel along the curve (see, e.g., [16], p. 43).

We say that a diffeomorphism $f\colon A_n \to \overline{A}_n$ is a *geodesic mapping* if any geodesic curve of A_n is mapped under f onto a geodesic curve in $\overline{A}_n$ (see [16], pp. 70–76), where A_n and $\overline{A}_n$ are manifolds with the affine connection ∇ and $\overline{\nabla}$, respectively.

According to [3,14–16], a necessary and sufficient condition for the mapping f of a space A_n onto a space $\overline{A}_n$ to be geodesic is that in the common coordinate system $x^1, x^2, \ldots, x^n$ the *deformation tensor* $P^h_{ij}(x)$ of the mapping f

$$P^h_{ij}(x) = \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x), \tag{17}$$

which has to satisfy the condition

$$P^h_{ij}(x) = \psi_i(x)\delta^h_j + \psi_j(x)\delta^h_i. \tag{18}$$

The symbols $\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are components of affine connections of the spaces A_n and $\overline{A}_n$, respectively, where $\psi_i(x)$ are components of a covariant vector.

A geodesic mapping is called *nontrivial* if $\psi_i(x) \neq 0$. It is obvious that any space A_n with an affine connection admits a nontrivial geodesic mapping onto some space $\overline{A}_n$ with an affine connection. It is not hard to construct an example of nontrivial geodesic mapping. Let A_n be a space with affine connection Γ^h_{ij}. Determininig in A_n an arbitrary vector field ψ_i, we construct the nontrivial geodesic mapping of the space A_n onto a space $\overline{A}_n$ with affine connection $\overline{\Gamma}^h_{ij}$. Using (17) and (18), we can calculate the components of $\overline{\Gamma}^h_{ij}$. However, in general, similar ideas concerning geodesic mappings of Riemannian spaces onto Riemannian spaces are wrong. In particular, there are Riemannian spaces that do not admit nontrivial geodesic mappings onto Riemannian spaces.

5. Geodesic Mappings of Spaces with Affine Connections onto Ricci-2-Symmetric Spaces

Let us consider the geodesic mapping of a space A_n with an affine connection onto a Ricci-2-symmetric space $\overline{A}_n$. Yet, in general, spaces with an affine connection, and especially Ricci-2-symmetric spaces are not (pseudo-)Riemannian spaces.

Suppose that the spaces A_n and $\overline{A}_n$ are referred to a coordinate system common to the mapping.

One knows [14,16] that a relationship between the Riemann tensors R^h_{ijk} and $\overline{R}^h_{ijk}$ of the spaces A_n and $\overline{A}_n$, respectively, is presented by the formulas:

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik} P^h_{\alpha j} - P^\alpha_{ij} P^h_{\alpha k} \,. \tag{19}$$

Taking into account that a deformation tensor P^h_{ij} of the connections is defined by (18), it follows from (19) that

$$\overline{R}^h_{ijk} = R^h_{ijk} - \delta^h_j \psi_{i,k} + \delta^h_k \psi_{i,j} - \delta^h_i \psi_{j,k} + \delta^h_i \psi_{k,j} + \delta^h_j \psi_i \psi_k - \delta^h_k \psi_i \psi_j \,. \tag{20}$$

Contracting the equations (20) for h and k, we get

$$\overline{R}_{ij} = R_{ij} + n\psi_{i,j} - \psi_{j,i} + (1-n)\psi_i \psi_j \,. \tag{21}$$

Alternating (21) with respect to the indices i and j, we obtain

$$\overline{R}_{[ij]} = R_{[ij]} + (n+1)\psi_{i,j} - (n+1)\psi_{j,i} \,. \tag{22}$$

Here, we denote by the brackets $[ij]$ an operation called antisymmetrization (or alternation) without division with respect to the indices i and j. Taking account of (21), from (22) it follows that

$$\psi_{i,j} = \frac{1}{n^2-1} \left[n\overline{R}_{ij} + \overline{R}_{ji} - (nR_{ij} + R_{ji}) \right] + \psi_i \psi_j \,. \tag{23}$$

Using the relation (7) and taking into account that the deformation tensor is defined by (18), we find

$$\overline{R}_{ij|k} = \overline{R}_{ij,k} - 2\psi_k \overline{R}_{ij} - \psi_i \overline{R}_{kj} - \psi_j \overline{R}_{ik} \,. \tag{24}$$

Differentiating (24) with respect to x^m in the space A_n, we obtain

$$(\overline{R}_{ij|k})_{,m} = \overline{R}_{ij,km} - 2\psi_{k,m}\overline{R}_{ij} - 2\psi_k \overline{R}_{ij,m} - \psi_{i,m}\overline{R}_{kj} - \psi_i \overline{R}_{kj,m} - \psi_{j,m}\overline{R}_{ik} - \psi_j \overline{R}_{ik,m} \,. \tag{25}$$

Taking account of the formulas (10) and (18), from (25) it follows that

$$\begin{aligned} \overline{R}_{ij|km} = {} & \overline{R}_{ij,km} - 2\psi_{k,m}\overline{R}_{ij} - 2\psi_k \overline{R}_{ij,m} - \psi_{i,m}\overline{R}_{kj} - \psi_i \overline{R}_{kj,m} - \psi_{j,m}\overline{R}_{ik} - \psi_j \overline{R}_{ik,m} - \psi_i \overline{R}_{mj|k} - \\ & - 3\psi_m \overline{R}_{ij|k} - \psi_j \overline{R}_{im|k} - \psi_k \overline{R}_{ij|m} \,. \end{aligned} \tag{26}$$

Suppose that the space $\overline{A}_n$ is Ricci-2-symmetric. Then, taking account of (12), (23), and (24), we have from (26) that

$$\begin{aligned} \overline{R}_{ijk,m} = {} & 2\rho_{km}\overline{R}_{ij} + 2\psi_k \overline{R}_{ijm} + \rho_{im}\overline{R}_{kj} + \psi_i \overline{R}_{kjm} + \rho_{jm}\overline{R}_{ik} + \psi_j \overline{R}_{ikm} + \\ & + \psi_i \rho_{mjk} + 3\psi_m \rho_{ijk} + \psi_j \rho_{imk} + \psi_k \rho_{ijm} \,, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \rho_{ij} = {} & \frac{1}{n^2-1}\left(n\overline{R}_{ij} + \overline{R}_{ji} - \left(R_{ij} + R_{ji}\right)\right) + \psi_i \psi_j, \\ \rho_{ijk} = {} & \overline{R}_{ijk} - 2\psi_k \overline{R}_{ij} - \psi_i \overline{R}_{kj} - \psi_j \overline{R}_{ik}. \end{aligned}$$

Obviously, in the space $\overline{A}_n$, the Equations (12), (27), and (23) form a closed system of Cauchy-type differential equations in covariant derivatives with respect to functions $\psi_i(x)$, $\overline{R}_{ij}(x)$ and $\overline{R}_{ijk}(x)$. Hence, we have proved that:

Theorem 2. *In order for a space A_n with an affine connection to admit geodesic mapping onto a Ricci-2-symmetric space $\overline{A}_n$, it is necessary and sufficient that the closed system of Cauchy-type differential* *equations in covariant derivatives* (12), (27), *and* (23) *has a solution with respect to functions* $\psi_i(x)$, $\overline{R}_{ij}(x)$ *and* $\overline{R}_{ijk}(x)$.

The general solution of the closed system of Cauchy-type differential equations in covariant derivatives (12), (27), and (23) depends on no more than $\frac{1}{2}n \cdot (n+1)^2 + n$ essential parameters.

It is obvious that, similarly to the case of conformal mappings, the main equations for geodesic mappings of spaces with affine connections onto a Ricci-m-symmetric space could be obtained in the form of a closed system of Cauchy-type equations in covariant derivatives.

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References

1. Denisov, V.I. Special conformal mappings in general relativity. *J. Soviet Math.* **1990**, *48*, 36–40. [CrossRef]
2. Eisenhart, L.P. Non-Riemannian geometry. In *Reprint of the 1927 Original. American Mathematical Society Colloquium Publications*; American Mathematical Society: Providence, RI, USA, 1990; Volume 8, p. viii+184, ISBN 0-8218-1008-1.
3. Petrov, A.Z. *New Methods in the General Theory of Relativity*; Nauka: Moscow, Russia, 1966; p. 496.
4. Schouten, J.A.; Struik, D.J. *Einfuhrung in Die Neueren Methoden der Differentialgeometrie. B. 1*; Noordhoff: Groningen, The Netherlands, 1935.
5. Schouten. J.A.; Struik, D.J. *Einfuhrung in Die Neueren Methoden der Differentialgeometrie. B. 2*; Noordhoff: Groningen, The Netherlands, 1938.
6. Brinkmann, H.W. Einstein spaces which are mapped conformally on each other. *Math. Ann.* **1925**, *94*, 119–145. [CrossRef]
7. Evtushik, L.E.; Hinterleitner, I.; Guseva, N.I.; Mikeš, J. Conformal mappings onto Einstein spaces. *Russ. Math.* **2016**, *60*, 5–9. [CrossRef]
8. Mikeš, J.; Gavril'chenko, M.L.; Gladysheva, E.I. On Conformal Mappings onto Einstein Spaces. *Vestnik Moscow Univ.* **1994**, *3*, 13–17.
9. Berezovskii, V.E.; Hinterleitner, I.; Guseva, N.I.; Mikeš, J. Conformal Mappings of Riemannian Spaces onto Ricci Symmetric Spaces. *Math. Notes* **2018**, *103*, 304–307. [CrossRef]
10. Kaigorodov, V.R. 2-symmetric and 2-recurrent gravitational fields. *Gravitacija i Teor. Otnositelnosti* **1973**, *9*, 25–32.
11. Kaigorodov, V.R. On Riemannian spaces D_n^m. *Tr. Geom. Semin. VINITI* **1974**, *5*, 359–373.
12. Kaigorodov, V.R. A structure of space-time curvature. *Ltogi Nauki i Tekhn. VINITI Probl. Geometrii* **1983**, *14*, 177–204. [CrossRef]
13. Lcvi-Civita, T. Sulle trasformazioni delle equazioni dinamiche. *Annali di Matematica Ser. 2* **1896**, *24*, 255–300. [CrossRef]
14. Mikeš, J.; Stepanova, E.; Vanžurová, A.; Bácsó, S.; Berezovski, V.E.; Chepurna, O.; Chodorová, M.; Chudá, H.; Gavrilchenko, M.L.; Haddad, M.; et al. *Differential Geometry of Special Mappings*; Palacky University: Olomouc, Czech Republic, 2015; p. 568, ISBN 978-80-244-4671-4.
15. Mikeš, J.; Vanžurová, A.; Hinterleitner, I. *Geodesic Mappings and Some Generalizations*; Palacky University Press: Olomouc, Czech Republic, 2009; p. 304, ISBN 978-80-244-2524-5.
16. Sinyukov, N.S. *Geodesic Mappings of Riemannian Spaces*; Nauka: Moscow, Russia, 1979; p. 256.
17. Mikeš, J.; Berezovskii, V.E. Geodesic mappings of affine-connected spaces onto Riemannian spaces. In Proceedings of the Differential Geometry and Its Applications, Eger, Hungary, 20–25 August 1989; pp. 491–494.

18. Mikeš, J. On geodesic mappings of 2-Ricci symmetric Riemannian spaces. *Math. Notes* **1980**, *28*, 622–624. [CrossRef]
19. Berezovski, V.E.; Hinterleitner, I.; Mikeš, J. Geodesic mappings of manifolds with affine connection onto the Ricci symmetric manifolds. *Filomat* **2018**, *32*, 379–385. [CrossRef]