A Lyapunov-Type Inequality for a Laplacian System on a Rectangular Domain with Zero Dirichlet Boundary Conditions

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Abstract: We consider a coupled system of partial differential equations involving Laplacian operator, on a rectangular domain with zero Dirichlet boundary conditions. A Lyapunov-type inequality related to this problem is derived. This inequality provides a necessary condition for the existence of nontrivial positive solutions.

Keywords: Lyapunov-type inequality; coupled system; Laplacian; rectangular domain

MSC: 35A23; 47F05; 34A12

1. Introduction

Consider the second order differential equation

$$-u''(x) = p(x)u(x), \quad a < x < b, \tag{1}$$

subject to the Dirichlet boundary condition

$$u(a) = u(b) = 0, \tag{2}$$

where $a, b \in \mathbb{R}$, $a < b$, and $p \in C([a, b])$. It is well known (see, e.g., [1,2]) that if $u \in C^2([a, b])$ is a nontrivial solution to Equations (1) and (2), then

$$\int_a^b |p(x)| \, dx > \frac{4}{b - a}. \tag{3}$$

The inequality in Equation (3) is known in the literature as Lyapunov inequality. It has many applications in the study of spectral properties of ODE (see, e.g., [3–8]).

In the multi-dimensional case, there are some important works dealing with Lyapunov-type inequalities for PDEs. In [9], the authors studied the Laplace equation

$$-\Delta u = q(x)u, \quad x \in \Omega,$$

under Neumann boundary conditions, where Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $q : \Omega \to \mathbb{R}$. In [10], the authors studied the p-Laplacian equation

$$-\Delta_p u = q(x)|u|^{p-2}u, \quad x \in \Omega,$$

under Dirichlet boundary conditions, where Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $p > 1$ and $q \in L^s(\Omega)$, for some s, which depends on p and N. In [11], the authors studied the fractional p–Laplacian equation

$$(-\Delta_p)^s u = q(x)|u|^{p-2}u, \quad x \in \Omega,$$

under the boundary conditions

$$u(x) = 0, \quad x \in \mathbb{R}^N \backslash \Omega,$$

where Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geq 2$), $p > 1$, $0 < s < 1$ and $q \in L^\infty(\Omega)$. In [12], the authors extended the obtained results in [11] to a fractional p-Laplacian system. In [13], the authors studied the partial differential equation

$$-G_\gamma u(x,y) = q(x)u(x,y), \quad (x,y) \in \Omega,$$

under Dirichlet boundary conditions, where $\Omega =]a, b[\times \mathcal{O}$, $(a,b) \in \mathbb{R}^2$, $a < b$, $\mathcal{O}$ is an open bounded subset in $\mathbb{R}^N$ ($N \geq 1$), $q \in C([a,b])$ and G_γ, $\gamma \geq 0$, is the differential operator given by

$$G_\gamma u(x,y) = \frac{\partial^2 u}{\partial x^2} + x^{2\gamma}\Delta_y u(x,y), \quad (x,y) \in \Omega.$$

The authors of [14] extended the obtained results in [13] to the differential operator

$$L_{\gamma,g}u(x,y) = G_\gamma u(x,y) + g(x)\frac{\partial u}{\partial x}(x,y), \quad (x,y) \in \Omega,$$

where $g \in C([a,b])$ and Δ_y is the Laplacian operator with respect to the variable "y".

Motivated by the above cited works, in this paper, we consider the two-dimensional coupled system of partial differential equations

$$\begin{cases} -\Delta u(x,y) = p_{11}(x)u(x,y) + \dfrac{p_{12}(x)}{\Gamma(\alpha)}\displaystyle\int_a^x (x-z)^{\alpha-1}v(z,y)\,dz, & (x,y) \in \Omega, \\[4mm] -\Delta v(x,y) = \dfrac{p_{21}(x)}{\Gamma(\beta)}\displaystyle\int_a^x (x-z)^{\beta-1}u(z,y)\,dz + p_{22}(x)v(x,y), & (x,y) \in \Omega, \end{cases} \tag{4}$$

under the Dirichlet boundary conditions

$$u(x,y) = v(x,y) = 0, \quad (x,y) \in \partial\Omega, \tag{5}$$

where $\Omega =]a, b[\times]c, d[$, $(a,b), (c,d) \in \mathbb{R}^2$, $a < b$, $c < d$, $\alpha > 0$, $\beta > 0$, $p_{ij} \in C([a,b])$, $1 \leq i, j \leq 2$, and Γ is the Gamma function. We derive a Lyapunov-type inequality, which provides a necessary condition for the existence of nontrivial positive solutions to Equations (4) and (5). Note that the used technique in this paper is different to that used in [11,12]. The approach used in this paper is based on an eigenvalue method from Kaplan [15]. Observe that the system in Equation (4) involves the nonlocal operators

$$\frac{1}{\Gamma(\kappa)}\int_a^x (x-z)^{\kappa-1}f(z)\,dz,$$

where $\kappa \in \{\alpha, \beta\}$ and $f \in \{u(\cdot,y), v(\cdot,y)\}$. Such operators are known in the literature as Riemann–Liouville fractional integrals of order κ. For more details on fractional operators and their applications, see, for example, [16,17].

The rest of the paper is organized as follows. In Section 2, we recall and prove some results on matrices theory that are used in the proof of our main result. In Section 3, we establish a Lyapunov-type inequality for Equations (4) and (5), and we discuss some special cases of Equations (4) and (5).

2. Preliminaries

Let $N \geq 1$ be a natural number. We denote by $\vec{0}_N$ the zero vector in $\mathbb{R}^N$. Let $\| \cdot \|_N$ be the Euclidean norm in $\mathbb{R}^N$, that is,

$$\|\vec{\mathcal{V}}\|_N = \sqrt{\mathcal{V}_1^2 + \mathcal{V}_2^2 + \cdots + \mathcal{V}_N^2}, \quad \vec{\mathcal{V}} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_N \end{pmatrix} \in \mathbb{R}^N.$$

We define in $\mathbb{R}^N$ the partial order $\leq_N$ given by

$$\vec{\mathcal{U}} = \begin{pmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \\ \vdots \\ \mathcal{U}_N \end{pmatrix} \leq_N \vec{\mathcal{V}} = \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \\ \vdots \\ \mathcal{V}_N \end{pmatrix} \iff \mathcal{U}_i \leq \mathcal{V}_i, \ i = 1, 2, \cdots, N.$$

It can be easily seen that

Lemma 1. *Let $\vec{\mathcal{U}}$ and $\vec{\mathcal{V}}$ be two vectors in $\mathbb{R}^N$ such that*

$$\vec{0}_N \leq_N \vec{\mathcal{U}} \leq_N \vec{\mathcal{V}}.$$

then

$$\|\vec{\mathcal{U}}\|_N \leq \|\vec{\mathcal{V}}\|_N.$$

We denote by $\mathcal{M}_N(\mathbb{R})$ the set of square matrices of size N with coefficients in $\mathbb{R}$, and by $\mathcal{M}_N(\mathbb{R}_+)$ the subset of $\mathcal{M}_N(\mathbb{R})$ with positive coefficients. We endow $\mathcal{M}_N(\mathbb{R})$ with the subordinate matrix norm

$$\|M\|_{\mathcal{M}_N} = \sup_{\vec{X} \in \mathbb{R}^N, \vec{X} \neq \vec{0}_N} \frac{\|M\vec{X}\|_N}{\|\vec{X}\|_N}, \quad M \in \mathcal{M}_N(\mathbb{R}).$$

For a given matrix $M \in \mathcal{M}_N(\mathbb{R})$, let $\rho(M)$ be its spectral radius, i.e.,

$$\rho(M) = \max\{|\lambda_i(M)| : i = 1, 2, \cdots, N\},$$

where $\lambda_i(M)$, $i = 1, 2, \cdots, N$, are the (real or complex) eigenvalues of M.

The following result is standard in the theory of matrices.

Lemma 2. *Let $M \in \mathcal{M}_N(\mathbb{R})$. Then*

$$\rho(M) < 1 \iff \lim_{n \to \infty} \|M^n\|_{\mathcal{M}_N} = 0.$$

Lemma 3. *Let $M \in \mathcal{M}_N(\mathbb{R}_+)$ and $\vec{X} \in \mathbb{R}^N$, $\vec{X} \neq \vec{0}_N$. If*

$$\vec{0}_N \leq_N \vec{X} \leq_N M\vec{X}, \tag{6}$$

then

$$\rho(M) \geq 1. \tag{7}$$

Proof. From Equation (6) and using the fact that $M \in \mathcal{M}_N(\mathbb{R}_+)$, for all natural number $n \geq 1$, we have

$$\vec{0}_N \leq_N \vec{X} \leq_N M^n \vec{X}.$$

Next, by Lemma 1, we obtain

$$\|\vec{X}\|_N \leq \|M^n \vec{X}\|_N \leq \|M^n\|_{\mathcal{M}_N} \|\vec{X}\|_N, \quad n \geq 1.$$

Since $\vec{X} \neq \vec{0}_N$, we get

$$\|M^n\|_{\mathcal{M}_N} \geq 1, \quad n \geq 1.$$

Finally, using Lemma 2, Equation (7) follows. $\square$

Lemma 4. *Let* $M = (m_{ij})_{1 \leq i,j \leq 2} \in \mathcal{M}_2(\mathbb{R}_+)$. *Then,*

$$\rho(M) = \frac{m_{11} + m_{22} + \sqrt{(m_{11} - m_{22})^2 + 4m_{21}m_{12}}}{2}.$$

Proof. Let P_M be the characteristic polynomial of the matrix M, that is,

$$P_M(\lambda) = \lambda^2 - \text{tr}(M)\lambda + \det(M), \quad \lambda \in \mathbb{C},$$

where $\text{tr}(M)$ is the trace of M and $\det(M)$ is its determinant. Then, the discriminant of P_M is given by

$$\Delta(P_M) = [\text{tr}(M)]^2 - 4\det(M),$$

i.e.,

$$\Delta(P_M) = (m_{11} - m_{22})^2 + 4m_{21}m_{12}.$$

Hence, M admits two eigenvalues

$$\lambda_1(M) = \frac{m_{11} + m_{22} + \sqrt{(m_{11} - m_{22})^2 + 4m_{21}m_{12}}}{2}$$

and

$$\lambda_2(M) = \frac{m_{11} + m_{22} - \sqrt{(m_{11} - m_{22})^2 + 4m_{21}m_{12}}}{2}.$$

Observe that

$$\lambda_1(M) \geq \lambda_2(M).$$

Therefore, if $\lambda_2(M) \geq 0$, then $\rho(M) = \lambda_1(M)$. Further, if $\lambda_2(M) < 0$, we obtain

$$|\lambda_2(M)| = -\lambda_2(M) = \frac{\sqrt{(m_{11} - m_{22})^2 + 4m_{21}m_{12}} - (m_{11} + m_{22})}{2} \leq \lambda_1(M).$$

Then, in all cases, we have $\rho(M) = \lambda_1(M)$, which proves the desired result. $\square$

3. Lyapunov-Type Inequalities

In this section, a Lyapunov-type inequality is derived for Equations (4) and (5) and some special cases are discussed.

Definition 1. *We say that* (u, v) *is a nontrivial positive solution to Equations* (4) *and* (5), *if and only if,*

(i) $(u, v) \in C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ *satisfies Equations* (4) *and* (5).
(ii) *For all* $(x, y) \in \Omega$, *we have*

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \geq_2 \vec{0}_2.$$

(iii) $u \not\equiv 0$ *and* $v \not\equiv 0$.

3.1. From PDEs to ODEs

In this subsection, we reduce the study of Equation (4) to a coupled system of ordinary differential equations.

Suppose that (u, v) is a nontrivial positive solution to Equations (4) and (5). Let us introduce the functions

$$\varphi(y) = \sin\left(\frac{\pi(y - c)}{d - c}\right), \quad c \le y \le d,$$

$$\overline{u}(x) = \int_c^d u(x, y)\varphi(y)\, dy, \quad a \le x \le b \tag{8}$$

and

$$\overline{v}(x) = \int_c^d v(x, y)\varphi(y)\, dy, \quad a \le x \le b. \tag{9}$$

Observe that due to the positivity of the function φ in $[c, d]$, (ii) and (iii), we have $\overline{u} \not\equiv 0$ and $\overline{v} \not\equiv 0$. Further, multiplying the first equation in Equation (4) by $\varphi(y)$ and integrating over (c, d), we obtain

$$-\frac{d^2}{dx^2}\int_c^d u(x, y)\varphi(y)\, dy - \int_c^d \frac{\partial^2 u}{\partial y^2}(x, y)\varphi(y)\, dy$$
$$= p_{11}(x)\int_c^d u(x, y)\varphi(y)\, dy + \frac{p_{12}(x)}{\Gamma(\alpha)}\int_c^d \int_a^x (x - z)^{\alpha - 1}v(z, y)\, dz\varphi(y)\, dy, \tag{10}$$

for all $a < x < b$. On the other hand, using an integration by parts, we obtain

$$\int_c^d \frac{\partial^2 u}{\partial y^2}(x, y)\varphi(y)\, dy = \left[\frac{\partial u}{\partial y}(x, y)\varphi(y)\right]_{y=c}^d - \frac{\pi}{d - c}\int_c^d \frac{\partial u}{\partial y}(x, y)\cos\left(\frac{\pi(y - c)}{d - c}\right)dy$$
$$= -\frac{\pi}{d - c}\int_c^d \frac{\partial u}{\partial y}(x, y)\cos\left(\frac{\pi(y - c)}{d - c}\right)dy.$$

Again, using an integration by parts and Equation (5), we obtain

$$\int_c^d \frac{\partial^2 u}{\partial y^2}(x, y)\varphi(y)\, dy = -\frac{\pi}{d - c}\left[u(x, y)\cos\left(\frac{\pi(y - c)}{d - c}\right)\right]_{y=c}^d - \left(\frac{\pi}{d - c}\right)^2\int_c^d u(x, y)\varphi(y)\, dy$$
$$= -\left(\frac{\pi}{d - c}\right)^2\int_c^d u(x, y)\varphi(y)\, dy. \tag{11}$$

Next, using Fubini's theorem, we have

$$\int_c^d \int_a^x (x - z)^{\alpha - 1}v(z, y)\, dz\varphi(y)\, dy = \int_a^x (x - z)^{\alpha - 1}\left(\int_c^d v(z, y)\varphi(y)\, dy\right)dz. \tag{12}$$

Combining Equations (10), (11) and (12), we obtain

$$-\overline{u}''(x) = \left(p_{11}(x) - \left(\frac{\pi}{d - c}\right)^2\right)\overline{u}(x) + \frac{p_{12}(x)}{\Gamma(\alpha)}\int_a^x (x - z)^{\alpha - 1}\overline{v}(z)\, dz,$$

for $a < x < b$. Similarly, multiplying the second equation in Equation (4) by $\varphi(y)$ and integrating over (c, d), we obtain

$$-\overline{v}''(x) = \frac{p_{21}(x)}{\Gamma(\beta)}\int_a^x (x - z)^{\beta - 1}\overline{u}(z)\, dz + \left(p_{22}(x) - \left(\frac{\pi}{d - c}\right)^2\right)\overline{v}(x),$$

for $a < x < b$. Moreover, using the boundary conditions in Equation (5), we have

$$\overline{u}(a) = \overline{u}(b) = 0 = \overline{v}(a) = \overline{v}(b).$$

Hence, we have the following result.

Proposition 1. *Let (u, v) be a nontrivial positive solution to Equations (4) and (5). Then, $(\overline{u}, \overline{v})$ is a nontrivial solution to the coupled system*

$$
\begin{cases}
-\overline{u}''(x) = \left(p_{11}(x) - \left(\dfrac{\pi}{d-c}\right)^2\right)\overline{u}(x) + \dfrac{p_{12}(x)}{\Gamma(\alpha)}\displaystyle\int_a^x (x-z)^{\alpha-1}\overline{v}(z)\,dz, & a < x < b, \\[4mm]
-\overline{v}''(x) = \dfrac{p_{21}(x)}{\Gamma(\beta)}\displaystyle\int_a^x (x-z)^{\beta-1}\overline{u}(z)\,dz + \left(p_{22}(x) - \left(\dfrac{\pi}{d-c}\right)^2\right)\overline{v}(x), & a < x < b,
\end{cases}
\tag{13}
$$

subject to the boundary conditions

$$\overline{u}(a) = \overline{u}(b) = 0 = \overline{v}(a) = \overline{v}(b), \tag{14}$$

where $\overline{u}$ and $\overline{v}$ are given by Equations (8) and (9).

3.2. Main Result

Let us introduce the matrix $M = (m_{ij})_{1\leq i,j\leq 2} \in \mathcal{M}_2(\mathbb{R}_+)$ given by

$$m_{ij} = \frac{(b-a)^{\gamma_{ij}+1}}{4\Gamma(\gamma_{ij}+1)}\int_a^b \left|p_{ij}(s) - \left(\frac{\pi}{d-c}\right)^2 \delta_{ij}\right| ds, \quad 1 \leq i,j \leq 2,$$

where

$$
\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}
$$

and

$$
\gamma_{ij} = \begin{cases} 0 & \text{if } i = j, \\ \alpha & \text{if } (i,j) = (1,2), \\ \beta & \text{if } (i,j) = (2,1). \end{cases}
$$

Now, we are able to state and prove our main result.

Theorem 1. *Let (u, v) be a nontrivial positive solution to Equations (4) and (5). Then,*

$$m_{11} + m_{22} + \sqrt{(m_{11} - m_{22})^2 + 4m_{21}m_{12}} \geq 2. \tag{15}$$

Proof. Let (u, v) be a nontrivial positive solution to Equations (4) and (5). By Proposition 1, we have

$$
\begin{cases}
-\overline{u}''(x) = f(x, \overline{u}, \overline{v}), & a < x < b, \\
\overline{u}(a) = \overline{u}(b) = 0,
\end{cases}
$$

where $\overline{u} \not\equiv 0$ and

$$f(x,\overline{u},\overline{v}) = \left(p_{11}(x) - \left(\frac{\pi}{d-c}\right)^2\right)\overline{u}(x) + \frac{p_{12}(x)}{\Gamma(\alpha)}\int_a^x (x-z)^{\alpha-1}\overline{v}(z)\,dz, \quad a < x < b.$$

Therefore, $\overline{u}$ is a solution to the integral equation

$$\overline{u}(x) = \int_a^b G(x,s)f(s,\overline{u},\overline{v})\,ds, \quad a \le x \le b, \tag{16}$$

where

$$G(x,s) = \frac{(x-a)(b-s)}{b-a}, \quad (x,s) \in [a,b] \times [a,b].$$

On the other hand, the arithmetic-geometric-harmonic mean inequality yields

$$\max_{a \le x,s \le b} |G(x,s)| \le \frac{b-a}{4}. \tag{17}$$

Further, let us estimate the term $|f(s,\overline{u},\overline{v})|$, $a < s < b$. We have

$$
\begin{aligned}
\left| \frac{p_{12}(s)}{\Gamma(\alpha)}\int_a^s (s-z)^{\alpha-1}\overline{v}(z)\,dz \right| &\le \frac{|p_{12}(s)|}{\Gamma(\alpha)}\left(\int_a^s (s-z)^{\alpha-1}\,ds\right)\|\overline{v}\|_\infty \\
&= \frac{|p_{12}(s)|}{\Gamma(\alpha+1)}(s-a)^\alpha\|\overline{v}\|_\infty \\
&\le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}|p_{12}(s)|\|\overline{v}\|_\infty,
\end{aligned}
\tag{18}
$$

where

$$\|h\|_\infty = \max_{a \le t \le b} |h(t)|, \quad h \in C([a,b]).$$

Moreover, we have

$$\left| \left(p_{11}(s) - \left(\frac{\pi}{d-c}\right)^2\right)\overline{u}(s) \right| \le \left|p_{11}(s) - \left(\frac{\pi}{d-c}\right)^2\right| \|\overline{u}\|_\infty. \tag{19}$$

Next, combining Equations (18) and (19), we get

$$|f(s,\overline{u},\overline{v})| \le \left|p_{11}(s) - \left(\frac{\pi}{d-c}\right)^2\right| \|\overline{u}\|_\infty + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}|p_{12}(s)|\|\overline{v}\|_\infty, \quad a < s < b. \tag{20}$$

Now, combining Equations (16), (17) and (20), we deduce that

$$\|\overline{u}\|_\infty \le m_{11}\|\overline{u}\|_\infty + m_{12}\|\overline{v}\|_\infty. \tag{21}$$

Similarly, by Proposition 1, we have

$$\begin{cases} -\overline{v}''(x) = g(x,\overline{u},\overline{v}), & a < x < b, \\ \overline{v}(a) = \overline{v}(b) = 0, \end{cases}$$

where $\overline{v} \not\equiv 0$ and

$$g(x,\overline{u},\overline{v}) = \frac{p_{21}(x)}{\Gamma(\beta)}\int_a^x (x-z)^{\beta-1}\overline{u}(z)\,dz + \left(p_{22}(x) - \left(\frac{\pi}{d-c}\right)^2\right)\overline{v}(x), \quad a < x < b.$$

Using a similar argument as above, we obtain

$$\|\overline{v}\|_\infty \le m_{21}\|\overline{u}\|_\infty + m_{22}\|\overline{v}\|_\infty. \tag{22}$$

Using Equations (21) and (22), we deduce that

$$\vec{0}_2 \le_2 \vec{X} \le_2 M\vec{X},$$

where

$$\vec{X} = \left(\begin{array}{c} \|\overline{u}\|_\infty \\ \|\overline{v}\|_\infty \end{array} \right) \ne \vec{0}_2.$$

Hence, by Lemma 3, we deduce that

$$\rho(M) \ge 1.$$

Finally, using Lemma 4, Equation (15) follows. □

3.3. Particular Cases

In this subsection, we discuss some special cases following from Theorem 1.

3.3.1. The Case $\alpha = \beta$

Let us consider the coupled system

$$\begin{cases} -\Delta u(x,y) = p_{11}(x)u(x,y) + \dfrac{p_{12}(x)}{\Gamma(\theta)} \displaystyle\int_a^x (x-z)^{\theta-1}v(z,y)\,dz, & (x,y) \in \Omega, \\[4mm] -\Delta v(x,y) = \dfrac{p_{21}(x)}{\Gamma(\theta)} \displaystyle\int_a^x (x-z)^{\theta-1}u(z,y)\,dz + p_{22}(x)v(x,y), & (x,y) \in \Omega, \end{cases} \tag{23}$$

where $\theta > 0$ and $p_{ij} \in C([a,b])$, $1 \le i,j \le 2$. Observe that Equation (23) is a special case of Equation (4) with $\alpha = \beta = \theta$. Using Theorem 1, we obtain the following Lyapunov-type inequality for Equations (23) and (5).

Corollary 1. *Let (u,v) be a nontrivial positive solution to Equations (23) and (5). Then,*

$$\int_a^b \sum_{i=1}^2 \left| p_{ii}(s) - \left(\frac{\pi}{d-c}\right)^2 \right| ds$$

$$+ \sqrt{\left(\int_a^b \sum_{i=1}^2 (-1)^{i+1} \left| p_{ii}(s) - \left(\frac{\pi}{d-c}\right)^2 \right| ds \right)^2 + \frac{4(b-a)^{2\theta}}{\Gamma(\theta+1)} \left(\int_a^b |p_{12}(s)|\,ds \right) \left(\int_a^b |p_{21}(s)|\,ds \right)} \tag{24}$$

$$\ge \frac{8}{b-a}.$$

3.3.2. The Limit Case $\theta \to 0^+$

In the limit case $\theta \to 0^+$, Equation (23) reduces to

$$\begin{cases} -\Delta u(x,y) = p_{11}(x)u(x,y) + p_{12}(x)v(x,y), & (x,y) \in \Omega, \\ -\Delta v(x,y) = p_{21}(x)u(x,y) + p_{22}(x)v(x,y), & (x,y) \in \Omega. \end{cases} \tag{25}$$

Therefore, passing to the limit as $\theta \to 0^+$ in Equation (24), we deduce the following Lyapunov-type inequality for Equations (25) and (5).

Corollary 2. *Let (u, v) be a nontrivial positive solution to Equations (25) and (5). Then,*

$$
\int_a^b \sum_{i=1}^2 \left| p_{ii}(s) - \left(\frac{\pi}{d-c} \right)^2 \right| ds
$$
$$
+ \sqrt{ \left(\int_a^b \sum_{i=1}^2 (-1)^{i+1} \left| p_{ii}(s) - \left(\frac{\pi}{d-c} \right)^2 \right| ds \right)^2 + 4 \left(\int_a^b |p_{12}(s)| \, ds \right) \left(\int_a^b |p_{21}(s)| \, ds \right) }
$$
$$
\geq \frac{8}{b-a}. \tag{26}
$$

3.3.3. The Case $p_{11} = p_{22}$

Let us consider the system

$$
\begin{cases}
-\Delta u(x,y) = p(x)u(x,y) + p_{12}(x)v(x,y), & (x,y) \in \Omega, \\
-\Delta v(x,y) = p_{21}(x)u(x,y) + p(x)v(x,y), & (x,y) \in \Omega,
\end{cases} \tag{27}
$$

where $p_{12}, p_{21}, p \in C([a,b])$. Observe that Equation (27) is a special case of Equation (25) with

$$
p_{11} = p_{22} = p.
$$

Therefore, taking in Equation (26) p_{11} and p_{22} as above, we deduce the following Lyapunov-type inequality for Equation (27) and (5).

Corollary 3. *Let (u, v) be a nontrivial positive solution to Equations (27) and (5). Then,*

$$
\int_a^b \left| p(s) - \left(\frac{\pi}{d-c} \right)^2 \right| ds + \sqrt{ \left(\int_a^b |p_{12}(s)| \, ds \right) \left(\int_a^b |p_{21}(s)| \, ds \right) } \geq \frac{4}{b-a}. \tag{28}
$$

3.3.4. The Case of a Single Equation

Taking in Equation (27) $p_{12} = p_{21} \equiv 0$ and $u = v$, we obtain the single equation

$$
-\Delta u(x,y) = p(x)u(x,y), \quad (x,y) \in \Omega. \tag{29}
$$

Hence, by Equation (28), we deduce the following Lyapunov-type inequality (see [13]) for Equation (29) under the Dirichlet boundary conditions

$$
u(x,y) = 0, \quad (x,y) \in \partial\Omega. \tag{30}
$$

Corollary 4. *Let u be a nontrivial positive solution to Equations (29) and (30). Then,*

$$
\int_a^b \left| p(s) - \left(\frac{\pi}{d-c} \right)^2 \right| ds \geq \frac{4}{b-a}.
$$

Remark 1. *In this paper, the system in Equation (4) is discussed under Dirichlet boundary conditions. It would be interesting to study other types of boundary conditions, as well as more general domains.*

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