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The Generalized Solutions of the nth Order Cauchy–Euler Equation

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Abstract: In this paper, we use the Laplace transform technique to examine the generalized solutions of the nth order Cauchy–Euler equations. By interpreting the equations in a distributional way, we found that whether their solution types are classical, weak or distributional solutions relies on the conditions of their coefficients. To illustrate our findings, some examples are exhibited.

Keywords: Cauchy–Euler equations; distributional solution; Dirac delta function; generalized solution; Laplace transform

1. Introduction

The nth order Cauchy–Euler equations

$$a_n t^n y^{(n)}(t) + a_{n-1} t^{n-1} y^{(n-1)}(t) + \cdots + a_1 t y'(t) + a_0 y(t) = 0, \tag{1}$$

where $a_0, a_1, \ldots, a_n$ are real constant coefficients and $t \in \mathbb{R}$, are often one of the first higher order ordinary homogeneous linear differential equations with variable coefficients introduced in an undergraduate level course. Naturally, we will discuss the second order Cauchy–Euler equations first. The appropriate form for its solution is $y = t^r$ where r is a parameter to be resolved. Replacing y with t^r in the Cauchy–Euler equations yields the characteristic polynomial whose roots determine the forms of the general solution (e.g., see the textbooks [1,2]). This same technique can be carried over to solve the higher order Cauchy–Euler equations.

In the framework of distribution theory, R. P. Kanwal [3] classifies solution the type of ordinary homogeneous linear differential equations

$$a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \cdots + a_1(t) y'(t) + a_0(t) y(t) = 0, \tag{2}$$

where the coefficient functions $a_0(t), a_1(t), \ldots, a_n(t)$ are infinitely differentiable and $t \in \mathbb{R}$. The type can be explained as follows. The solution is a classical solution if it is at least n times continuously differentiable so that the differentiation in Equation (2) can be achieved in the ordinary sense with an identity result. The solution is a weak solution if it is less than n times continuously differentiable and thus it does not satisfy Equation (2) in the ordinary sense but in the weak or distributional sense. The solution is a distributional solution if it is a singular distribution satisfying Equation (2) in the weak sense. All of these are referred to as generalized solutions. It is widely known that the normal form of Equation (2) does not have weak or distributional, but classical solutions. Of particular interest are the singular distributions appearing as a finite series of the Dirac delta function and its derivatives. They can arise as a distributional solution for certain classes of ordinary differential equations with singular coefficients (see J. Wiener [4] in 1982). The applications of the distribution theory to differential equations have been examined by L. Schwartz [5] and A. H. Zemanian [6]. In 1983, J. Wiener and S. M. Shah [7] provided an overview of research in the distributional field and proposed a unified way in the investigation of both distributional and entire solutions to some classes of linear ordinary differential equations. Many mathematicians have also studied the distributional solutions in the field of theory of distributions, as can be seen in [8–12].

A. Kananthai [13] in 1999 considered certain third order Cauchy–Euler equations

$$t^3y'''(t) + t^2y''(t) + ty'(t) + my(t) = 0, \tag{3}$$

where m is an integer and $t \in \mathbb{R}$. He constructed a formula for m corresponding to each type of generalized solution of Equation (3), which are Laplace transformable. In 2017, A. Liangprom and K. Nonlaopon [14] extended the same study to certain fourth order Cauchy–Euler equations, a natural extension of Equation (3). The result for the general nth order Cauchy–Euler equations in this form was finally established by A. Sangsuwan, K. Nonlaopon and S. Orankitjaroen [15] one year after.

In 2018, P. Jodnok and K. Nonlaopon [16] presented the generalized solutions of the fifth order Cauchy–Euler equations of the form

$$t^5y^5(t) + a_4t^4y^4(t) + a_3t^3y'''(t) + a_2t^2y''(t) + a_1ty'(t) + a_0y(t) = 0, \tag{4}$$

where $a_0, a_1, \ldots, a_4$ are real constants and $t \in \mathbb{R}$. Depending on the values of $a_0, a_1, \ldots, a_4$, they showed that the solutions of Equation (4) are either the weak solutions or the distributional solutions.

In 2015, S. Nanta [17] studied the distributional solutions of the nth order Cauchy–Euler

$$a_nt^ny^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_1ty'(t) + a_0y(t) = 0, \tag{5}$$

where $a_i, i = 0, 1, \ldots, n$ are real constants using Fourier transform. She found that the type of solutions of Equation (5) depend on the conditions of a_i.

Here we aim to seek the generalized solutions of the nth order Cauchy–Euler equations of the form of Equation (5) in the space of right-sided distributions. The solutions are obtained by applying the Laplace transform technique. Our work is an improved version of that of A. Sangsuwan et al. [15].

The present paper is arranged into three sections. In Section 2, we provide related definitions and lemmas necessary to obtain our main results. We then proceed to prove our results together with supported examples in Section 3.

2. Preliminaries

The space $\mathscr{D}'$ (the space of distributions) is the dual space of $\mathscr{D}$, the space of testing functions.

Definition 1. *A distribution $T \in \mathscr{D}'$ is a continuous linear functional on $\mathscr{D}$. The value of T acting on a testing function $\phi(t)$ is written as $\langle T, \phi \rangle$ or $\langle T, \phi(t) \rangle$, and $\langle T, \phi(t) \rangle \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers.*

Distributions that are most useful are those generated by locally integrable functions. In fact, every locally integrable function $f(t)$ generates a distribution, which is defined by

$$\langle f, \phi \rangle = \int_{\mathbb{R}} f(t)\phi(t)\, dt.$$

Definition 2. *The kth order derivative of a distribution T is defined by*

$$\left\langle T^{(k)}, \phi(t) \right\rangle = (-1)^k \langle T, \phi(t) \rangle$$

for all $\phi(t) \in \mathscr{D}$.

Definition 3. *Let $f(t)$ be a locally integrable function which satisfies the following conditions:*

(i) $f(t) = 0$ *for all* $t < 0$;
(ii) *there exists a real number c such that $e^{-ct}f(t)$ is absolutely integrable over $\mathbb{R}$.*

Then the Laplace transform of $f(t)$ is

$$F(s) = \mathscr{L}\{f(t)\} = \langle f(t), e^{-st} \rangle, \tag{6}$$

where s is a complex variable.

Furthermore, if f is continuous, then its Laplace transform $F(s)$ is analytic on the half-plane $\mathcal{R}e(s) > \sigma_a$, where σ_a is an abscissa of absolute convergence for $\mathscr{L}\{f(t)\}$.

Definition 4. *Let $f(t)$ be a function satisfying the conditions in Definition 3, and $\mathscr{L}\{f(t)\} = F(s)$. The inverse Laplace transform of $F(s)$ is defined by*

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{c-i\omega}^{c+i\omega} F(s)e^{st} ds, \tag{7}$$

where $\mathcal{R}e(s) > \sigma_a$.

Recall that the Laplace transform $G(s)$ of a locally integrable function $g(t)$ that satisfies the conditions of Definition 3 is

$$G(s) = \mathscr{L}\{g(t)\} = \langle g(t), e^{-st} \rangle, \tag{8}$$

where $\mathcal{R}e(s) > \sigma_a$.

Definition 5. *Let $f(t)$ be a distribution satisfying the following properties:*

(i) *f is a right-sided distribution, that is, $f \in \mathscr{D}'_R$.*
(ii) *There exists a real number c for which $e^{-ct}f(t)$ is a tempered distribution.*

The Laplace transform of a right-sided distribution $f(t)$ satisfying (ii) is defined by

$$F(s) = \mathscr{L}\{f(t)\} = \left\langle e^{-st}f(t), X(t)e^{-(s-c)t} \right\rangle, \tag{9}$$

where $X(t)$ is an infinitely differentiable function with support bounded on the left, which equals to 1 over the neighborhood of the support of $f(t)$.

For $\mathcal{R}e(s) > c$, $X(t)e^{-(s-c)t}$ is a testing function in the space S of testing functions of rapid descent and $e^{-ct}f(t)$ is in the space S' of tempered distributions. Equation (9) can be deduced to

$$F(s) = \mathscr{L}\{f(t)\} = \langle f(t), e^{-st} \rangle, \tag{10}$$

then Equation (10) posses the sense given by the right-hand side of Equation (9). Now, $F(s)$ is a function of s defined over the right half-plane $\mathcal{R}e(s) > c$. A. H. Zemanian [6] proved that $F(s)$ is an analytic function in the region of convergence $\mathcal{R}e(s) > \sigma_1$, where σ_1 is the abscissa of convergence for which $e^{-ct}f(t) \in S'$ for some real $c > \sigma_1$. For more details about the Laplace transform of distributions, see [18,19] and the references therein.

Example 1. *Let $H(t)$ be the Heaviside function, $\delta(t)$ be the Dirac delta function and $f(t)$ be a Laplace-transformable distribution in $\mathscr{D}'_R$. If k is a positive integer, then*

(i) $\mathscr{L}\{(t^{k-1}H(t))/(k-1)!\} = 1/s^k, \quad \mathcal{R}e(s) > 0.$
(ii) $\mathscr{L}\{\delta(t)\} = 1, \quad -\infty < \mathcal{R}e(s) < \infty.$

(iii) $\mathscr{L}\left\{\delta^{(k)}(t)\right\} = s^k, \quad -\infty < \mathcal{R}e(s) < \infty.$

(iv) $\mathscr{L}\left\{t^k f(t)\right\} = (-1)^k F^{(k)}(s), \quad \mathcal{R}e(s) > \sigma_1.$

(v) $\mathscr{L}\left\{f^{(k)}(t)\right\} = s^k F(s), \quad \mathcal{R}e(s) > \sigma_1.$

Lemma 1. *Let $\psi(t)$ be an infinitely differentiable function. Then*

$$\begin{aligned}\psi(t)\delta^{(m)}(t) =&(-1)^m\psi^{(m)}(0)\delta(t) + (-1)^{m-1}m\psi^{(m-1)}(0)\delta'(t)\\ &+(-1)^{m-2}\frac{m(m-1)}{2!}\psi^{(m-2)}(0)\delta''(t) + \cdots + \psi(0)\delta^{(m)}(t).\end{aligned} \tag{11}$$

and

$$\begin{aligned}[\psi(t)H(t)]^{(m)} =&\psi^{(m)}(t)H(t) + \psi^{(m-1)}(0)\delta(t) + \psi^{(m-2)}(0)\delta'(t)\\ &+ \cdots + \psi(0)\delta^{(m-1)}(t).\end{aligned} \tag{12}$$

We refer the reader to [3] for a proof of Lemma 1.

A useful formula that follows from Equation (11), for any monomial $\psi(t) = t^n$, is

$$t^n\delta^{(m)}(t) = \begin{cases} 0, & \text{for } m < n; \\ (-1)^n \frac{m!}{(m-n)!}\delta^{(m-n)}(t), & \text{for } m \geq n. \end{cases} \tag{13}$$

Lemma 2. *If the equation*

$$\sum_{i=0}^{n} a_i(t)t^i y^{(i)}(t) = 0 \tag{14}$$

with coefficients $a_i(t) \in C^n$ and $a_n(0) \neq 0$ has a solution

$$y(t) = \sum_{i=0}^{k} b_i\delta^{(i)}(t), \quad b_k \neq 0, \tag{15}$$

of order k (order of distribution Equation (15)), then we have

$$\sum_{i=0}^{n}(-1)^i a_i(0)(k+i)! = 0. \tag{16}$$

Conversely, if k is the smallest non-negative integer root of Equation (16), there exists a kth order solution of Equation (15) at $t = 0$.

We refer the reader to [4] for a proof of Lemma 2.

3. Main Results

Equipped with the Laplace transform technique, we are now ready to prove our main results.

Theorem 1. *Consider the nth order Cauchy–Euler equations of the form*

$$a_n t^n y^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_1 t y'(t) + a_0 y(t) = 0, \tag{17}$$

where $a_i, i = 0, 1, 2, \ldots, n$ are real constants, $a_n \neq 0$, n is any integers with $n \geq 2$ and $t \in \mathbb{R}$. The types of Laplace transformable solutions in $\mathscr{D}'_{\mathbb{R}}$ of Equation (17) depend on the value of a_i, and are given by the following cases:

(i) If there exists a non-negative integer k such that

$$\sum_{i=0}^{n}(-1)^i(k+i)!a_i = 0, \tag{18}$$

then there exists a distributional solution of Equation (17), which is a singular distribution of the Dirac delta function and its derivatives.

(ii) If there exists a non-negative integer k less than or equal to n such that

$$\sum_{i=0}^{k}\frac{a_i}{(k-i)!} = 0, \tag{19}$$

then there exists a weak solution of Equation (17). Moreover, the solution is continuous if k is greater than or equal to 1.

(iii) If there exists a positive integer k such that

$$\sum_{i=0}^{n}\frac{a_i}{(n+k-i)!} = 0, \tag{20}$$

then there exists a classical solution of Equation (17).

Proof. We rewrite Equation (17) in brief as

$$\sum_{i=0}^{n} a_i t^i y^{(i)}(t) = 0, \tag{21}$$

where $a_n = 1$. Applying Laplace transform to Equation (21) with a notation $\mathscr{L}\{y\} = Y(s)$, we now refer to properties (iv) and (v) in Example 1 to get

$$\sum_{i=0}^{n}(-1)^i a_i \frac{d^i}{ds^i}\left[s^i Y(s)\right] = 0,$$

that is,

$$\sum_{i=0}^{n}(-1)^i a_i \left\{\sum_{m=0}^{i}\left[\binom{i}{m}^2 m! s^{i-m} Y^{(i-m)}(s)\right]\right\} = 0, \tag{22}$$

or

$$\sum_{i=0}^{n}(-1)^i a_i i! \left\{\sum_{j=0}^{i}\left[\binom{i}{j}\frac{1}{j!}s^j Y^{(j)}(s)\right]\right\} = 0. \tag{23}$$

Consider a solution of Equation (23) in a simple form $Y(s) = s^r$, where r is a real constant that must be determined. Replacing $Y^{(i)}(s)$ for $i = 1, 2, 3, \ldots, n$ in Equation (23) gives

$$\sum_{i=0}^{n}(-1)^i a_i i! \sum_{j=0}^{i}\left[\binom{i}{j}\binom{r}{j}\right] s^r = 0.$$

The identity

$$\sum_{j=0}^{i}\left[\binom{i}{j}\binom{r}{j}\right] = \binom{r+i}{i}$$

and $s^r \neq 0$ imply that

$$\sum_{i=0}^{n}(-1)^i i! \binom{r+i}{i} a_i = 0. \tag{24}$$

We now examine the proposed three cases of the value r.

Case (i). If r is a non-negative integer, then substituting $r = k$ for $k \in \mathbb{N} \cup \{0\}$ into Equation (24), we obtain Equation (18). Thus, if the condition of Equation (18) holds, then the solution of Equation (21) is $Y(s) = s^k$. Obviously $Y(s)$ is analytic over the whole s-plane. Taking inverse Laplace transform to $Y(s)$ and applying property (iii) in Example 1, we obtain the distributional solutions of Equation (17) in the form

$$y(t) = \delta^{(k)}(t). \tag{25}$$

Case (ii). If r is a negative integer which is no less than $-(n+1)$, then substituting $r = -(k+1)$ for $k \in \{0, 1, 2, \ldots, n\}$ into Equation (24), we obtain

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i i! \binom{-(k+1)+i}{i} a_i \\
&= \sum_{i=0}^{k} (-1)^i i! \binom{-(k+1)+i}{i} a_i \\
&= \sum_{i=0}^{k} (-1)^i i! \frac{(-(k+1)+i)(-(k+1)+i-1)\cdots(-(k+1)+i-i+1)}{i!} a_i \\
&= \sum_{i=0}^{k} \frac{k!}{(k-i)!} a_i = 0.
\end{aligned}$$

Thus, if the condition of Equation (19) holds, then the solution of Equation (21) is $Y(s) = s^{-(k+1)}$. Now we take the inverse Laplace transform to $Y(s)$, applying property (i) in Example 1, and we obtain the weak solutions of Equation (17), since $k \leq n$ in the form

$$y(t) = H(t)\frac{t^k}{k!}. \tag{26}$$

Observe that the solution is continuous for $k \geq 1$.

Case (iii). If r is a negative integer less than $-(n+1)$, then a substitution of $r = -(n+k+1)$ for $k \in \mathbb{N}$ into Equation (24), gives

$$\begin{aligned}
&\sum_{i=0}^{n} (-1)^i i! \binom{-(n+k+1)+i}{i} a_i \\
&= \sum_{i=0}^{n} (-1)^i i! \frac{(-(n+k+1)+i)(-(n+k+1)+i-1)\cdots(-(n+k+1)+i-i+1)}{i!} a_i \\
&= \sum_{i=0}^{k} \frac{(n+k)!}{(n+k-i)!} a_i = 0.
\end{aligned}$$

Thus, if the condition of Equation (20) holds, then the solution of Equation (21) is $Y(s) = s^{-(n+k+1)}$. Now we take the inverse Laplace transform to $Y(s)$ and, applying property (i) in Example 1, we obtain the classical solutions of Equation (17) because $k \geq 1$ and the solutions are

$$y(t) = H(t)\frac{t^{n+k}}{(n+k)!}. \tag{27}$$

□

Theorem 2. *The distributional solution of the nth order Cauchy–Euler equations of the form*

$$a_n t^n y^{(n)}(t) + a_{n-1} t^{n-1} y^{(n-1)}(t) + \cdots + a_1 t y'(t) + a_0 y(t) = 0,$$

where $a_i, i = 0, 1, 2, \dots, n$ are real constants, n is any integer and $t \in \mathbb{R}$, depends on the values of a_i, $i = 0, 1, \dots, n$ of the form

$$\sum_{i=0}^{n} (-1)^i (k+i)! a_i = 0, \tag{28}$$

for any $k \in \mathbb{N}$, being the order of the distribution.

Proof. Using Lemma 2 and substituting $a_i(0) = a_i$ for $i = 0, 1, \dots, n-1$ and $a_n(0) = 1$ into Equation (16), we have Equation (28) as required. □

Example 2. *When $n = 2$, Equation (17) is just*

$$t^2 y''(t) + a_1 t y'(t) + a_0 y(t) = 0, \tag{29}$$

where $t \in \mathbb{R}$. Various values of a_1 and a_0 lead to various types of solutions as mentioned in Theorem 1.

If a_1, a_0, k are chosen according to Equation (18), for example $a_1 = 3$, $a_0 = 1$ and $k = 1$, then Equation (29) becomes

$$t^2 y''(t) + 3t y'(t) + y(t) = 0, \tag{30}$$

and Equation (25) implies that the distributional solution of Equation (30) is $y(t) = \delta(t)$. Using Equation (13), we can verify easily that Equation (30) is true for $y(t) = \delta(t)$.

If a_1, a_0, k are chosen according to Equation (20), for example $a_1 = 7/4$, $a_0 = -45/4$ and $k = 1$, then Equation (29) becomes

$$t^2 y''(t) + \frac{7}{4} t y'(t) - \frac{45}{4} y(t) = 0, \tag{31}$$

and Equation (27) implies that the classical solution of Equation (31) is $y(t) = t^3 H(t)/3!$. We can verify easily that Equation (31) is true for $y(t) = t^3 H(t)/3!$.

Example 3. *When $n = 3$, Equation (17) becomes*

$$t^3 y'''(t) + a_2 t^2 y''(t) + a_1 t y'(t) + a_0 y(t) = 0, \tag{32}$$

where $t \in \mathbb{R}$. Various values of a_2, a_1 and a_0 lead to various types of solutions, as mentioned in Theorem 1.

If a_2, a_1, a_0, k are chosen according to Equation (18), for example $a_2 = 1/12, a_1 = 2/3$, $a_0 = 61$ and $k = 2$, then Equation (32) becomes

$$t^3 y'''(t) + \frac{1}{12} t^2 y''(t) + \frac{2}{3} t y'(t) + 61 y(t) = 0, \tag{33}$$

and Equation (25) implies that the distributional solution of Equation (33) is $y(t) = \delta''(t)$. Using Equation (13), we can verify easily that Equation (33) is true $y(t) = \delta''(t)$.

If a_2, a_1, a_0, k are chosen according to Equation (19), for example $a_2 = 1, a_1 = -2/3$, $a_0 = -2/3$ and $k = 2$, then Equation (32) becomes

$$t^3 y'''(t) + t^2 y''(t) - \frac{2}{3} t y'(t) - \frac{2}{3} y(t) = 0, \tag{34}$$

and Equation (27) implies that the weak solution of Equation (34) is $y(t) = t^2 H(t)/2!$. Using Equations (12) and (13), we can verify easily that Equation (34) is true for $y(t) = t^2 H(t)/2!$.

Remark 1. *If $n = 3$, $a_1 = a_2 = a_3 = 1$ and $a_0 = m$, then Theorem 1 is reduced to the condition in [13].*

Remark 2. *If $n = 5$, then Theorem 1 is reduced to the case of the fifth order Cauchy–Euler equation, as appears in theorem 3.1 of [16].*

Remark 3. *If $a_1 = a_2 = \cdots = a_n = 1$ and $a_0 = m$, then Theorem 1 is identical to the condition in [15].*

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