mathematics

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The Fixed Point Property of Non-Retractable Topological Spaces

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Received: 2 September 2019; Accepted: 16 September 2019; Published: 21 September 2019

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Abstract: Unlike the study of the fixed point property (*FPP*, for brevity) of retractable topological spaces, the research of the *FPP* of non-retractable topological spaces remains. The present paper deals with the issue. Based on order-theoretic foundations and fixed point theory for Khalimsky (*K*-, for short) topological spaces, the present paper studies the product property of the *FPP* for *K*-topological spaces. Furthermore, the paper investigates the *FPP* of various types of connected *K*-topological spaces such as non-*K*-retractable spaces and some points deleted *K*-topological (finite) planes, and so on. To be specific, after proving that not every one point deleted subspace of a finite *K*-topological plane X is a *K*-retract of X, we study the *FPP* of a non-retractable topological space Y, such as one point deleted space $Y \setminus \{p\}$.

Keywords: Khalimsky topology; *K*-retraction; non-*K*-retractable space; fixed point property

MSC: 06A06; 54A10; 54C05; 55R15; 54C08; 54F65

1. Introduction

To make the paper self-contained, we recall that a mathematical object X has the *FPP* if every well-behaved mapping f from X to itself has a point $x \in X$ such that $f(x) = x$. Reference [1] studied a certain fixed point Theorem in semimetric spaces and further, Reference [2] explored a coincidence point and common fixed point theorems in the product spaces of quasi-ordered metric spaces. Unlike the study of the fixed point property (*FPP*, for short) for retractable topological spaces, since the research of the *FPP* of non-retractable topological spaces still remains, the present paper addresses the issue.

A well-known connection exists between Alexandroff topology with T_0-separation axiom and order theory via the so-called specialization order and the down-sets [3,4]. More precisely, Alexandroff topologies on X, denoted by (X, T_O) which are induced by the preordered sets $(X, \leq)$ (for more details, see Section 3), are in one-to-one correspondence with preorders on X [3]. This approach is often used to describe a topological space on which every continuous mapping has a fixed point. But another approach is in order theory, where a partially ordered set (or a poset for brevity) P is said to have the *FPP* if every increasing function on P has a fixed point.

The paper will study the *FPP* from the viewpoints of both order theory and Alexandroff topology and adopt the results into the study of the *FPP* for Khalimsky topological spaces, which can play an important role in both pure and applied mathematics. In fixed point theory from the viewpoint of order theory, the well-known issue existed, as follows: If X and Y are ordered sets with the *FPP*, does $X \times Y$ have *FPP*? This was of interest in the theory of ordered sets [5]. It was conjectured for a long time (see Reference [5]) that the product of two finite ordered sets with the *FPP* has the *FPP*. This has been referred to as the *Product Problem* or the *Product Conjecture*. Motivated by the results in Reference [6], the conjecture was settled positively [7] if P is finite so that this became a theorem when Roddy [7] proved the conjecture true in 1994.

Let us recall the well-known theorem [8,9] that a lattice L has the *FPP* if and only if L is a complete lattice. Under this situation, for each order-preserving self-map g of L, the set $Fix(g) := \{x \in L \,|\, g(x) = x\}$ is a complete lattice. Furthermore, motivated by the Tarski-Davis theorem [8,9] on a lattice and Kuratowski's question [10,11] on the product property of the *FPP* on a peano continuum (or a compact, connected and locally connected metric space), many works dealt with the *FPP* for ordered sets and topological spaces. Some of these include References [3,10,12–18]. Rival [19] considered irreducible points in arbitrary ordered sets, as follows: For a poset $(P, \leq)$, consider two distinct points $x, y \in P$. If $x < y$ and there is no $z \in P$ such that $x < z < y$, then y is said to be an upper cover of x and x is called a lower cover of y. Given a finite poset $(P, \leq)$, a point in P is called irreducible if and only if it has exactly one upper cover or exactly one lower cover. Let $x \in P$ be irreducible and assume that x has a unique lower cover. Then Rival [19] proved the following:

[Rival theorem] For a poset P and let $x \in P$ be irreducible. Then P has the *FPP* if and only if $P \setminus \{x\}$ has the *FPP*.

In poset theory, consider a poset P and two points $a, b \in P$. Then a is called retractable to b (see Definition 3.1 of Reference [20]) if and only if $a \neq b$ and $(\downarrow a) \setminus \{a\} \subseteq \downarrow b$ and $(\uparrow a) \setminus \{a\} \subseteq \uparrow b$, where under the Hasse diagram of P, $\downarrow a$ and $\uparrow b$ mean the down set of a and the upper set of b in the poset P, respectively.

Besides, Schröder (see Theorem 3.3 of Reference [20]) proved the following:

[Schröder theorem] For a poset $(P, \leq)$, assume that $a (\in P)$ is retractable to $b \in P$. P has the *FPP* if and only if (1) $P \setminus \{a\}$ has the *FPP* and (2) One of $\{p \in P \,|\, p > a\}$ and $\{p \in P \,|\, p < a\}$ has the *FPP*.

Motivated by these results, we can study the *FPP* for some Alexandroff (topological) spaces with T_0-separation axiom (T_0-A-space, for brevity if there is no danger of ambiguity) because a T_0-A-space induces a poset and vice versa, as mentioned above. At the moment we have the following query.

[Question] How can we study the *FPP* of the poset (or the T_0-A-space) in case a given poset (or a T_0-A-space) is related to neither the Rival Theorem nor the Schrder theorem?

In Section 4, we will address the question (see Theorem 3). In topology, it is well known that in general the product property of the *FPP* does not hold [16,17,21]. Comparing with the *FPP* in References [5,16,17,21], its Khalimsky topological version has its own feature. Since the term "Khalimsky" will be often used in this paper, hereafter we will use the terminology 'K-' instead of "Khalimsky" if there is no danger of confusion. To study the product property of the *FPP* for K-topological spaces, we need to recall basic notions associated with both K-topology and fixed point theory.

Comparing a K-topological space with spaces dealt with in earlier papers [16,17,21], we can obtain a poset derived from the given K-topological space. Every two points of a poset $(X, \leq)$ need not be retractable point and further, each element of $(X, \leq)$ need not be irreducible either (see Property 1). Moreover, this poset need not be a lattice (see Lemma 1 in the present paper). Hence we cannot use the Tarski-Davis [8,9], Rival [19] and Schrder [20] theorems to study the *FPP* for K-topological spaces. Henceforth, we need to study fixed point theory for K-topological spaces in some different approaches from those of References [16,17,21] (see Theorem 3).

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of natural numbers, integers, and real numbers, respectively. Owing to Borsuk's or Brouwer's fixed point theorem [22], it is well known that a compact Euclidean n-dimensional cube $X \subset \mathbb{R}^n$ has the *FPP*. However, in digital topology it is clear that any digital plane (X, k) followed from the Rosenfeld model on $\mathbb{Z}^n$ does not have the *FPP* related to digitally k-continuous maps [23] (for more details, see References [24–27]). Besides, it turns out that not every T_0-A-space has the *FPP* [28]. For instance, not every K-topological space has the *FPP* [28]. That is why in this paper we give particular attention to the *FPP* for K-topological spaces and its product property and further, to the further study of the *FPP* for some points deleted K-topological planes.

The rest of this paper is organized as follows. Section 2 provides some terminology from K-topology. Section 3 investigates a poset structure derived from an n-dimensional K-topological space and studies some properties of a K-continuous map from the viewpoint of order theory. Section 4 studies the product property of the *FPP* for K-topological spaces and further, investigates various properties of fixed points of K-continuous self-maps of a non-K-retractable space. Section 5 concludes the paper with summary and a further work.

2. Preliminaries

A poset consists of a set $X(\neq \emptyset)$ and a reflexive, antisymmetric and transitive binary relation $\leq$, denoted by $(X; \leq)$. In this paper we will refer to the underlying set X as the ordered set and many notions are followed from References [29,30]. Subsets $S \subset X$ inherit the order relation from X in terms of a restriction to S. A homomorphism between ordered sets P and Q is an order-preserving map, that is, a map $f : P \to Q$ having the following property: $x \leq y$ (in P) implies $f(x) \leq f(y)$ (in Q). In addition, we say that an ordered set $(X; \leq)$ has the fixed point property if every order-preserving self-map f of X has a point $x \in X$ such that $f(x) = x$ and further, we denote $Fix(f) = \{x \in X \,|\, x = f(x)\}$. Besides, for a poset L, L is called a lattice if and only if any two elements of L have a supremum and an infimum. L is called a complete lattice if and only if any subset of L has a supremum and an infimum [3].

An Alexandroff (A-, for brevity) topological space (X, T) is said to be a topological space of which each point $x \in X$ has the smallest open neighborhood in (X, T) [31]. As an example of an A-topological space is the Khalimsky (K-,for short) line topology. To be precise, *K-line topology* κ on $\mathbb{Z}$, denoted by $(\mathbb{Z}, \kappa)$, is induced by the set $\{[2n-1, 2n+1]_{\mathbb{Z}} \,|\, n \in \mathbb{Z}$ as a subbase [31] (see also Reference [4]), where for $a, b \in \mathbb{Z}$, $[a, b]_{\mathbb{Z}} := \{x \in \mathbb{Z} \,|\, a \leq x \leq b\}$. Furthermore, the product topology on $\mathbb{Z}^n$ induced by $(\mathbb{Z}, \kappa)$ is said to be the *Khalimsky product topology* on $\mathbb{Z}^n$ (or the Khalimsky n-dimensional space), denoted by $(\mathbb{Z}^n, \kappa^n)$. Hereafter, for a subset $X \subset \mathbb{Z}^n$ we will denote by (X, κ^n_X), $n \geq 1$, a subspace induced by $(\mathbb{Z}^n, \kappa^n)$ and it is called a K-topological space. The study of these spaces includes References [4,28,32–39].

Let us examine the structure of $(\mathbb{Z}^n, \kappa^n)$ more precisely. A point $x = (x_i)_{i \in [1,n]_{\mathbb{Z}}} \in \mathbb{Z}^n$ is *pure open* if all coordinates are odd and *pure closed* if each of the coordinates is even [4] and the other points in $\mathbb{Z}^n$ is called *mixed* [4]. These points are showed like following symbols: The symbols ■, a black jumbo dot, • mean a pure closed point, a pure open point and a mixed point (see Figures 1–5), respectively.

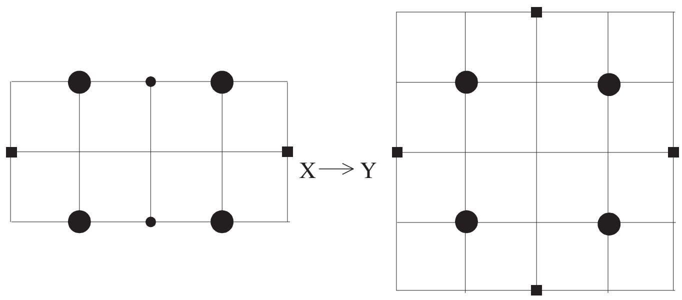

Figure 1. Explanation of a K-homeomorphism with different numbers of pure closed points.

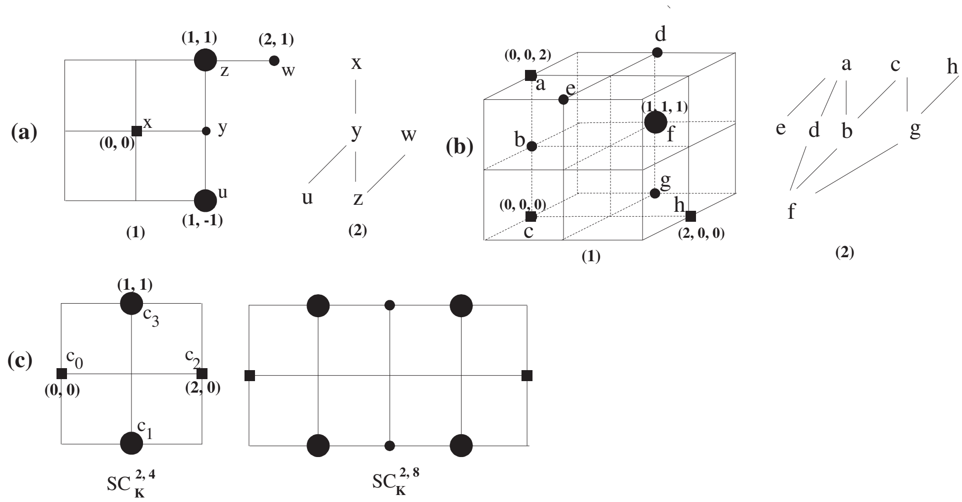

Figure 2. (a) (1) (X, κ_X^2), where $X = \{x, y, z, w, u\}$; (2) The Hasse diagram illustrating the poset derived from (X, κ_X^2); (b) (1) (Y, κ_Y^3), where $Y = \{a, b, c, d, e, f, g, h\}$; (2) The Hasse diagram representing the poset derived from (Y, κ_Y^3) and further, it is not a lattice; (c) Some examples for simple closed K-curves on $\mathbb{Z}^2$ [28].

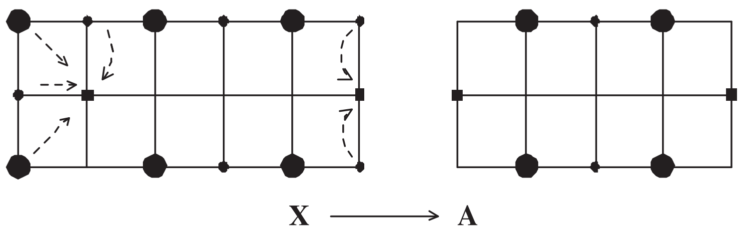

Figure 3. Explanation of a K-retract of $X := (X, \kappa_X^2)$ onto $A := (A, \kappa_A^2)$.

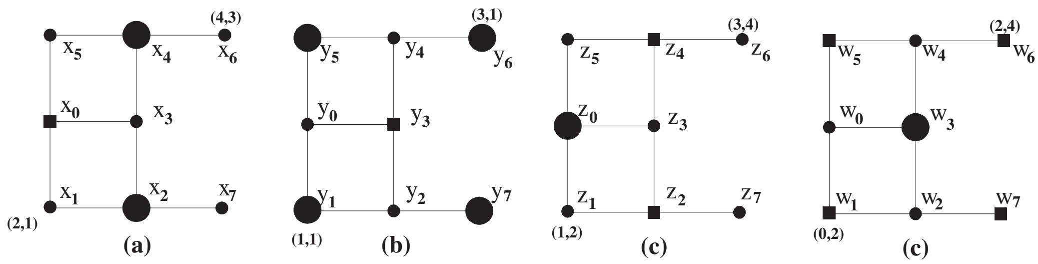

Figure 4. (**a**) A non-K-retractable space from the given K-topological plane X; (**b**) A K-retractable space from the given K-topological plane Y; (**c**) A K-retractable space from the given K-topological plane Z; (**d**) A K-retractable space from the given K-topological plane W.

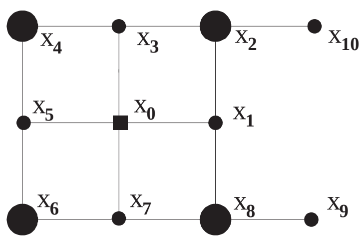

Figure 5. Fixed Point Property (*FPP*) for a non-K-retractable space (A, κ_A^2).

In relation to the further statement of a mixed point in $(\mathbb{Z}^2, \kappa^2)$, for the points $p = (2m, 2n+1)$(resp. $p = (2m+1, 2n)$), we call the point p *closed-open* (resp. *open-closed*) [37].

In terms of this perspective, we clearly observe that the *smallest (open) neighborhood* of the point $p := (p_1, p_2)$ of $\mathbb{Z}^2$, denoted by $SN_K(p) \subset \mathbb{Z}^2$, is the following:

$$SN_K(p) = \left\{ \begin{array}{l} \{p\} \text{ if } p \text{ is pure open}, \\ \{(p_1 - 1, p_2), p, (p_1 + 1, p_2)\} \text{ if } p \text{ is closed-open}, \\ \{(p_1, p_2 - 1), p, (p_1, p_2 + 1)\} \text{ if } p \text{ is open-closed}, \\ N_8(p) \text{ if } p := (2m, 2n), m, n \in \mathbb{Z}, \text{ is pure closed}, \\ where N_8(p) := [2m-1, 2m+1]_{\mathbb{Z}} \times [2n-1, 2n+1]_{\mathbb{Z}}. \end{array} \right\} \quad (1)$$

Similarly, for (X, κ_X^n) and $x \in X$, we can also consider $SN_K(x)$ in (X, κ_X^n).

Definition 1 ([4,33]). *For (X, κ_X^n), we say that two distinct points x and y in X are K-adjacent in (X, κ_X^n) if $y \in SN_K(x)$ or $x \in SN_K(y)$.*

Let us recall the following terminology for studying K-topological spaces.

Definition 2 ([33,40]). *For $(X, \kappa_X^n) := X$ we define the followings.*

(1) *Two distinct points $x, y \in (X, \kappa_X^n) := X$ are called K-path connected (or K-connected) if there is the sequence (or a path) $(x_0, x_1, \cdots, x_l)$ on X with $\{x_0 = x, x_1, \cdots, x_l = y\}$ such that x_i and x_{i+1} are K-adjacent, $i \in [0, l-1]_{\mathbb{Z}}, l \geq 1$. This sequence is called a K-path. Furthermore, the number l is called the length of this K-path.*
(2) *A simple K-path in X is the K-path $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ in X such that x_i and x_j are K-adjacent if and only if $|i - j| = 1$.*
(3) *We say that a simple closed K-curve with l elements $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ in X, denoted by $SC_K^{n,l}, l \geq 4$, is the K-path such that x_i and x_j are K-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.*

Let us now recall the K-continuity of a map between two K-topological spaces [32] as follows: For two spaces $(X, \kappa_X^{n_0}) := X$ and $(Y, \kappa_Y^{n_1}) := Y$, a function $f : X \to Y$ is said to be *K-continuous* at a point $x \in X$ if the following property holds

$$f(SN_K(x)) \subset SN_K(f(x)). \quad (2)$$

Furthermore, we say that a map $f : X \to Y$ is K-continuous if it is K-continuous at every point $x \in X$. This approach can be reasonable because each subspace of $(\mathbb{Z}^n, \kappa^n)$ is an Alexandroff space.

In addition, we recall the notion of *K-homeomorphism* (see Figure 1) as follows: For two spaces $(X, \kappa_X^{n_0})$ and $(Y, \kappa_Y^{n_1})$, a map $h : X \to Y$ is called a K-homeomorphism if h is a K-continuous bijection and further, $h^{-1} : Y \to X$ is K-continuous.

Using the K-continuity of the map f, we obtain the *K-topological category*, denoted by *KTC*, consisting of the following two data [39]:

- The set of objects (X, κ_X^n), denoted by $Ob(KTC)$;
- For every ordered pair of objects $(X, \kappa_X^{n_0})$ and $(Y, \kappa_Y^{n_1})$, the set of all K-continuous maps $f : (X, \kappa_X^{n_0}) \to (Y, \kappa_Y^{n_1})$ as morphisms.

Assume that $(X, \kappa_X^{n_0}) := X$ has the *FPP*. If $h : X \to (Y, \kappa_Y^{n_1}) := Y$ is a K-homeomorphism, then Y is also proved to have the *FPP* [28].

3. An Ordered Space Derived from a Khalimsky Topological Space

Recently, it turns out that not every K-topological space $(X, \kappa_X^n), n \in \mathbb{N}$ has the *FPP* such as $(\mathbb{Z}^n, \kappa^n)$ and $SC_K^{n,l}$ [28]. Hereafter, we assume that a K-topological space (X, κ_X^n) is finite, K-connected and further, not a singleton. Besides, we will study the *FPP* of K-topological spaces by using an order-theoretical approach. In a Hasse diagram for a finite ordered set, elements x and y satisfy the relation $x \leq y$ if and only if $x = y$ or there is an upward path from x to y that may go through other elements of the set, but for which all segments are traversed in the upward direction (see Figure 2a,b). Let $(X; \leq)$ be a poset. For $x \in X$, put

$$S_x := \{t \in X \,|\, t \leq x\} :=\downarrow x \tag{3}$$

as a down set of x in $(X; \leq)$.

Using the set of (3), we develop an induced topology T_O on X generated by the set $\{S_x \,|\, x \in X\}$ as a base and we denoted by (X, T_O) the topological space.

In this paper we say that an order-topological space (ordered space, for short) is a set which is both a topological space and a poset. In *KTC*, as special cases of the results in References [5,7,16,17,21], as follows:

Remark 1. *For two posets $(X; \leq), (Y; \leq')$ and a function $f : X \to Y$, f is an order-preserving map if and only if f is continuous between the induced Alexandroff topological spaces $(X, T_O), (Y, T'_O)$.*

Proof. Assume that f preserves order from $(X; \leq)$ to $(Y, \leq')$ and let $A = S_y \subset Y$ be an open base set, where S_y means the down set of y in the given poset (for more details, see the property (3)). We need to show that $f^{-1}(A)$ is open in X. Let us now take any $x \in f^{-1}(A)$. In case $z \leq x$, we have $f(z) \leq f(x)$ so that $f(z) \in A$. Therefore $z \in f^{-1}(A)$. Hence, for any $x \in f^{-1}(A)$ we have $S_x \subset f^{-1}(A)$ and further,

$$f^{-1}(A) = \cup_{x \in f^{-1}(A)} S_x,$$

which implies that $f^{-1}(A)$ is open.

Conversely, assume that f is continuous between the induced Alexandroff topologies $(X, T_O), (Y, T'_O)$ and let $z \leq x$ for some $x, z \in X$. Suppose $f(z) \not\leq f(x)$. Let $A = S_{f(x)}$. Therefore $f(z) \notin A$ but A is open, due to the continuity of f, we obtain that $f^{-1}(A)$ is open. Hence $S_x \subset f^{-1}(A)$. But $z \leq x$, so $z \in f^{-1}(A)$, which implies that $f(z) \in A$ so that it invokes a contradiction. □

For $X(\subset \mathbb{Z}^n)$, we ask if the ordered space $(X, (T_O)_X)$ induced by $(\mathbb{Z}^n, \kappa^n)$ is equal to the K-topological space (X, κ^n_X). To be specific, based on the poset $(X, \leq)$ derived from $(\mathbb{Z}^n, \kappa^n)$, to establish the topology $(T_O)_X$ on X, we may have the following set as a subbase for the topology of $\mathbb{Z}^n$ (see the property (1))

$$S := \{S_{(2m_1, 2m_2, \dots, 2m_n)} \,|\, m_i \in \mathbb{Z}, n \in \mathbb{N}, i \in [1, n]_{\mathbb{Z}}\},$$

where $S_{(2m_1, 2m_2, \dots, 2m_n)} = \prod_{i=1}^n [2m_i - 1, 2m_i + 1]_{\mathbb{Z}}$. For instance, for $X(\subset \mathbb{Z})$, to establish the topology $(T_O)_X$ induced by the poset $(\mathbb{Z}, \leq)$ derived from $(\mathbb{Z}, \kappa)$, we have the set $S = \{S_{2m} \,|\, S_{2m} := [2m-1, 2m+1]_{\mathbb{Z}}, m \in \mathbb{Z}\}$ as a subbase. Hence it is clear that the topology $(T_O)_X$ on $X(\subseteq \mathbb{Z}^n)$ is equal to the K-topological space $(X, (\kappa^n)_X)$.

By using this approach, for an integral interval $I := [a, b]_{\mathbb{Z}}$, the ordered space (I, T_O) induced by $(\mathbb{Z}, \kappa)$ is equal to the K-interval denoted by (I, κ_I).

According to this approach, for $X \subset \mathbb{Z}^n$ we obtain the following:

Corollary 1. *For $X \subset \mathbb{Z}^n$, $(X, (T_O)_X)$ is equal to (X, κ^n_X), where $(X, (T_O)_X)$ is a subspace for X relative to the topology $(\mathbb{Z}^n, T_O)$.*

Lemma 1. *A K-topological space (X, κ^n_X) is an ordered space which need not be a lattice.*

Proof. Under the Khalimsky n-dimensional space $(\mathbb{Z}^n, \kappa^n)$, we define a relation derived from $(\mathbb{Z}^n, \kappa^n)$, as follows:

For two points x, y of $(\mathbb{Z}^n, \kappa^n)$ we say that

$$x \leq y \text{ if and only if } x \in SN_K(y). \tag{4}$$

Apart from this approach (4), we can also consider this relation " $\leq$" in terms of the specialization order in Reference [4]. More precisely, we say that $x \leq y$ if and only if $y \in Cl(\{x\})$, where the notation "Cl" means the closure operator. However, if we take the specialization order for establishing a relation "$\leq$" for a poset, then this relation is different from the relation of (4). Since $(\mathbb{Z}^n, \kappa^n)$ satisfies the T_0-separation axiom, the relation "$\leq$" of (4) is reflexive, antisymmetric(due to the T_0-separation axiom) and transitive. Furthermore, it is clear that the subspace (X, κ^n_X) also induces a poset structure. However, the induced ordered space (X, κ^n_X) need not be a lattice. For instance, consider the K-topological spaces (X, κ^2_X) and (Y, κ^2_Y) in Figure 2a,b. Then, according to the property (4), we obtain their Hasse diagrams of which the induced posets $(X; \leq)$ and $(Y; \leq)$ are not lattices (see Figure 2a,b(2)). □

The following is obtained by using the property of (3) and the relation of (4), which can be effectively used in studying the *FPP* of ordered spaces in Section 4.

Lemma 2. *The ordered space (X, T_O) is a T_0-A-space and further, a semi-$T_{\frac{1}{2}}$ space.*

Proof. Let us prove that the ordered space (X, T_O) supports the T_0-separation axiom. Consider any two distinct points $x, y \in X$. Note that, owing to the antisymmetric property of the relation $\leq$, we have $x \not\leq y$ or $y \not\leq x$. Therefore $x \notin S_y$ or $y \notin S_x$. Thus (X, T_O) satisfies the T_0-separation axiom because for every point $x \in X$ we have $x \in S_x$.

Let us now prove that (X, T_O) is Alexandroff. Indeed, we suffice to show that an arbitrary intersection of base open sets is open. So assume that $\{x_i\}_{i \in M}$ is a subset of X such that $A = \cap_{i \in M} S_{x_i} \neq \emptyset$ and let $y \in A$. Then, owing to the transitive property of the relation $\leq$, it is clear that $S_y \subset A$ and thus

$$\cap_{i \in M} S_{x_i} = A = \cup_{y \in A} S_y$$

and therefore the intersection A is open, which implies that (X, T_O) is an Alexandroff space. □

By using the notion of (4), Remark 1 and Lemma 2, we obtain the following:

Corollary 2. *A K-continuous map between two K-topological spaces is equivalent to an order preserving map between posets derived from the given K-topological spaces.*

4. The *FPP* of Non-*K*-Retractable Spaces

There was the well-known conjecture [41] wondering if the fact that X and Y have the *FPP* implies that the product space $X \times Y$ has the property. This note contains an affirmative answer in the case that X and Y are compact ordered space *(with the order topology)* [7]. Besides, Cohen [29] rephrased the result as follows.

$$\left\{ \begin{array}{l} \text{If } X \text{ and } Y \text{ are compact connected ordered spaces,} \\ \quad \textit{then } X \times Y \text{ with the order topology has the } \textit{FPP} \text{ .} \end{array} \right\} \tag{5}$$

As referred to in the above, since the present paper deals with only finite and connected spaces which is not a singleton, in *KTC* all spaces (X, κ^n_X) are assumed to be both compact and connected. Motivated by the property (5) (or an immediate consequence of Reference [7]), we have the following:

Remark 2. *Assume that both $(X, \kappa^{n_1}_X)$ and $(Y, \kappa^{n_2}_Y)$ have the FPP. Then the product space $(X \times Y, \kappa^{n_1+n_2}_{X \times Y})$ has the FPP.*

Proof. Under the hypothesis, it is clear that both $(X, \kappa_X^{n_1})$ and $(Y, \kappa_Y^{n_2})$ are compact (i.e., finite) connected ordered spaces. Owing to the results of Reference [7], since both $(X, \leq)$ and $(Y, \leq')$ derived from the given K-topological spaces are finite posets, they have the *FPP*. Hence, by Corollary 1, the product space $(X \times Y, \kappa_{X\times Y}^{n_1+n_2})$ has the *FPP* because it is equal to the ordered space $(X \times Y, (T_O)_{X\times Y})$. By Corollary 1 and Lemma 1, the proof is completed (see the property (5)). □

Theorem 1 ([28,37])**.** *A simple K-path in (X, κ_X^n) has the FPP.*

Owing to Theorem 1, it is clear that a K-interval obviously has the *FPP* because a simple K-path in (X, κ_X^n) is K-homeomorphic to a K-interval (I, κ_I) (see the last part of Section 2).

By Theorem 1, since the K-interval $([a, b]_{\mathbb{Z}} := I, \kappa_I)$ has the *FPP* and further, (I, κ_I) is compact and connected, we have the following:

Remark 3. *(1) References [35,37] dealt with the FPP of a K-interval and further, Reference [28] proved the FPP of a K-path. Besides, Reference [35] proved the FPP for finite K-topological plane in terms of an implicit function Theorem (see Theorem 9.2.4 on page 78 [35]).*

(2) Let I^n be a finite n-dimensional cube as a K-topological subspace. Then, by Remark 2 and Theorem 1, $(I^n, \kappa_{I^n}^n)$ has the FPP.

The notion of retraction in order theory has been used in studying the *FPP* [19]. More precisely, let $(P; \leq)$ be a poset. As mentioned in Section 1, a point $a \in P$ is called retractable to $b \in P \setminus \{a\}$ [20] if and only if for all $x \in P, x > a$ implies $x \geq b$ and $x < a$ implies $x \leq b$ [20]. In other words, for an ordered set, we say that an order-preserving self-map r of P is called a retraction [20] if and only if $r^2 = r$. We will say that $R \subset P$ is a retract of P if and only if there is a retraction $r : P \to P$ with $r(P) = R$ [20].

Based on the Schrder Theorem in Section 1, Reference [30] further studied the *FPP* for product spaces in terms of the retractability from an order theoretical view. Since an ordered space is both a poset and a topological space, we will use the K-retraction in Reference [27] for studying the *FPP* for K-topological spaces.

Definition 3 ([27])**.** *In KTC, a K-continuous map $r : (X', \kappa_{X'}^n) := X' \to (X, \kappa_X^n) := X$ is a K-retraction if*

(1) (X, κ_X^n) is a K-topological subspace of $(X', \kappa_{X'}^n)$ and
(2) $r(x) = x$ for all $x \in X$.

Then we call X a K-retract of X'.

Example 1. *Consider the K-topological spaces (X, κ_X^2) and (A, κ_A^2) in Figure 3. Then we observe that (A, κ_A^2) is a K-retract of (X, κ_X^2).*

As a special case of the Theorem in Reference [7], we see the following: Let (X, κ_X^n) be a K-topological space having the *FPP*. For $(A, \kappa_A^n) := A \subset (X, \kappa_X^n) := X$, if a map $r : X \to A$ is a K-retraction, then (A, κ_A^n) has the *FPP*.

In this paper a K-topological plane is defined, as follows:

Definition 4. *A space (X, κ_X^2) of $(\mathbb{Z}^2, \kappa^2)$ is said to be a finite K-topological plane if (X, κ_X^2) is K-homeomorphic to (D, κ_D^2), where $D = [-m_1, m_1']_{\mathbb{Z}} \times [-m_2, m_2']_{\mathbb{Z}}$ for some $m_i, m_i' \in \mathbb{N}, i \in \{1, 2\}$.*

The following exploration involving a K-retract will be essentially used to prove main theorems of this paper.

Theorem 2. *Let X be a finite K-topological plane. Not every one point $p(\in X)$ deleted subspace of X is a K-retract of X.*

Proof. We may consider $X := [-m_1, m_1']_{\mathbb{Z}} \times [-m_2, m_2']_{\mathbb{Z}}$ for some $m_i, m_i' \in \mathbb{N}, i \in \{1,2\}$ such as

$$X := \left\{\begin{array}{l} [-2m_1, 2m_1']_{\mathbb{Z}} \times [-2m_2+1, 2m_2'+1]_{\mathbb{Z}} \text{ or} \\ [-2m_1+1, 2m_1'+1]_{\mathbb{Z}} \times [-2m_2, 2m_2'+1]_{\mathbb{Z}} \text{ or} \\ [-2m_1+1, 2m_1'+1]_{\mathbb{Z}} \times [-2m_2+1, 2m_2'+1]_{\mathbb{Z}} \text{ or} \\ [-2m_1+1, 2m_1'+1]_{\mathbb{Z}} \times [-2m_2, 2m_2']_{\mathbb{Z}} \text{ or} \\ [-2m_1, 2m_1']_{\mathbb{Z}} \times [-2m_2, 2m_2']_{\mathbb{Z}}, \text{ and so forth.} \end{array}\right\} \tag{6}$$

Based on this situation, we may examine the K-retract property of a spaces X with the following case according to the first case of (6).

Assume $X := [2,4]_{\mathbb{Z}} \times [1,3]_{\mathbb{Z}}$ and $A := X \setminus \{(4,2)\} := \{x_i \,|\, i \in [1,7]_{\mathbb{Z}}\}$ in Figure 4a. Let us consider the K-topological spaces (X, κ_X^2) and (A, κ_A^2). Then we observe that (A, κ_A^2) is not a K-retract of (X, κ_X^2). □

Thus this example guarantees that not every one point $p(\in X)$ deleted subspace of a finite K-topological plane X is a K-retract of X.

Using Remark 3(2), owing to the *FPP* of a retraction, we obtain the following:

Corollary 3. *Consider a K-topological space (X, κ_X^n) contained in a K-cube $(I^n, \kappa_{I^n}^n)$ as a subspace such that (X, κ_X^n) is a K-retract of $(I^n, \kappa_{I^n}^n)$. Then (X, κ_X^n) has the FPP.*

Example 2. *(1) As referred to in Theorem 2, since the space (Y, κ_Y^2) is K-retractable onto (B, κ_B^2) in Figure 4b, owing to the K-retract property of the FPP (see Corollary 3), it turns out that (Y, κ_Y^2) has the FPP. Hence we conclude that (B, κ_B^2) has the FPP.*

(2) As referred to in Theorem 2, since the space (Z, κ_Z^2) is K-retractable onto (C, κ_C^2) in Figure 4c and further, owing to the K-retract property of the FPP, it turns out that (Z, κ_Z^2) has the FPP. Hence we conclude that (C, κ_C^2) has the FPP.

In relation to the two theorems initiated by Rival and Schröder referred to in Section 1, we can obviously have the following property.

Property 1. *Consider the set $X := [2,4]_{\mathbb{Z}} \times [1,3]_{\mathbb{Z}}$ in Figure 4a with the K-topological structure denoted by (X, κ_X^2) or a poset $(X, \leq)$ induced by the space (X, κ_X^2). Then the point $a := (4,2)$ in the poset $(X, \leq)$ is neither irreducible point nor retractable to any point $b \in X \setminus \{a\}$.*

Proof. We first prove that the point $a := (4,2)$ is not irreducible in $(X, \leq)$. Assume an element $x \in X$ such that $x < a$. Then we may take $x \in SN_K(a)$ in (X, κ_X^2), for example, $x \in \{a, x_2, x_3, x_4, x_6, x_7\}$. In case $x := x_4$, there is an element $x_3 \in X$ such that $x < x_3 < a$, which implies that a is not irreducible in $(X, \leq)$. Next, we prove that a is not retractable to any point $x(\neq a) \in X$. To be precise, in $(X, \leq)$ since $\downarrow a = SN_K(a) = \{a, x_2, x_3, x_4, x_6, x_7\}$, for any $b \in X$ we obtain

$$(\downarrow a) \setminus \{a\} \not\subseteq \downarrow b, \tag{7}$$

which implies that a is not retractable to the point b. To be specific, for any element $b \in X \setminus \{a, x_6, x_7\}$ the set $\downarrow b$ is not comparable with $(\downarrow a) \setminus \{a\}$. In case $b \in \{x_6, x_7\}$, the element a is not retractable to b either because the points a and b also satisfies the property (7). □

In view of Property 1, we cannot address the question posed in Section 1 by using the two theorems in Section 1. Thus, by using the *FPP* for the K-topological category, let us now investigate the *FPP* for some points deleted K-topological spaces which are not K-retracts of some K-topological planes.

Theorem 3. *Consider the K-topological plane* $X := [-2m_1, 2m_1']_{\mathbb{Z}} \times [-2m_2+1, 2m_2'+1]_{\mathbb{Z}}$. *Let* $A := X \setminus \{p\}$, *where p is a pure closed point in* $\{2m_1'\} \times [-2m_2+1, 2m_2'+1]_{\mathbb{Z}} \subset X$. *While the space* (X, κ_X^2) *is not K-retractable onto* (A, κ_A^2), (A, κ_A^2) *has the FPP.*

Proof. Without loss of generality, we may consider $X := [2,4]_{\mathbb{Z}} \times [1,3]_{\mathbb{Z}}$ and $A := X \setminus \{(4,2)\} := \{x_i \mid i \in [1,7]_{\mathbb{Z}}\}$ in Figure 4a. As referred to in Theorem 2, the given space (A, κ_A^2) is not a K-retract of any K-topological plane such as (X, κ_X^2). Thus we cannot adopt Corollary 3 into the study of the *FPP* of (A, κ_A^2). Hence we need to prove the *FPP* for (A, κ_A^2) from the viewpoint of Khalimsky topological category. Let us consider any K-continuous self-map of (A, κ_A^2). In particular, consider the point $x_0 \in A$ (see Figure 4a) and assume any K-continuous self-map f sending the point $x_0 \in A$.

In case $f(x_0) = x_0$, the proof is completed.

In case $f(x_0)$ is mapped into a pure open point in A such as $f(x_0) \in \{x_2, x_4\}$. Then, owing to the K-continuity of f (see the property (2)), we have $f(f(x_0)) = f(x_0)$ so that $f(x_0)$ is a fixed point of f. In case $f(x_0)$ is mapped into a mixed point in A such as $f(x_0) := t \in \{x_1, x_3, x_5, x_6, x_7\}$. Then, owing to the K-continuity of f, we should have $f(t) = t$ or $f(t) \in \{x_2, x_4\}$. If it happens to the former, the point t is a fixed point of f, and if it happens to the latter, the point x_2 or x_4 is a fixed point of f because $f(f(t)) = f(t)$. □

Example 3. *By using the method similar to the proof of Theorem 3, while the given K-topological space* $(A, \kappa_A^2) := \{x_i \mid i \in [0,10]_{\mathbb{Z}}\}$ *is not a K-retract of any K-topological plane such as* (X, κ_X^2), *where* $X := [1,4]_{\mathbb{Z}} \times [1,3]_{\mathbb{Z}}$, *we observe that* (A, κ_A^2) *has the FPP.*

5. Summary and Future Work

Using both an order-theoretical and an Alexandroff topological approaches, we have studied the *FPP* of Khalimsky topological spaces. In particular, we developed the product property of the *FPP* for K-topological spaces. Furthermore, we studied the *FPP* of non-K-retractable spaces and some points deleted K-topological planes.

As an extension of the results in the present paper, we need to establish various types of T_0-A-spaces. Besides, let *ETC* be the category of Euclidean topological space [22]. Then we need to develop a functor from *ETC* into *KTC* preserving the *FPP* of spaces in *ETC* into that of spaces in *KTC*. In addition, we can extend the results in this paper to the study of n-dimensional K-topological spaces.

Author Contributions: Conceptualization, S.-E.H. and J.M.K.; methodology, S.-E.H. and J.M.K.; validation, S.-E.H., J.M.K. and S.L.; formal analysis, S.-E.H., J.M.K. and S.L.; investigation, S.-E.H., J.M.K. and S.L.; writing—original draft preparation, S.-E.H. and J.M.K.; writing—review and editing, S.-E.H.; visualization, S.-E.H. and J.M.K.; supervision, S.-E.H.; funding acquisition, S.-E.H.

Funding: The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2019R1I1A3A03059103).

Conflicts of Interest: The authors declare no conflict of interest.

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