mathematics

MDPI

Article

Approximation Properties in Felbin Fuzzy Normed Spaces

Ju Myung Kim † and Keun Young Lee *,†

Department of Mathematics and Statistics, Sejong University, Seoul 05006, Korea; kjm21@sejong.ac.kr
* Correspondence: bst21@sejong.ac.kr; Tel.: +82-2-6935-2504
† These authors contributed equally to this work.

Received: 5 September 2019; Accepted: 17 October 2019; Published: 22 October 2019

Abstract: In this paper, approximation properties in Felbin fuzzy normed spaces are considered. These approximation properties are new concepts in Felbin fuzzy normed spaces. Definitions and examples of such properties are given and we make a comparative study among approximation properties in Bag and Samanta fuzzy normed spaces and Felbin fuzzy normed spaces. We develop the representation of finite rank bounded operators in our context. By using this representation, characterizations of approximation properties are established in Felbin fuzzy normed spaces.

Keywords: felbin-fuzzy normed space; bag and samanta fuzzy normed space; approximation property; bound approximation property; finite rank operator

1. Introduction

The concept of a fuzzy norm on a vector space was first introduced by Katsaras [1]. After his works, Felbin [2] introduced an alternative definition of a fuzzy norm (namely, the Felbin fuzzy norm) related to a fuzzy metric of Kaleva–Seikkala's type [3]. Another fuzzy norm (namely, the B-S fuzzy norm) was defined by Bag and Samanta [4]. Bag and Samanta also conducted a comparative study of the relationship between their fuzzy norms and the fuzzy norms defined by Felbin [5]. Recently, topological properties including an inner product, fuzzy sets, and a boundedness have been studied according to Felbin type fuzzy norms and B-S type fuzzy norms [6–8]. Cho et al. systemically provided classical and recent results of fuzzy normed spaces and fuzzy operators in their book [9].

The approximation property (AP) is a key notion for the research of functional analysis. The AP indicates that the identity operator on an Banach space can be approximated in the compact open topology by finite rank operators [10–13]. The AP has been applied to study Shauder basis and operator theory. In 2010, Yilmaz introduced the approximation property in B-S fuzzy normed spaces [14]. The second author [15] modified Yilmaz's definitions and introduced the approximation property and the bounded approximation property in B-S fuzzy normed spaces. Related works have emerged from fuzzy theory. We would refer to intuitionistic fuzzy Banach space theory [16].

In this paper we establish approximation properties in Felbin fuzzy normed spaces. Moreover, we will conduct a comparative study among approximation properties in B-S fuzzy normed spaces and Felbin fuzzy normed spaces. We characterize approximation properties in Felbin fuzzy normed spaces. The advantage of our context is to make tools for operators in fuzzy analysis since we develop the representation of finite rank operators.

Our paper is organized as follows. Section 2 comprises some preliminary results. In Section 3, we define approximation properties and bounded approximation properties in Felbin fuzzy normed spaces. Furthermore, we provide several examples related to these properties. In Section 4, we give relations by making a comparative study of the approximation properties in fuzzy normed spaces defined by Bag and Samanta and Felbin. Section 5 is devoted to developing the representation of a finite rank operator as tools for analyzing approximation properties. In Section 6, we apply this representation to establish characterizations of approximation properties in Felbin fuzzy normed spaces.

2. Preliminaries

Definition 1. *(See [5].) A mapping $\eta : \mathbb{R} \to [0,1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geqslant \alpha\}$, if it satisfies the following conditions:*

(i) there exists a $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$

(ii) for each $\alpha \in (0,1]$, there exist real numbers $\eta_\alpha^- \leqslant \eta_\alpha^+$ such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta_\alpha^-, \eta_\alpha^+]$.

The set of all fuzzy real numbers is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ denoted by

$$\tilde{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r, \end{cases}$$

hence it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$ (See [5]).

Definition 2. *(See [5].) Let X be a vector space over $\mathbb{R}$. Assume the mappings $L, R : [0,1] \times [0,1] \to [0,1]$ are symmetric and non-decreasing in both arguments, and that $L(0,0) = 0$ and $R(1,1) = 1$. Let $\|\cdot\| : X \to F^*(\mathbb{R})$. The quadruple $(X, \|\cdot\|, L, R)$ is called a Felbin fuzzy normed space with the fuzzy norm $\|\cdot\|$, if the following conditions are satisfied:*

(F1) if $x \neq 0$, then $\inf_{0<\alpha\leqslant 1} \|x\|_\alpha^- > 0$,

(F2) $\|x\| = \tilde{0}$ if and only if $x = 0$,

(F3) $\|rx\| = |\tilde{r}|\|x\|$ for $x \in X$ and $r \in \mathbb{R}$,

(F4) for all $x, y \in X$,

(F4L) $\|x+y\|(s+t) \geqslant L(\|x\|(s), \|y\|(t))$ whenever $s \leqslant \|x\|_1^-, t \leqslant \|y\|^-$ and $s + t \leqslant \|x+y\|_1^-$,

(F4R) $\|x+y\|(s+t) \leqslant R(\|x\|(s), \|y\|(t))$ whenever $s \geqslant \|x\|_1^-, t \geqslant \|y\|^-$ and $s + t \geqslant \|x+y\|_1^-$.

We assume that

(F5) for any sequence (α_k) in $(0,1]$ such that $\alpha_k \downarrow \alpha \in (0,1]$ such that $\|x\|_{\alpha_k}^+ \uparrow \|x\|_\alpha^+$ for all $x \in X$. In this paper we fix $L(s,t) = \min(s,t)$ and $R(s,t) = \max(s,t)$ for all $s, t \in [0,1]$ and we write $(X, \|\cdot\|)$.

Definition 3. *(See [5].) Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space. A sequence $\{x_n\}$ of X is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} \|x_n - x\|_\alpha^+ = 0$ for all $\alpha \in (0,1]$. A subset A of X is called compact in $(X, \|\cdot\|)$ if each sequence of elements of A has a convergent subsequence in $(X, \|\cdot\|)$.*

Definition 4. *(See [17].) Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. The linear operator $T : X \to Y$ is said to be strongly fuzzy bounded if there is a real number $M > 0$ such that $\|Tx\|^\sim \preceq \tilde{M} \otimes \|x\|$ for all $x \in X$. We will denote the set of all strongly fuzzy bounded operators from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|^\sim)$ by $F(X,Y)$. Then $F(X,Y)$ is a vector space. For all $M > 0$ we denote $F(X,Y,M)$ by*

$$\{T \in F(X,Y) : \|Tx\|^\sim \preceq \tilde{M} \otimes \|x\|, \forall x \in X, \forall t \in \mathbf{R}\}$$

where M is a positive real number.

$\mathcal{A}$ is called bounded in $F(X,Y)$ if $\mathcal{A} = F(X,Y,M)$ for some $M > 0$. Moreover, we denote the set of all finite rank strongly fuzzy bounded operators from $(X, \|\cdot\|)$ to $(Y, \|\cdot\|^\sim)$ by $\mathcal{F}(X,Y)$. Then $\mathcal{F}(X,Y)$ is a subspace of $F(X,Y)$. We similarly define $\mathcal{F}(X,Y,M)$ for some $M > 0$. Now, we provide definitions of the approximation properties in Felbin fuzzy normed spaces. For the definition and properties of the α-level set ($\alpha \in (0,1]$), see [3,18].

Definitions of a B-S fuzzy norm and a B-S fuzzy antinorm are well mentioned in [3]. Thus, we only give additional properties related to them.

Definition 5. *(See [17].) Let (X, N) be a B-S fuzzy normed space and (X, N^*) be a B-S fuzzy antinormed space. We assume that*
(N6) $N(x, t) > 0$ for all $t > 0$ implies that $x = 0$.
(N7) For $x \neq 0$ $N(x, \cdot)$ is continuous on $\mathbf{R}$ and strictly increasing on $\{t : 0 < N(x, t) < 1\}$.
Moreover, we assume that
(N^*6) $N^*(x, t) < 1$ for all $t > 0$ implies that $x = 0$.
(N^*7) For $x \neq 0$ $N(x, \cdot)$ is continuous on $\mathbf{R}$ and strictly decreasing on $\{t : 0 < N^*(x, t) < 1\}$.

Also, we need the new definition of compactness in given a B-S fuzzy norm and a B-S fuzzy antinorm as follows.

Definition 6. *Let N be a B-S fuzzy norm and N^* be a B-S fuzzy antinorm on a vector space X (briefly, (X, N, N^*)). A subset A of X is called compact in (X, N, N^*) if each sequence of elements of A has a convergent subsequence in (X, N, N^*) where A sequence $\{x_n\}$ of X is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$) if for each $t > 0$,*

$$\lim_{n\to\infty} N(x_n - x, t) = 1, \lim_{n\to\infty} N^*(x_n - x, t) = 0.$$

Definition 7. *(See [18,19].)*
(a) Let (X, N_1) and (Y, N_2) be B-S-fuzzy normed spaces. The linear operator $T : (X, N_1) \to (Y, N_2)$ is said to be a strongly fuzzy bounded if there exists a positive real number M such that $N_2(T(x), t) \geq N_1(x, \frac{t}{M})$ for all $x \in X$ and $t \in \mathbf{R}$.
(b) Let (X, N_1, N_1^*) and (Y, N_2, N_2^*) be given. The linear operator $T : (X, N_1) \to (Y, N_2)$ is said to be a strongly fuzzy bounded if there exists a positive real number M such that $N_2(T(x), t) \geq N_1(x, \frac{t}{M})$ and $N^*(Tx, t) \leqslant N^*(x, t/M)$ for all $x \in X$ and $t \in \mathbf{R}$.

Lemma 1. *(See [5]) Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space and $[\|x\|]_\alpha = [\|x\|^1_\alpha, \|x\|^2_\alpha]$, $\alpha \in (0,1]$. Let N and N^* be two functions in $X \times \mathbb{R}$ defined by*

$$N(x,t) = \begin{cases} \sup\{\alpha \in (0,1] : \|x\|^1_\alpha \leqslant t\}, & (x,t) \neq (\underline{0}, 0) \\ 0, & (x,t) = (\underline{0}, 0). \end{cases}$$

and

$$N^*(x,t) = \begin{cases} \inf\{\alpha \in (0,1] : \|x\|^2_\alpha \leqslant t\}, & (x,t) \neq (\underline{0}, 0) \\ 1, & (x,t) = (\underline{0}, 0). \end{cases}$$

Then N is a B-S fuzzy norm satisfying (N6) and N^* is a B-S fuzzy antinorm satisfying (N^*6),
(i) N satisfies (N6),
(ii) N^* satisfies (N^*6),
(iii) for each $x \neq \underline{0}, \exists r > 0 s.t. N(x, t) = 1, \forall t \geqslant r$,
(iv) for each $x \neq \underline{0}, \exists t_1 > 0 s.t. N(x, t_1) = 0$,
(v) $N^*(x, t) < 1 \Rightarrow N(x, t+) = 1$, where $N(x, t+) = \lim_{s\to t+} N(x, s)$.

Lemma 2. *(See [5]) Let N be a B-S fuzzy norm and N^* be a B-S fuzzy antinorm on a linear vector space X satisfying the conditions (i)–(v) of Lemma 1. Define*

$$\|x\|^*_\alpha = \inf\{t > 0 : N^*(x, t) < \alpha\},$$

and

$$\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geqslant \alpha\}.$$

Then there is a Felbin fuzzy norm $\|\cdot\|$ *on* X *such that* $[\|x\|]_\alpha = [\|x\|_\alpha, \|x\|^*_\alpha]$, $\alpha \in (0,1]$ *and* $x \in X$.

Lemma 3. *(See [5]) Let* $(X, \|\cdot\|)$ *be a Felbin fuzzy normed space such that* $\|\cdot\|$ *satisfies the condition (F5),* N *and* N^* *be two functions on* $X \times \mathbb{R}$ *defined in Lemma 1 and* $\|\cdot\|'$ *be the fuzzy norm defined in Lemma 2. Then we have* $\|\cdot\| = \|\cdot\|'$.

Note that, if $Y = \mathbb{R}$, the linear space of all real numbers , we define a function $\|r\|^\sim : \mathbb{R} \to [0,1]$ by

$$\|r\|^\sim(t) = \begin{cases} 1, & t = |r| \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|\cdot\|^\sim$ is a fuzzy norm on $\mathbb{R}$ and α-level sets of $\|r\|^\sim$ are given by $[\|r\|^\sim]_\alpha = [|r|, |r|]$ for all $0 < \alpha \leqslant 1$.

Definition 8. *(See [19].) A strongly fuzzy bounded linear operator defined from a Felbin fuzzy normed space* $(X, \|\cdot\|)$ *to* $(\mathbb{R}, \|\cdot\|^\sim)$ *is called a strongly fuzzy bounded linear functional. Denote by* $(X, \|\cdot\|)^*$ *the set of all strongly fuzzy bounded linear functionals over* $(X, \|\cdot\|)$. *Define*

$$\|f\|^{*-}_\alpha = \sup_{x \in X, x \neq 0} \frac{|f(x)|}{\|x\|^+_\alpha}, \|f\|^{*+}_\alpha = \sup_{x \in X, x \neq 0} \frac{|f(x)|}{\|x\|^-_\alpha}$$

for all $f \in (X, \|\cdot\|)^*$.

Remark 1. *Definition 4.1 came from Bag and Samanta [18]. Although they defined a strongly fuzzy bounded linear operator differently from this paper, the two definitions are the same in the case of functionals.*

The following lemma is the Hahn–Banach theorem on fuzzy normed spaces ([19], Theorem 7.1).

Lemma 4. *Let* $(X, \|\cdot\|)$ *be a Felbin fuzzy normed space and* Z *be a subspace of* X. *Let* f *be a strongly fuzzy bounded linear functional defined on* $(Z, \|\cdot\|)$. *Then there exists a strongly fuzzy bounded linear functional* $\widehat{f}$ *on* X *such that* $\widehat{f}|_Z = f$ *and* $\sup_{\alpha \in (0,1]} \|f\|^{*+}_\alpha = \sup_{\alpha \in (0,1]} \|\widehat{f}\|^{*+}_\alpha$.

Now we provide the definitions of approximation property in B-S fuzzy normed spaces [15].

Definition 9. *Let* (X, N) *be a B-S fuzzy normed space. A fuzzy normed space* (X, N) *is said to have the approximation property, if for every compact set* K *in* (X, N) *and for each* $\alpha \in (0,1)$ *and* $\varepsilon > 0$, *there exists a strongly fuzzy bounded* $T : X \to X$ *such that*

$$N(T(x) - x, \varepsilon) \geq 1 - \alpha, \forall x \in K.$$

3. Approximation Properties

In this section, we introduce definitions of approximation properties in Felbin fuzzy normed spaces and several examples.

Definition 10. *A Felbin fuzzy normed space* $(X, \|\cdot\|)$ *is said to have the approximation property (AP), if for every compact set* K *in* $(X, \|\cdot\|)$ *and for each* $\alpha \in (0,1]$ *and* $\varepsilon > 0$, *there exists an operator* $T \in \mathcal{F}(X, X)$ *such that*

$$\|T(x) - x\|^+_\alpha \leqslant \varepsilon$$

for every $x \in K$.

Definition 11. *Let λ be a positive real number. A Felbin fuzzy normed space $(X, \|\cdot\|)$ is said to have the λ-bounded approximation property (λ-BAP), if for every compact set K in $(X, \|\cdot\|)$ and for each $\alpha \in (0,1]$ and $\varepsilon > 0$, there exists an operator $T \in \mathcal{F}(X, X, \lambda)$ such that*

$$\|T(x) - x\|_\alpha^+ \leqslant \varepsilon$$

for every $x \in K$. Also we say that $(X, \|\cdot\|)$ has the BAP if $(X, \|\cdot\|)$ has the λ-BAP for some $\lambda > 0$.

By definition, we can have the following proposition.

Proposition 1. *The following are equivalent for a Felbin fuzzy normed space $(X, \|\cdot\|)$.*

(a) $(X, \|\cdot\|)$ has the AP.

(b) If $(Y, \|\cdot\|^\sim)$ is a Felbin fuzzy normed space, then for every $T \in F(X, Y)$, every compact set K in (X, N) and for each $\alpha \in (0,1)$ and $\varepsilon > 0$, there exists an operator $S \in \mathcal{F}(X, Y)$ such that

$$\|S(x) - T(x)\|_\alpha^{\sim +} \leqslant \varepsilon$$

for every $x \in K$.

(c) If $(Y, \|\cdot\|^\sim)$ is a Felbin fuzzy normed space, then for every $T \in F(Y, X)$, every compact set K in $(Y, \|\|^\sim)$ and for each $\alpha \in (0,1)$ and $\varepsilon > 0$, there exists an operator $S \in \mathcal{F}(Y, X)$ such that

$$\|S(y) - T(y)\|_\alpha^+ \leqslant \varepsilon$$

for every $x \in K$.

Proposition 2. *Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space and $M \geqslant 1$. Suppose that there exists a sequence $(T_n) \subset \mathcal{F}(X, X, M)$ such that $T_n(x) \longrightarrow x$ for every $x \in X$. Then $(X, \|\cdot\|)$ has the AP.*

Given a Felbin fuzzy normed space $(X, \|\cdot\|)$, we recall

$$B(x, \alpha, \varepsilon) = \{y \in X : \|x - y\|_\alpha^+ < \varepsilon\}.$$

By the proof of ([15], Lemma 4.2), we have the following.

Lemma 5. *Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space and K be a compact subset in $(X, \|\cdot\|)$. Then there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ in K such that for $x \in K$, we have $x \in B(x_i, \alpha, \varepsilon)$ for some x_i.*

Proof of Proposition 2. Let (T_n) be a sequence in $\mathcal{F}(X, X, M)$ such that $T_n \longrightarrow x$ for ever $x \in X$. Let K be a compact in $(X, \|\cdot\|)$ and $\alpha \in (0,1)$ and $\epsilon > 0$. By Lemma 5, there exists a finite set $\{x_1, x_2, \cdots, x_n\} \subset K$ such that for $x \in K$, we have $x \in B(x_i, \alpha, \epsilon/3M)$ for some x_i. Then there exists $N \in \mathbb{N}$ such that if $n \geqslant N$, then

$$\|T_n(x_i) - x_i\|_\alpha^+ < \epsilon/3,$$

for each i. Let $x \in K$ and take i such that $x \in B(x_i, \alpha, \epsilon/3M)$. Then for $n \geqslant N$, we have,

$$\begin{aligned} \|T_n(x) - x\|_\alpha^+ &\leqslant \|T_n(x) - T_n(x_i)\|_\alpha^+ + \|T_n(x_i) - x_i\|_\alpha^+ + \|x_i - x\|_\alpha^+ \\ &\leqslant M\|x - x_i\|_\alpha^+ + \|T_n(x_i) - x_i\|_\alpha^+ + \|x_i - x\|_\alpha^+ \\ &\leqslant M\|x - x_i\|_\alpha^+ + \frac{2\epsilon}{3} \\ &\leqslant \epsilon. \end{aligned} \tag{1}$$

Hence $(X, \|\cdot\|)$ has the AP. $\square$

By Proposition 2, we have the following.

Corollary 1. *Let $M \geqslant 1$ be given. Suppose a Felbin fuzzy normed space $(X, \|\cdot\|)$ has a basis $\{x_n\}$ and every natural projection $P_n : (X, \|\cdot\|) \to (X, \|\cdot\|)$ is in $\mathcal{F}(X, X, M)$. Then $(X, \|\cdot\|)$ has the AP.*

The converse of the above corollary may not be true.

Example 1. *There exists a Felbin fuzzy normed space $(X, \|\cdot\|)$ which has the AP (even MAP) but does not have a basis.*

Proof. Let us consider Banach space ℓ_∞ with $\|x\|_\infty = \sup_n |x_n|$. Moreover, $\|x\|_0 = \sup_n |\frac{x_n}{n}|$ is another norm on ℓ_∞. Now let us define

$$\|x\|(t) = \begin{cases} 1, & t = \|x\|_0 \\ 1/2, & \|x\|_0 \leqslant t < \|x\|_\infty \\ 0, & \text{otherwise.} \end{cases}$$

Then it can be easily shown that $(\ell_\infty, \|\|)$ is a fuzzy normed space. Also, we have,

$$[\|x\|]_\alpha = \begin{cases} [\|x\|_0, \|x\|_0], & \frac{1}{2} < \alpha \leqslant 1 \\ [\|x\|_0, \|x\|_\infty], & 0 < \alpha \leq \frac{1}{2}. \end{cases}$$

Then it follows that

$$\|x\|^+_\alpha = \begin{cases} \|x\|_0, & \frac{1}{2} < \alpha \leqslant 1 \\ \|x\|_\infty, & 0 < \alpha \leq \frac{1}{2}. \end{cases}$$

Then $(\ell_\infty, \|\cdot\|)$ cannot have a basis because the Banach space $(\ell_\infty, \|\cdot\|_\alpha)$, for $\frac{1}{2} \geqslant \alpha > 0$, is not separable. By the argument of ([14], Example 1), $(\ell_\infty, \|\cdot\|)$ has the AP.
□

Example 2. *There exists a Felbin fuzzy normed space $(X, \|\cdot\|)$ which does not have the AP.*

Proof. Let us say that a Banach space $(X, \|\cdot\|)$ does not have the approximation property [11]. Let us define

$$\|x\|(t) = \begin{cases} 1, & t = \|x\| \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\|x\|^+_\alpha = \|x\|$ for all $\alpha \in (0, 1]$. Then $(X, \|\cdot\|)$ does not have the AP. □

Example 3. *There exists a Felbin fuzzy normed space $(X, \|\cdot\|)$ which has the AP but fails the BAP.*

Proof. ([15], Example 4.9) indicates that the Banach space $X = (\sum \oplus X_n)_{\ell_2}$ has the AP but does not have the BAP, where for each n, X_n has the AP but does not have the BAP, and its norm is notated by $\|\cdot\|_n$. Let us define $\|x\|_2 = (\sum_{n=1}^\infty \|x_n\|_n^2)^{1/2}$ and $\|x\|_\infty = \sup_n \|x_n\|_n$ for all $x = (x_1, x_2, \ldots) \in X$. Now let us define

$$\|x\|(t) = \begin{cases} 1, & t = \|x\|_\infty \\ 1/2, & \|x\|_\infty < t \leqslant \|x\|_2 \\ 0, & \text{otherwise.} \end{cases}$$

Then it can be easily shown that $(X, \|\cdot\|)$ is a fuzzy normed space. Also, we have,

$$[\|x\|]_\alpha = \begin{cases} [\|x\|_\infty, \|x\|_\infty], & \frac{1}{2} < \alpha \leqslant 1 \\ [\|x\|_\infty, \|x\|_2], & 0 < \alpha \leq \frac{1}{2}. \end{cases}$$

Hence we obtain that

$$\|x\|_\alpha^+ = \begin{cases} \|x\|_\infty & \frac{1}{2} < \alpha \leqslant 1 \\ \|x\|_2, & 0 < \alpha \leq \frac{1}{2}. \end{cases}$$

Now we suppose that $(X, \|\cdot\|)$ has the BAP. Let K be a compact subset in $(X, \|\|_2)$. Then it is clear that K is a compact in $(X, \|\|)$. Let us take $\varepsilon > 0$ and $\alpha \in (0, 1/2]$. Then, by the assumption, there exist $\lambda > 0$ and $T \in \mathcal{F}(X, X, \lambda)$ such that

$$\|T(x) - x\|_\alpha^+ \leqslant \varepsilon$$

for every $x \in K$. Then we have $\|T(x) - x\|_2 < \varepsilon$ for all $x \in K$ and $\|T(x)\|_2 \leq \lambda\|x\|_2$ hence it is a contradiction.

To show that $(X, \|\cdot\|)$ has the AP, let K be a compact subset in $(X, \|\cdot\|)$ and $\varepsilon > 0$. By using the argument of ([15], Example 4.9), there exists a natural number $N \in \mathbf{N}$ and a finite rank operator $T_0 : (\sum_{n=1}^N \oplus X_n)_{\ell_2} \to (\sum_{n=1}^N \oplus X_n)_{\ell_2}$ such that

$$\|jT_0P_N(x) - x\|_2 < \varepsilon$$

for every $x \in K$ where $j : (\sum_{n=1}^N \oplus X_n)_{\ell_2} \to X$ defined by

$$j(x_1, x_2, \ldots, x_N) = (x_1, x_2, \ldots, x_N, 0, \ldots)$$

and $P_N : X \to (\sum_{n=1}^N \oplus X_n)_{\ell_2}$ is the projection given by $P_N((x_n)) = (x_1, x_2, \ldots, x_N)$. Put $T = jT_0P_N$. So we have

$$\|T(x) - x\|_\alpha^+ \leqslant \epsilon,$$

for every $\alpha \in (0, 1]$. Finally, we shall show that T is a strongly fuzzy bounded. Since $(\sum_{n=1}^N \oplus X_n)_{\ell_2}$ and $(\sum_{n=1}^N \oplus X_n)_{\ell_\infty}$ are equivalent, there exists $M_0 > 1$ such that

$$(\sum_{n=1}^N \|x_n\|_n^2)^{1/2} \leq M_0 \sup_{1 \leq n \leq N} \|x_n\|_n.$$

Now we put $M = \max\{\|T\|, \|jT_0\|M_0\}$. Then, for every $\alpha \in (0, 1]$, we obtain

$$\begin{aligned} \|T(x)\|_\alpha^- &= \|T(x)\|_\infty \\ &\leq \|T(x)\|_2 = \|jT_0P_N(x)\|_2 \\ &\leq \|jT_0\|(\sum_{n=1}^N \|x_n\|_n^2)^{1/2} \\ &\leq \|jT_0\|M_0 \sup_{1 \leq n \leq N} \|x_n\|_n \\ &\leq \|jT_0\|M_0\|x\|_\alpha^- \leqslant M\|x\|_\alpha^-. \end{aligned} \tag{2}$$

Also, for every $\alpha \in (0, 1/2]$, we obtain

$$\|T(x)\|_\alpha^+ = \|T(x)\|_2 \leqslant M\|x\|_2 = M\|x\|_\alpha^+.$$

For $\alpha \in (1/2, 1]$, by (2), we have

$$\|T(x)\|_\alpha^+ = \|T(x)\|_\infty \leqslant M\|x\|_\alpha^- \leqslant M\|x\|_\alpha^+.$$

□

4. Relations Between APs in Fellbin Fuzzy Normed Spaces and APs in B-S Fuzzy Normed Spaces

In this section, we establish relationships between approximation properties in Felbin fuzzy normed spaces and approximation properties in B-S fuzzy normed spaces.

Proposition 3. *Let X be a linear space. If X has the AP (BAP) with respect to any Felbin fuzzy norm, then it has the AP (BAP) with respect to any B-S fuzzy norm satisfying condition (N6) and (N7).*

Proof. Let N be a B-S fuzzy norm on X satisfying (N6) and (N7). Put $N^* = 1 - N$. Clearly, N^* is a fuzzy antinorm on X satisfying (N^*6) and (N^*7). For $0 < \alpha \leqslant 1$, we define

$$\|x\|_\alpha^* = \inf\{t > 0 : N^*(x,t) < \alpha\}.$$

Then, by ([5], Theorem 3.2), for each $\alpha \in (0,1]$, $\|x\|_\alpha^*$ is a norm on X. Now we define

$$N'(x,t) = \begin{cases} 1, & t > \|x\|_1^* \\ 0, & t \leqslant \|x\|_1^*. \end{cases}$$

It is clear that N' is a B-S-fuzzy norm satisfying (N6). Moreover it can be easily shown that N' and N^* satisfy (i)–(v) of Lemma 1. Let $\|x\|'_\alpha = \inf\{t > 0 : N'(x,t) \geqslant \alpha\}$ for every $\alpha \in (0,1]$. Indeed, we have $\|x\|'_\alpha = \|x\|_1^*$ for every $\alpha \in (0,1]$. Then, by Lemma 2, there is a Felbin fuzzy norm $\|\cdot\|$ on X such that $[\|x\|]_\alpha = \left[\|x\|_1^*, \|x\|_\alpha^*\right]$, $\alpha \in (0,1]$ and $x \in X$. By ([17], Theorem 15), we have

$$N^*(x,t) = \inf\{\alpha \in (0,1] : \|x\|_\alpha^* \leqslant t\}.$$

Take a compact subset K of (X, N). Then we claim that K is a compact subset of $(X, \|\cdot\|)$. Indeed, take any sequence (x_n) in K. Then there exist a subsequence (x_{n_j}) and x in K such that $x_{n_j} \to x$ in (X, N). Then $N(x_{n_j} - x, t) \to 1$ as $j \to \infty$, $\forall t > 0$. i.e. $N^*(x_{n_j} - x, t) \to 0$ as $j \to \infty$, $\forall t > 0$. So we have

$$\|x_{n_j} - x\|_\alpha^* \to 0$$

as $j \to \infty$, $\forall \alpha \in (0,1]$. Then (x_{n_j}) is a convergent sequence in $(X, \|\cdot\|)$, hence K is a compact subset of $(X, \|\cdot\|)$. Let $\alpha \in (0,1]$ and $\varepsilon > 0$. By definition of the AP in B-S fuzzy normed spaces, we shall show that there exists a strongly fuzzy bounded $T : X \to X$ such that

$$N(T(x) - x, \varepsilon) \geq 1 - \alpha$$

for every $x \in K$. By the assumption, there exists an operator $T \in \mathcal{F}(X, X)$ such that

$$\|T(x) - x\|_\alpha^* < \varepsilon$$

for every $x \in K$. By definition of $\|\cdot\|_\alpha^*$, there exists $0 < t_0 < \epsilon$ such that

$$N^*(Tx - x, t_0) < \alpha.$$

Since N^* is non-increasing, we have

$$N^*(Tx - x, \epsilon) \leqslant N^*(Tx - x, t_0) < \alpha,$$

so we have $N(Tx - x, \epsilon) > 1 - \alpha$. Finally, we shall show that T is sf -bounded operator on (X, N). By definition, we show that there exists a positive real number M such that $N(T(x), t) \geq N(x, \frac{t}{M})$ for all $x \in X$ and $t \in \mathbf{R}$. Indeed, since $T \in \mathcal{F}(X, X)$, there exists a $M > 0$ such that $\|Tx\|_1^* \leqslant M\|x\|_1^*$ and $\|Tx\|_\alpha^* \leqslant M\|x\|_\alpha^*$ for all $\alpha \in (0, 1]$. Then, for all $t > 0$, we have

$$\begin{aligned} N^*(Tx, t) &= \inf\{\alpha \in (0,1] : \|Tx\|_\alpha^* \leqslant t\} \\ &\leq \inf\{\alpha \in (0,1] : \|x\|_\alpha^* \leqslant \frac{t}{M}\} \\ &= N^*(x, \frac{t}{M}), \end{aligned} \tag{3}$$

so we have $N(Tx, t) \geqslant N(x, \frac{t}{M})$, hence T is sf-bounded.
□

We do not know whether the converse of Proposition 3 is true. However, we obtain the following.

Theorem 1. *Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space such that $\|\cdot\|$ satisfies the condition (F5). Let N and N^* be two functions in $X \times \mathbb{R}$ defined in Lemma 1. Then $(X, \|\cdot\|)$ has the AP, if and only if, for every a compact K in (X, N, N^*) and $\alpha \in (0, 1]$ and $t > 0$, there exists a strongly fuzzy bounded $T : (X, N, N^*) \to (X, N, N^*)$ such that*

$$N(T(x) - x, t) \geqslant 1 - \alpha, N^*(T(x) - x, t) < \alpha$$

for every $x \in K$.

Proof. Sufficiency. Put $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $\alpha \in (0, 1]$. Let N and N^* be two functions in $X \times \mathbb{R}$ defined in Lemma 1. By Lemma 3, we have

$$\|x\|_\alpha^- = \inf\{t > 0 : N(x, t) \geqslant \alpha\},$$

and

$$\|x\|_\alpha^+ = \inf\{t > 0 : N^*(x, t) < \alpha\}.$$

Take any a compact K in (X, N, N^*). By ([17], Theorem 24), K is a compact in $(X, \|\cdot\|)$. Let $\alpha \in (0, 1]$ and $\varepsilon > 0$. By the assumption, there exists an operator $T \in \mathcal{F}(X, X)$ such that

$$\|T(x) - x\|_\alpha^+ < \varepsilon$$

for every $x \in K$. By the argument of Proposition 3, we have

$$N^*(T(x) - x, t) < \alpha$$

for every $x \in K$. Also, we can observe that $N(T(x) - x, t) \geqslant 1 - \alpha$ for all $x \in K$ and $\alpha \in (0, 1)$. Indeed, fix $x \in X$. If $0 < \alpha < \frac{1}{2}$, then we have

$$\|x\|_{1-\alpha}^- \leqslant \|x\|_{1-\alpha}^+ \leqslant \|x\|_\alpha^+.$$

If $1 > \alpha > \frac{1}{2}$, then we have

$$\|x\|_{1-\alpha}^- \leqslant \|x\|_\alpha^- \leqslant \|x\|_\alpha^+,$$

hence we have $\|x\|_{1-\alpha}^- \leqslant \|x\|_\alpha^+$ for all $\alpha \in (0, 1)$. Then we have $\|T(x) - x\|_{1-\alpha}^- < \varepsilon$. By definition of $\|\cdot\|_{1-\alpha}^-$, there exists $0 < t_0 < \epsilon$ such that

$$N(Tx - x, t_0) \geqslant 1 - \alpha.$$

Since N is non-decreasing, we have

$$N(Tx - x, \epsilon) \geqslant N(Tx - x, t_0) \geqslant 1 - \alpha,$$

so we have $N(Tx - x, \epsilon) \geqslant 1 - \alpha$. Finally, by ([17], Theorem 17), T is strongly fuzzy bounded in (X, N, N^*).

Necessity. Let us take any compact K in $(X, \|\cdot\|)$. By ([17], Theorem 24), K is a compact in (X, N, N^*). Let $\alpha \in (0,1]$ and $\epsilon > 0$. By the assumption, there exists a strongly fuzzy bounded $T : (X, N, N^*) \to (X, N, N^*)$ such that

$$N(T(x) - x, \epsilon) \geqslant 1 - \alpha, N^*(T(x) - x, \epsilon) < \alpha$$

for every $x \in K$. Then there exists $0 < \alpha_0 \leqslant \alpha$ such that $\|Tx - x\|^+_{\alpha_0} \leqslant \epsilon$. Then we have $\|Tx - x\|^+_{\alpha} \leqslant \epsilon$. Finally, by ([17], Theorem 18), T is strongly fuzzy bounded in $(X, \|\cdot\|)$. □

The following example shows a partial relationship between the AP in Felbin fuzzy normed spaces and the AP in B-S fuzzy normed spaces.

Example 4. *There exists a linear space X such that a Felbin fuzzy normed space $(X, \|\cdot\|)$ does not have the AP and a B-S fuzzy normed space (X, N) also does not have the AP.*

Proof. We use Example 4. Let us define

$$N(x,t) = \begin{cases} 1, & t = \|x\| \\ 0, & \text{otherwise} \end{cases}$$

Clearly, we obtain that (X, N) has the condition (N6) and $\|x\|_\alpha = \|x\|$ for each $\alpha \in (0,1)$. Then, it is obvious that (X, N) does not have the AP. □

Question. Is there a linear space X such that a Felbin-fuzzy normed space $(X, \|\cdot\|)$ has the AP but a B-S fuzzy normed space (X, N) does not have the AP for some a felbin fuzzy norm $\|\cdot\|$ and a B-S fuzzy norm N?

5. The Representation of Finite Rank Strongly Fuzzy Bounded Operators

In this section, we develop the representation of finite rank strongly fuzzy bounded operators.

Theorem 2. *Let Y be a subspace of Felbin fuzzy normed space $(X, \|\cdot\|)$ such $\forall \alpha \in (0,1]$, Y is closed in $(X, \|\cdot\|^-_\alpha)$. Suppose that $x \in X \backslash Y$. Then there is a strongly fuzzy bounded linear functional f on X such that $\sup_{\alpha \in (0,1]} \|f\|^{*+}_\alpha = 1$ and $Y \subseteq \ker(f)$.*

Proof. Denote

$$d(x, Y) := \inf_{y \in Y} \inf_{\alpha \in (0,1]} \|x - y\|^-_\alpha.$$

We claim that $d(x, Y) > 0$. Suppose that $d(x, Y) = 0$. Take any $\epsilon > 0$. Then there exists $y \in Y$ such that $\inf_{\alpha \in (0,1]} \|x - y\|^-_\alpha < \epsilon$. So, there exists $\alpha \in (0,1]$ such that $0 < \|x - y\|^-_\alpha < \epsilon$. Then we have $x \in \overline{Y}^{\|\cdot\|^-_\alpha}$. Since Y is a closed subspace of $(X, \|\cdot\|^-_\alpha)$, hence we have $x \in Y$, it is a contradiction.

Let $f_0(y + \beta x) = \beta \cdot d(x, Y)$ for each $y \in Y$ and each scalar β. Then f_0 is a linear functional on $Y + \langle \{x\} \rangle$ such that $f_0(x) = d(x, Y)$ and $f_0(y) = 0$ for each $y \in Y$. For each $y \in Y, \alpha \in (0,1]$ and $\beta \neq 0$, we have

$$|f_0(y + \beta x)| = |\beta| \cdot d(x, Y) \leqslant |\beta| \|x - (-\beta^{-1} y)\|^-_\alpha = \|y + \beta x\|^-_\alpha,$$

so f_0 is a strongly fuzzy bounded linear functional on $Y + \langle \{x\} \rangle$ and $\sup_{\alpha \in (0,1]} \|f_0\|^{*+}_\alpha \leqslant 1$. Moreover, for each $\alpha \in (0,1]$, we have

$$\|f_0\|^{*+}_\alpha \|x - y\|^-_\alpha \geqslant |f_0(x - y)| = d(x, Y)$$

for all $y \in Y$, so we have

$$\sup_{\alpha \in (0,1]} \|f_0\|_\alpha^{*+} \cdot d(x,Y) = \sup_{\alpha \in (0,1]} \|f_0\|_\alpha^{*+} \inf_{\alpha \in (0,1]} \inf_{y \in Y} \|x-y\|_\alpha^- \geqslant d(x,Y).$$

Since $d(x,Y) > 0$, we obtain that $\sup_{\alpha \in (0,1]} \|f_0\|_\alpha^{*+} \geqslant 1$, so $\sup_{\alpha \in (0,1]} \|f_0\|_\alpha^{*+} = 1$. By Lemma 4, we can finish our proof.
□

The following corollary gives the representation of finite rank strongly fuzzy bounded operators.

Corollary 2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. If $T : X \to Y$ is a finite rank strongly fuzzy bounded linear operator, then there exist $(y_n)_{n=1}^k \subset Y$ and $(f_n)_{n=1}^k \subset (X, \|\cdot\|)^*$ such that*

$$T(x) = \sum_{n=1}^k f_n(x) y_n$$

for all $x \in X$.

Proof. Since $T(X)$ is a finite dimensional subspace of Y, there exists a basis $(y_n)_{n=1}^k \subset Y$ of $T(X)$. Now fix $n \in \{1, 2, \cdots, k\}$. Consider $Z = \langle y_{j \neq n} \rangle$. Then Z is a finite dimensional subspace of Y and $y_n \in Y \backslash Z$. Since, $\forall \alpha \in (0,1]$, Z is closed in $(Y, \|\cdot\|_\alpha^{\sim -})$, by Theorem 2, there is a strongly fuzzy bounded linear functional g_n on Y such that $g_n(y_n) \neq 0$ and $Z \subseteq \ker(g_n)$. We may assume that $g_n(y_n) = 1$. Put $f_n := g_n \circ T$. Then f_n is also a strongly fuzzy bounded linear functional on X. Finally, we take any $x \in X$ and write $T(x) = \sum_{n=1}^k a_n(x) y_n$ where for each n $a_n(x)$ is a scalar depending on x. By properties of g_n, it is clear that $f_n(x) = a_n(x)$ for each n.
□

6. Characterizations of Approximation Properties in Felbin Fuzzy Normed Spaces

In this section, we establish characterizations of approximation properties in Felbin fuzzy normed spaces. To do this, we develop topological methods in spaces of strongly fuzzy bounded linear operators.

Definition 12. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. For a compact $K \subset (X, \|\cdot\|)$, $\varepsilon > 0$, $\alpha \in (0,1]$, and $T \in F(X,Y)$ we put*

$$Ne(T, K, \alpha, \varepsilon) = \{R \in F(X,Y) : \sup_{x \in K} \|Tx - Rx\|_\alpha^{\sim +} < \varepsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $Ne(T, K, \alpha, \varepsilon)'s$. Then the τ-topology on $F(X,Y)$ is the topology generated by $\mathcal{S}$.

For a net $(T_\beta) \subset F(X,Y)$ and $T \in F(X,Y)$ we have $T_\beta \to T$ in $(F(X,Y), \tau)$ if and only if for every compact $K \subset (X, \|\cdot\|)$ and $\alpha \in (0,1]$,

$$\sup_{x \in K} \|T_\beta(x) - T(x)\|_\alpha^+ \to 0.$$

Definition 13. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin-fuzzy normed spaces. For $x \in X$, $\varepsilon > 0$, $\alpha \in (0,1]$, and $T \in F(X,Y)$ we put*

$$Ne(T, x, \alpha, \varepsilon) = \{R \in F(X,Y) : \|Tx - Rx\|_\alpha^{\sim 2} < \varepsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $Ne(T, x, \alpha, \varepsilon)'s$. Then the τ_{sto}-topology on $F(X, Y)$ is the topology generated by $\mathcal{S}$.

For a net $(T_\beta) \subset F(X, Y)$ and $T \in F(X, Y)$ we have $T_\beta \to T$ in $(F(X, Y), \tau_{sto})$ if and only if for every $x \in X$ and $\alpha \in (0, 1]$,

$$\|T_\beta(x) - T(x)\|_\alpha \to 0.$$

From ([20], Notation 3.6), we denote by $(X_\alpha^{*1}, \|\cdot\|_\alpha^{*1})$ and $(X_\alpha^{*2}, \|\cdot\|_\alpha^{*2})$ the dual space of $(X, \|\cdot\|_\alpha^1)$ and $(X, \|\cdot\|_\alpha^2)$ respectively.

Definition 14. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. Let $\mathcal{Z}$ be the linear span of all linear functionals f on $F(X, Y)$ of the form $f(T) = y^*Tx$ for $x \in X$ and $y^* \in Y_\alpha^{*2}$ where $\alpha \in (0, 1]$. Then τ_{wo}-topology on $F(X, Y)$ is the topology generated by $\mathcal{Z}$.*

For a net $(T_\beta) \subset F(X, Y)$ and $T \in F(X, Y)$ we have $T_\beta \to T$ in $(F(X, Y), \tau_{wo})$ if and only if for every $x \in X$, $\alpha \in (0, 1]$ and $y^* \in Y_\alpha^{*2}$

$$y^*T_\beta x \to y^*Tx.$$

Then we provide the following simple proposition. The proof is clear.

Proposition 4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces.*
(a) τ, τ_{sto} and τ_{wo} are locally convex topologies.
(b) τ is stronger than τ_{sto} and τ_{sto} is stronger than τ_{wo}.

By the argument of ([15], Proposition 5.6) and Lemma 5, we have the following.

Proposition 5. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. If $\mathcal{A}$ is a bounded in $F(X, Y)$, then we have $\tau = \tau_{sto}$ on $\mathcal{A}$.*

To show a relation between τ_{sto} and τ_{wo} on $F(X, Y)$, we need the following two lemmas. We recall that $(F(X, Y), \tau_{sto})^*((F(X, Y), \tau_{wo})^*)$ is the vector space of all τ_{sto}-continuous (τ_{wo}-continuous) linear functionals on $F(X, Y)$.

Lemma 6. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^\sim)$ be Felbin fuzzy normed spaces. If $f \in (F(X, Y), \tau_{sto})^*$, then there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and $\alpha \in (0, 1]$ and $\varepsilon > 0$ such that $T \in \bigcap_{i=1}^n \{T \in F(X, Y) : \|Tx_i\|_\alpha^{\sim 2} < \varepsilon\}$ implies $|f(T)| < 1$.*

Proof. By Definition 13, there exists a finite set $\{x_1, x_2, \ldots, x_n\}$ of X and a set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and a set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ such that $|f(T)| < 1$ for all $T \in \bigcap_{i=1}^n \{R \in F(X, Y) : \|Rx_i\|_{\alpha_i}^{\sim 2} < \varepsilon_i\}$ where $\alpha_i \in (0, 1]$ and $\varepsilon_i > 0$ for all $i = 1, 2, \ldots, n$. Now we put $\alpha = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. We consider a set $\bigcap_{i=1}^n \{T \in F(X, Y) : \|Tx_i\|_\alpha < \varepsilon\}$. Since $\{\|\cdot\|_\beta^{\sim 2} : \beta \in (0, 1]\}$ is a descending family of norms on Y, we obtain

$$\bigcap_{i=1}^n \{T \in F(X, Y) : \|Tx_i\|_\alpha^{\sim 2} < \varepsilon\} \subseteq \bigcap_{i=1}^n \{T \in F(X, Y) : \|Tx_i\|_{\alpha_i}^{\sim 2} < \varepsilon_i\}.$$

Hence if $T \in \bigcap_{i=1}^n \{T \in F(X, Y) : \|Tx_i\|_\alpha^{\sim 2} < \varepsilon\}$, then $|f(T)| < 1$.
□

By the proof of ([15], Lemma 5.8) and Lemma 6, we derive the following lemma.

Lemma 7. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^{\sim})$ be Felbin fuzzy normed spaces. Then*

$$(F(X,Y), \tau_{sto})^* = (F(X,Y), \tau_{wo})^*$$

and the form of the continuous linear functionals f on $F(X,Y)$ is $f(T) = \sum_{i=1}^{n} y_i^*(Tx_i)$, $(x_i)_{i=1}^{n} \subset X$ and $(y_i^*)_{i=1}^{n} \subset Y_\alpha^{*2}$ for some $\alpha \in (0,1]$.

Proposition 6. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|^{\sim})$ be Felbin fuzzy normed spaces.*
(a) If $\mathcal{C}$ is a convex set in $F(X,Y)$, then $\overline{\mathcal{C}}^{\tau_{sto}} = \overline{\mathcal{C}}^{\tau_{wo}}$.
(b) If $\mathcal{C}$ is a bounded convex set in $F(X,Y)$, then $\overline{\mathcal{C}}^{\tau} = \overline{\mathcal{C}}^{\tau_{wo}}$.

Proof.
(a) By Lemma 6 and ([21], Corollary 2.2.29), we derive (a).
(b) By Proposition 5 and (a), we prove (b).
□

Now, using the results so far, we provide characterizations of a Felbin fuzzy normed space to have the BAP. To prove these characterizations, we need the following lemma. For the proof, we refer to [22].

Lemma 8. *Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space. Suppose that $\mathcal{C}$ is a balanced convex subset of $F(X,X)$. Let $T \in F(X,X)$. Then the following are equivalent.*
(a) T belongs to $\overline{\mathcal{C}}^{\tau_{wo}}$.
(b) For every $f \in (F(X,Y), \tau_{wo})^*$ such that $|f(S)| \le 1$ for all $S \in \mathcal{C}$, we have $|f(T)| \le 1$.

Theorem 3. *Let $(X, \|\cdot\|)$ be a Felbin fuzzy normed space. Then the following are equivalent.*
(a) $(X, \|\cdot\|)$ has λ-BAP.
(b) There exists a net (T_β) in $\mathcal{F}(X,X,\lambda)$ such that $x^*T_\beta x \to x^*x$ for each $x \in X$, $\alpha \in (0,1]$ and $x^* \in X_\alpha^{*2}$.
(c) For every $\alpha \in (0,1]$, $(x_i)_{i=1}^{n} \subset X$ and $(x_i^*)_{i=1}^{n} \subset X_\alpha^{*2}$, if $|\sum_{i=1}^{n} x_i^*(Sx_i)| \le 1$ for all $S \in \mathcal{F}(X,X,\lambda)$, then $|\sum_{i=1}^{n} x_i^*(x_i)| \le 1$.
(d) For every $\alpha \in (0,1]$, $(x_i)_{i=1}^{n} \subset X$ and $(x_i^*)_{i=1}^{n} \subset X_\alpha^{*2}$, if

$$|\sum_{i=1}^{n}\sum_{k=1}^{\ell} x_i^*(z_k) f_k(x_i)| \le 1$$

for all $(f_k)_{k=1}^{\ell} \in (X, \|\cdot\|)^*$ and $(z_k)_{k=1}^{\ell} \in X$ with $\sum_{k=1}^{\ell} f_k(\cdot)z_k \in \mathcal{F}(X,X,\lambda)$, then $|\sum_{i=1}^{n} x_i^*(x_i)| \le 1$.

Proof. (a)⇔(b)⇔(c) By Lemma 7, Proposition 6, Lemma 8, and the argument of ([15], Theorem 6.3), it can be deduced,
(c)⇔(d). By Corollary 2, it is clear.
□

Remark 2. *Theorem 3 (d) implies that the BAP in Felbin fuzzy normed spaces has better characterizations compared with the BAP in B-S fuzzy normed spaces (cf. [15], Theorem 6.3).*

7. Conclusions and Further Works

In this paper we have introduced approximation properties in Felbin fuzzy normed spaces and investigated several examples. We have established a comparative study among approximation properties in B-S fuzzy normed spaces and Felbin fuzzy normed spaces. The representation of finite rank operators has been developed and, by using this, we provided characterizations of approximation properties in Felbin fuzzy normed spaces. We hope that our approach may provide a key role in fuzzy analysis by application to fuzzy function spaces, for example, spaces of fuzzy continuous functions. Moreover, many kinds of approximation properties can be introduced.

Author Contributions: The individual contributions of the authors are as follows: conceptualization, K.Y.L.; methodology, K.Y.L. and J.M.K.; validation K.Y.L. and J.M.K.; formal analysis, K.Y.L.; investigation, K.Y.L. and J.M.K.; resources J.M.K.; writing—original draft preparation, K.Y.L. and J.M.K.; writing—review and editing, K.Y.L.; project administration, K.Y.L. and J.M.K.; funding acquisition, K.Y.L. and J.M.K.

Funding: The first author was supported by NRF-2018R1D1A1B07043566 funded by the Korean Government. The corresponding author was supported by NRF-2017R1C1B5017026 funded by the Korean Government.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Katsaras, A.K. Fuzzy topological vector spaces. *Fuzzy Sets Syst.* **1984**, *12*, 143–154. [CrossRef]
2. Felbin, C. Finite dimensional fuzzy normed linear spaces. *Fuzzy Sets Syst.* **1992**, *48*, 239–248. [CrossRef]
3. Kaleva, O.; Seikkala, S. On fuzzy metric spaces. *Fuzzy Sets Syst.* **1984**, *12*, 215–229. [CrossRef]
4. Bag, T.; Samanta, S.K. Finite dimenstional fuzzy normed linear spaces. *J. Fuzzy Math.* **2003**, *11*, 687–705.
5. Bag, T.; Samanta, S.K. A comparative study of fuzzy norms on a linear space. *Fuzzy Sets Syst.* **2008**, *159*, 685–707. [CrossRef]
6. Hasankhani, A.; Nazari, A.; Saheli, M. Some properties of fuzzy Hilbert spaces and norm of operators. *Iran. J. Fuzzy Syst.* **2010**, *7*, 129–157.
7. Binazr, T.; Peter, F.; Nadanban, S. A study of boundedness in fuzzy normed linear spaces. *Symmetry* **2018**, *11*, 923. [CrossRef]
8. Wu, H.C. Duality in fuzzy sets and dual arithmetics of fuzzy sets. *Mathematics* **2019**, *7*, 11. [CrossRef]
9. Cho, Y.J.; Rassias, T.M.; Saadati, R. Fuzzy normed spaces and fuzzy metric spaces. In *Fuzzy Operator Theory in Mathematical Analysis*; Springer: Cham, Geramny, 2018; pp. 11–43.
10. Banach, S. Théorie des opérations linéaires; Monografje Matematyczne: Warsaw, Poland, 1932.
11. Enflo, P. A counter example to the approximation problem. *Acta. Math.* **1973**, *130*, 309–317. [CrossRef]
12. Grothendieck, A. Produits tensoriels topologiques et espaces nucleires. *Mem. Amer. Math. Soc.* **1955**, *16*, 1–140.
13. Lindenstrauss, J.; Tzafriri, L. The approximation property. In *Classical Banach Spaces I, Sequence Spaces*; Springer: Berlin, Germany, 1977; pp. 29–42.
14. Yilmaz, Y. Schauder bases and approximation property in fuzzy normed spaces. *Comput. Math. Appl.* **2010**, *59*, 1957–1964. [CrossRef]
15. Lee, K.Y. Approximation properties in fuzzy normed spaces. *Fuzzy Sets Syst.* **2016**, *282*, 115–130. [CrossRef]
16. Konwar, N.; Davvaz, B.; Debnath, P. Approximation of new bounded operators in intuitionistic fuzzy n-Banach spaces. *J. Intell. Fuzzy Syst.* **2018**, *35*, 6301–6312. [CrossRef]
17. Saheli, M. A comparative study of fuzzy norms of linear operators on a fuzzy normed linear spaces. *J. Math. Modeling.* **2015**, *2*, 217–234.
18. Bag, T.; Samanta, S.K. Fuzzy bounded linear operators. *Fuzzy Sets Syst.* **2005**, *151*, 513–547. [CrossRef]
19. Bag, T.; Samanta, S K. Fuzzy bounded linear operators in Felbin's type fuzzy normed linear spaces. *Fuzzy Sets Syst.* **2008**, *159*, 685–707. [CrossRef]
20. Bag, T.; Samanta, S.K. Fuzzy reflexivity of Felbin's type fuzzy normed linear spaces and fixed point theorems in such spaces. *Iran. J. Fuzzy Syst.* **2011**, *8*, 103–115.
21. Megginson, R.E. Vector topologies. In *An Introduction to Banach Space Theory*; Springer: New York, USA, 1988; pp. 180–181.
22. Kim, J.M. Characterizations of bounded approximation properties. *Taiwanese J. Math.* **2008**, *12*, 179–190. [CrossRef]