mathematics

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Article

Appell-Type Functions and Chebyshev Polynomials

Pierpaolo Natalini [1] and Paolo Emilio Ricci [2,*]

[1] Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy
[2] Section of Mathematics, International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39, 00186 Roma, Italy
* Correspondence: paoloemilioricci@gmail.com

Received: 13 June 2019; Accepted: 26 July 2019; Published: 30 July 2019

Abstract: In a recent article we noted that the first and second kind Cebyshev polynomials can be used to separate the real from the imaginary part of the Appell polynomials. The purpose of this article is to show that the same classic polynomials can also be used to separate the even part from the odd part of the Appell polynomials and of the Appell–Bessel functions.

Keywords: Bessel functions; Appell–Bessel functions; generating functions; Chebyshev polynomials

MSC: AMS 2010 Mathematics Subject Classifications: 33C99–11B83–12E10

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1. Introduction

In a recent article [1] we noted that the first and second type Chebyshev polynomials can be used to separate the real from the imaginary part of the Appell polynomials. This is just one of the countless applications of these classic polynomials in Function theory. The purpose of this article is to highlight another of their applications, which is, in some way, analogous to the previous one. In fact everybody knows that the even and the odd part of a function $F(x)$ are derived, in a trivial form, by the equations $\frac{1}{2}[F(x)+F(-x)]$ and $\frac{1}{2}[F(x)-F(-x)]$.

However, in the case of more complicated expressions of the F function, for example in relation to the Appell-type functions, the result is not so obvious, and it will be shown here that it can be obtained using another property of the first and second kind Chebyshev polynomials.

The article is organized as follows. In the first section we use a formula, similar to that of Euler, to separate the even part from the odd part of a binomial power containing hyperbolic functions, showing a connection with the Chebyshev polynomials considered outside their orthogonality interval. Then the results are applied to the case of the Appell polynomials and, in the following sections, to the case of the first kind Bessel functions and to the recently introduced Appell–Bessel functions [2].

It is noteworthy that the study of Appell's polynomials, and of their various extensions, has been considered in both earlier [3] and recent times [4–6]. In these articles have been shown applications to difference equations, expansions in polynomial series and has been analyzed the relative internal structure.

2. Recalling the Chebyshev Polynomials

It is well known that the first kind Chebyshev polynomias $T_n(x)$ can be defined outside the interval $[-1,1]$, by using their expression in terms of hyperbolic functions [7].

Putting $x = \cosh\theta$, and $f(\theta) = \cosh\theta + \sinh\theta$, we can write:

$$\begin{aligned}
[f(\theta)]^n = (\cosh\theta + \sinh\theta)^n = &\sum_{h=0}^{[n/2]} \binom{n}{2h} [\cosh\theta]^{n-2h} ([\cosh\theta]^2 - 1)^h + \\
&+ \sinh\theta \sum_{h=0}^{[(n-1)/2]} \binom{n}{2h+1} [\cosh\theta]^{n-2h-1} ([\cosh\theta]^2 - 1)^h = \\
&= \mathcal{E}[f(\theta)]^n + \mathcal{O}[f(\theta)]^n \,,
\end{aligned} \tag{1}$$

where $\mathcal{E}[f(\theta)]^n$ and $\mathcal{O}[f(\theta)]^n$ denote the even and odd part of the $[f(\theta)]^n$ function.

Therefore, using the explicit expression of Chebyshev polynomias, we find

$$[f(\theta)]^n = (\cosh\theta + \sinh\theta)^n = T_n(\cosh\theta) + \sinh\theta\, U_{n-1}(\cosh\theta)\,, \tag{2}$$

and

$$\mathcal{E}[f(\theta)]^n = T_n(\cosh\theta)\,, \qquad \mathcal{O}[f(\theta)]^n = \sinh\theta\, U_{n-1}(\cosh\theta)\,. \tag{3}$$

Equations (2) and (3) can be interpreted as an Euler-type formula, owing the analogy with the classical one:

$$\begin{aligned}
[\exp(\mathrm{i}\,\theta)]^n = (\cos\theta + \mathrm{i}\,\sin\theta)^n = &\sum_{h=0}^{[n/2]} \binom{n}{2h} [\cos\theta]^{n-2h} ([\cos\theta]^2 - 1)^h + \\
&+ \mathrm{i}\,\sin\theta \sum_{h=0}^{[(n-1)/2]} \binom{n}{2h+1} [\cos\theta]^{n-2h-1} ([\cos\theta]^2 - 1)^h = \\
&= \Re[(\exp(\mathrm{i}\,\theta))^n] + \mathrm{i}\,\Im[(\exp(\mathrm{i}\,\theta))^n]\,.
\end{aligned} \tag{4}$$

Consequences of the Euler-Type Formula

Using the expansions proven in [8] (but in the case of hyperbolic functions), we find:

Theorem 1. *The Taylor expansions hold:*

$$\begin{aligned}
e^{x\,\tau} \cosh(y\,\tau) &= \sum_{n=0}^{\infty} \tilde{C}_n(x,y)\, \frac{\tau^n}{n!}\,, \\
e^{x\,\tau} \sinh(y\,\tau) &= \sum_{n=0}^{\infty} \tilde{S}_n(x,y)\, \frac{\tau^n}{n!}\,,
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
\tilde{C}_n(x,y) &= \sum_{j=0}^{[n/2]} \binom{h}{2j} x^{n-2j}\, y^{2j} \\
\tilde{S}_n(x,y) &= \sum_{j=0}^{[(n-1)/2]} \binom{h}{2j+1} x^{n-2j-1}\, y^{2j+1}\,.
\end{aligned} \tag{6}$$

Proof. The result follows by using the product of series:
I. Cauchy product involving an even function.

$$\sum_{k=0}^{\infty} c_k \frac{\tau^k}{k!} \sum_{k=0}^{\infty} d_k \frac{\tau^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \left[\sum_{h=0}^{\left[\frac{k}{2}\right]} \binom{k}{2h} c_{k-2h}\, d_h \right] \frac{\tau^k}{k!}. \tag{7}$$

II. Cauchy product involving an odd function.

$$\sum_{k=0}^{\infty} a_k \frac{\tau^k}{k!} \sum_{k=0}^{\infty} b_k \frac{\tau^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \left[\sum_{h=0}^{\left[\frac{k-1}{2}\right]} \binom{k}{2h+1} a_{k-2h-1}\, b_h \right] \frac{\tau^k}{k!}. \tag{8}$$

As a consequence of Equations (5) and (6) we find:

$$e^{x\,\tau}\left[\cosh(y\,\tau) + \sinh(y\,\tau)\right] = e^{(x+y)\,\tau} = \sum_{n=0}^{\infty} \left[\tilde{C}_n(x,y) + \tilde{S}_n(x,y)\right] \frac{\tau^n}{n!}, \tag{9}$$

and putting $x = \cosh\theta$, $y = \sinh\theta$

$$\begin{aligned} \exp[(\cosh\theta + \sinh\theta)\,\tau] &= \sum_{n=0}^{\infty} (\cosh\theta + \sinh\theta)^n \frac{\tau^n}{n!} = \\ = \sum_{n=0}^{\infty} &\left[\tilde{C}_n(\cosh\theta, \sinh\theta) + \tilde{S}_n(\cosh\theta, \sinh\theta)\right] \frac{\tau^n}{n!}. \end{aligned} \tag{10}$$

Therefore, we conclude that

$$\begin{aligned} \tilde{C}_n(\cosh\theta, \sinh\theta) &= T_n(\cosh\theta) \\ \tilde{S}_n(\cosh\theta, \sinh\theta) &= \sinh\theta\, U_{n-1}(\cosh\theta), \end{aligned} \tag{11}$$

and furthermore:

$$\tilde{C}_n(x, \sqrt{x^2-1}) = T_n(x), \qquad \tilde{S}_n(x, \sqrt{x^2-1}) = \sqrt{x^2-1}\, U_{n-1}(x). \tag{12}$$

□

3. The Even and Odd Part of Appell Polynomials

In this section we show how to represent the even and odd part of Appell polynomials. Consider the Appell polynomials [9–11], defined by the generating function [12]

$$A(t)\, e^{x\,t} = \sum_{n=0}^{\infty} \alpha_n(x) \frac{t^n}{n!}, \tag{13}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_k \frac{t^k}{k!}. \tag{14}$$

Putting $y := \sqrt{x^2-1}$, we have

$$A(t)\, e^{(x+y)\,t} = \sum_{n=0}^{\infty} A_n(x,y) \frac{t^n}{n!}, \tag{15}$$

where

$$A_n(x,y) = A_n(x, \sqrt{x^2-1}) = \alpha_n(x+\sqrt{x^2-1})\,. \tag{16}$$

By using the Cauchy product we find:

$$A_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,(x+y)^k = \mathcal{E}[A_n(x,y)] + \mathcal{O}[A_n(x,y)]\,, \tag{17}$$

and putting $x = \cosh\theta$, $y = \sqrt{x^2-1} = \sinh\theta$,

$$A_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,(\cosh\theta + \sinh\theta)^k = \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,[\tilde{C}_k(x,y) + \tilde{S}_k(x,y)] =$$

$$= \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,[T_k(x) + y\,U_{k-1}(x)]\;.$$

Therefore, by recalling (2) and (3), we conclude that

$$\begin{aligned} \mathcal{E}[A_n(x,\sqrt{x^2-1})] &= \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,T_k(x)\,, \\ \mathcal{O}[A_n(x,\sqrt{x^2-1})] &= \sqrt{x^2-1}\,\sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,U_{k-1}(x)\,. \end{aligned} \tag{18}$$

Remark 1. *Note that Equation (18) can be applied in general, since, when $|x| > 1$, the position $x + \sqrt{x^2-1} = u$ is equivalent to $x = \frac{u^2-1}{2u}$, and $\sqrt{x^2-1} = \frac{u^2-1}{2u}$, so that Equation (13) becomes:*

$$A(t)\,e^{u\,t} = \sum_{n=0}^{\infty} \alpha_n(u)\,\frac{t^n}{n!}\,. \tag{19}$$

Therefore, we can conclude with the theorem.

Theorem 2. *The even and odd part of the Appell polynomials $\alpha_n(u)$ defined by the generating function (19) can be represented, in terms of the first and second kind Chebyshev polynomials, by the equations:*

$$\begin{aligned} \mathcal{E}[\alpha_n(u)] &= \sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,T_k\left(\tfrac{u^2+1}{2u}\right)\,, \\ \mathcal{O}[\alpha_n(u)] &= \tfrac{u^2-1}{2u}\,\sum_{k=0}^{n} \binom{n}{k} a_{n-k}\,U_{k-1}\left(\tfrac{u^2+1}{2u}\right)\,. \end{aligned} \tag{20}$$

4. 1st Kind Bessel Functions

We consider here the first kind of Bessel functions with integer order [13], defined by the generating function [12]:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\,t^n\,. \tag{21}$$

Putting, for shortness:

$$\tilde{J}_0(x) := \tfrac{1}{2}\,J_0(x)\,, \qquad \tilde{J}_k(x) := J_k(x)\,, \quad (k \geq 1)\,. \tag{22}$$

Equation (21) writes:

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} \tilde{J}_n(x)\, t^n + \sum_{n=0}^{\infty} (-1)^n \tilde{J}_n(x)\, t^{-n}\,. \tag{23}$$

Note that, using the notation

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \mathcal{E}\left[e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right] + \mathcal{O}\left[e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right], \tag{24}$$

the even and odd part must be understood with respect to both t and x, and we find:

$$\begin{aligned} \mathcal{E}\left[e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right] &= \sum_{n=0}^{\infty} \tilde{J}_{2n}(x)\, \left(t^{2n} + t^{-2n}\right), \\ \mathcal{O}\left[e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}\right] &= \sum_{n=0}^{\infty} J_{2n+1}(x)\, \left(t^{2n+1} - t^{-(2n+1)}\right). \end{aligned} \tag{25}$$

Representation by Chebyshev Polynomials

Inverting the equation $\tau = \frac{1}{2}\left(t - \frac{1}{t}\right)$, we have:

$$t = \tau + \sqrt{\tau^2 + 1}\,. \tag{26}$$

Theorem 3. *The generating function* (21) *can be represented in terms of Chebyshev polynomials by:*

$$e^{x\,\tau} = 2 \sum_{n=0}^{\infty} \tilde{J}_{2n}(x)\, T_{2n}(\tau) + 2\sqrt{\tau^2+1} \sum_{n=0}^{\infty} J_{2n+1}(x)\, U_{2n}(\tau)\,, \tag{27}$$

so that the even and odd part of the Bessel functions defined by Equation (21) *are given by*

$$\begin{aligned} \mathcal{E}\left[\sum_{k=-\infty}^{\infty} J_k(x)\, t^k\right] &= 2 \sum_{k=0}^{\infty} \tilde{J}_{2k}(x)\, T_{2k}(\tau)\,, \\ \mathcal{O}\left[\sum_{k=-\infty}^{\infty} J_k(x)\, t^k\right] &= 2\sqrt{\tau^2+1} \sum_{k=0}^{\infty} J_{2k+1}\, U_{2k}(\tau)\,. \end{aligned} \tag{28}$$

Proof. Equation (21) writes:

$$e^{x\,\tau} = \sum_{n=0}^{\infty} \tilde{J}_{2n}(x)\, \left(t^{2n} + t^{-2n}\right) + \sum_{n=0}^{\infty} J_{2n+1}(x)\, \left(t^{2n+1} - t^{-(2n+1)}\right), \tag{29}$$

so that

$$\left(t^{2n} + t^{-2n}\right) = 2\, T_{2n}(\tau)\,, \qquad \left(t^{2n+1} - t^{-(2n+1)}\right) = 2\sqrt{\tau^2+1}\, U_{2n}(\tau)\,, \tag{30}$$

and the result is proven. □

5. Appel–Bessel Functions

Several mixed-type (or hybrid) functions have been recently considered. The starting point of these type of special functions can be found in [14,15]. In this section we consider the Appel–Bessel functions introduced in [2].

Definition 1. *The Appel–Bessel functions [2] are defined by generating function:*

$$G(x,t) = A\left[\tfrac{x}{2}(t-\tfrac{1}{t})\right] \exp\left[\tfrac{x}{2}(t-\tfrac{1}{t})\right] = \sum_{k=-\infty}^{\infty} [{}_A J_k(x)]\, t^k\,, \tag{31}$$

where

$$A(\tau) = \sum_{k=0}^{\infty} a_k \, \frac{\tau^k}{k!}\,, \quad (a_0 \neq 0)\,. \tag{32}$$

Since

$$G(x,t) = \sum_{k=-\infty}^{\infty} [{}_A J_k(x)]\, t^k = \sum_{k=-\infty}^{\infty} [{}_A J_{-k}(x)]\, t^{-k}\,, \tag{33}$$

we find

$$[{}_A J_{-k}(x)] = (-1)^k\, [{}_A J_k(x)]\,. \tag{34}$$

Furthermore, from

$$G(x,-t) = G(-x,t) = \sum_{k=-\infty}^{\infty} [{}_A J_k(x)]\, (-1)^k\, t^k = \sum_{k=-\infty}^{\infty} [{}_A J_k(-x)]\, t^k\,, \tag{35}$$

we find

$$[{}_A J_k(-x)] = (-1)^k\, [{}_A J_k(x)]\,. \tag{36}$$

That is, the same symmetry properties of the ordinary 1st *kind Bessel functions still hold for the Appell–Bessel functions.*

5.1. Representation of the Appell–Bessel Functions

Even in this case we put, for shortness:

$$[{}_A \tilde{J}_0(x)] := \tfrac{1}{2}\,[{}_A J_0(x)]\,, \qquad [{}_A \tilde{J}_k(x)] := [{}_A J_k(x)]\,, \quad (k \geq 1)\,. \tag{37}$$

Theorem 4. *The generating function (31) can be represented in terms of Chebyshev polynomials by:*

$$A(x\,\tau)\, e^{x\,\tau} = 2\sum_{n=0}^{\infty} [{}_A \tilde{J}_{2n}(x)]\, T_{2n}(\tau) + 2\sqrt{\tau^2+1}\sum_{n=0}^{\infty} [{}_A J_{2n+1}(x)]\, U_{2n}(\tau)\,, \tag{38}$$

so that the even and odd part of the Appell-Bessel functions are given by

$$\begin{aligned} \mathcal{E}\left[\sum_{k=-\infty}^{\infty} [{}_A J_k(x)]\, t^k\right] &= 2\sum_{k=0}^{\infty} [{}_A \tilde{J}_{2k}(x)]\, T_{2k}(\tau)\,, \\ \mathcal{O}\left[\sum_{k=-\infty}^{\infty} [{}_A J_k(x)]\, t^k\right] &= 2\sqrt{\tau^2+1}\sum_{k=0}^{\infty} [{}_A J_{2k+1}]\, U_{2k}(\tau)\,. \end{aligned} \tag{39}$$

Proof. Using the symmetry properties (34) and (36), the same technique applied in Section 3 gives the result in the present case. □

5.2. Connection with the Appel–Bessel Functions

Theorem 5. *The following equation holds:*

$$\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h} a_{k-h}\,\frac{(x\,\tau)^k}{k!} = \\ = 2\sum_{n=0}^{\infty}[{}_A\tilde{J}_{2n}(x)]\,T_{2n}(\tau) + 2\sqrt{\tau^2+1}\,\sum_{n=0}^{\infty}[{}_A J_{2n+1}(x)]\,U_{2n}(\tau)\,. \tag{40}$$

Proof. By using the Cauchy product we find:

$$A(x\,\tau)\,e^{x\,\tau} = \sum_{k=0}^{\infty} a_k\,\frac{(x\,\tau)^k}{k!}\,\sum_{k=0}^{\infty}\frac{(x\,\tau)^k}{k!} = \sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h} a_{k-h}\,\frac{(x\,\tau)^k}{k!}\,. \tag{41}$$

Therefore, the result follows by comparing Equations (38) and (41). □

6. Conclusions

It has been shown that the first and second kind Chebyshev polynomials play an important role in separating the even part from the odd part of several polynomials and special functions, which include the Appell polynomials, the first kind Bessel functions and the recently introduced Appell–Bessel functions [2]. This is another remarkable property of Chebyshev's classic polynomials within Function theory, which seems to be the counterpart of another, highlighted in [1], which showed its role in separating the real from the imaginary part of Appell's polynomials.

Author Contributions: Review and editing, P.N.; investigation, original draft preparation, P.E.R.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have not received funds from any institution and that they have no conflict of interest.

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