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New Quantum Estimates of Trapezium-Type Inequalities for Generalized ϕ-Convex Functions

Miguel J. Vivas-Cortez [1,*,†], Rozana Liko [2,†], Artion Kashuri [2,†] and Jorge E. Hernández Hernández [3,†]

1 Facultad de Ciencias Exactas y Naturales, Escuela de Matemáticas y Físicas, Pontificia Universidad Católica del Ecuador, Av. 12 de Octubre 1076. Apartado: 17-01-2184, Quito 170143, Ecuador
2 Department of Mathematics, Faculty of Technical Science, University Ismail Qemali, 1001 Vlora, Albania; rozanaliko86@gmail.com (R.L.); artionkashuri@gmail.com (A.K.)
3 Decanato de Ciencias Económicas y Empresariales, Universidad Centroccidental Lisandro Alvarado, Barquisimeto 3001, Venezuela; jorgehernandez@ucla.edu.ve or jhernandezh.jh@gmail.com
* Correspondence: mjvivas@puce.edu.ec
† These authors contributed equally to this work.

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Abstract: In this paper, a quantum trapezium-type inequality using a new class of function, the so-called generalized ϕ-convex function, is presented. A new quantum trapezium-type inequality for the product of two generalized ϕ-convex functions is provided. The authors also prove an identity for twice $q-$differentiable functions using Raina's function. Utilizing the identity established, certain quantum estimated inequalities for the above class are developed. Various special cases have been studied. A brief conclusion is also given.

Keywords: Hermite–Hadamard inequality; Hölder's inequality; power mean inequality; quantum estimates; Raina's function; convex function

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

1. Introduction

The convexity of a function has played a very important role as a tool in the development of inequalities. The relationship of this concept is always present in branches such as functional analysis [1], harmonic analysis (specifically in interpolation theory) [2] and control theory and optimization [3]. This property is defined by Jensen in [4,5] as follows.

Definition 1. *A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be convex on I, if:*

$$f((1-\imath)\ell_1 + \imath\ell_2) \le (1-\imath)f(\ell_1) + \imath f(\ell_2)$$

holds for every $\ell_1, \ell_2 \in I$ and $\imath \in [0,1]$.

The famous Hermite–Hadamard inequality, which involves convex functions, appears in the literature regarding the study of inequalities. Its name was derived from the works of Hermite, Ch. [6] and Hadamard J. [7], and it was established as follows.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $\ell_1, \ell_2 \in I$ with $\ell_1 < \ell_2$. Then the following inequality holds:*

$$f\left(\frac{\ell_1 + \ell_2}{2}\right) \leq \frac{1}{\ell_2 - \ell_1} \int_{\ell_1}^{\ell_2} f(x)dx \leq \frac{f(\ell_1) + f(\ell_2)}{2}. \tag{1}$$

Inequality (1) is also known as trapezium inequality.

The trapezium-type inequality has remained a subject of great interest due to its broad application in the field of mathematical analysis. For other recent results which generalize, improve and extend inequality (1) through various classes of convex functions, interested readers may consult [8–13].

Quantum calculus, also known as calculus with no limits, was begun by Euler in the eighteenth century (1707–1783). In 1910, Jackson F.H. [14] began a symmetric study of q-calculus and introduced q-definite integrals. He was also the first to develop q-calculus in a systematic fashion. Some branches of mathematics and physics, such as number theory, orthogonal polynomials, combinatorics, basic hypergeometric functions, quantum theory, mechanics and the theory of relativity, have been enriched by the research work of various authors such as Ernst T. [15], Gauchman H. [16] and Kac V. [17].

Motivated by the growing body of work on the development of the concept of convexity, its relationships with integral inequalities and its connection with quantum analysis, as is addressed in the work mentioned above, in this work we seek to establish certain quantum estimates of trapezium-type inequalities for generalized ϕ-convex functions.

2. Preliminaries

Let K be a non-empty closed set in $\mathbb{R}^n$ and $\phi : K \to \mathbb{R}$ a continuous function.

In [10], Noor M.A. introduced a new class of non-convex functions, the so-called ϕ-convex, as follows:

Definition 2. *The function $f : K \to \mathbb{R}$ on the ϕ-convex set K is said to be ϕ-convex, if:*

$$f(\ell_1 + \imath e^{i\phi}(\ell_2 - \ell_1)) \leq (1 - \imath)f(\ell_1) + \imath f(\ell_2), \quad \forall\, \ell_1, \ell_2 \in K,\ \imath \in [0, 1].$$

The function f is said to be ϕ-concave if $(-f)$ is ϕ-convex. Note that every convex function is ϕ-convex but in general the converse does not hold.

In [18], Raina R.K. introduced a class of functions defined by:

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(z) = \sum_{k=0}^{+\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} z^k, \tag{2}$$

where $\rho, \lambda > 0, |z| < R$ and:

$$\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$$

is a bounded sequence of positive real numbers. Note that, if we take in (2) $\rho = 1, \lambda = 0$ and:

$$\sigma(k) = \frac{((\alpha)_k (\beta)_k)}{(\gamma)_k} \ \text{ for } \ k = 0, 1, 2, \ldots,$$

where α, β and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0, -1, -2, \ldots$), and the symbol $(a)_k$ denotes the quantity:

$$(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = a(a + 1) \ldots (a + k - 1), \quad k = 0, 1, 2, \ldots,$$

and restrict its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, which is:

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(z) = F(\alpha, \beta; \gamma; z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

In addition, if $\sigma = (1, 1, \ldots)$ with $\rho = \alpha, (Re(\alpha) > 0), \lambda = 1$ and restricting its domain to $z \in \mathbb{C}$ in Equation (2), then we have the classical Mittag-Leffler function:

$$E_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{1}{\Gamma(1+\alpha k)} z^k.$$

Finally, let us recall the new class of set and new class of functions involving Raina's function introduced by Vivas-Cortez et al. in [13], the so-called generalized ϕ-convex set as well as the generalized ϕ-convex function.

Definition 3. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. A non-empty set K is said to be a generalized ϕ-convex set, if:*

$$\ell_1 + \imath \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1) \in K, \quad \forall\, \ell_1, \ell_2 \in K \;\; \text{and} \;\; \imath \in [0, 1], \tag{3}$$

where $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$ is Raina's function.

Definition 4. *Let $\rho, \lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ be a bounded sequence of positive real numbers. If a function $f : K \to \mathbb{R}$ satisfies the following inequality:*

$$f(\ell_1 + \imath \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \leq (1 - \imath) f(\ell_1) + \imath f(\ell_2), \tag{4}$$

for all $\imath \in [0, 1]$ and $\ell_1, \ell_2 \in K$, then f is called generalized ϕ-convex.

Remark 1. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1) = \ell_2 - \ell_1 > 0$ in Definition 4, we then obtain Definition 1. Moreover, under suitable choices of $\mathcal{F}^{\sigma}_{\rho,\lambda}(\cdot)$, we get Definition 2.*

Recently, several authors have utilized quantum calculus as a strong tool in establishing new extensions of trapezium-type and other inequalities, see [17,19–24] and the references therein.

We now recall some concepts from quantum calculus. Let $I = [\ell_1, \ell_2] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant.

Definition 5 ([23])**.** *Let $f : I \to \mathbb{R}$ be a continuous function and $x \in I$. Then q-derivative of f on I at x is defined as:*

$$_{\ell_1}\mathcal{D}_q f(x) = \frac{f(x) - f(qx + (1-q)\ell_1)}{(1-q)(x - \ell_1)}, \; x \neq \ell_1, \;\; _{\ell_1}\mathcal{D}_q f(\ell_1) = \lim_{x \to \ell_1} {}_{\ell_1}\mathcal{D}_q f(x). \tag{5}$$

We say that f is q-differentiable on I provided $_{\ell_1}\mathcal{D}_q f(x)$ exists for all $x \in I$. Note that if $\ell_1 = 0$ in Equation (5), then $_{\ell_1}\mathcal{D}_q f = \mathcal{D}_q f$, where $\mathcal{D}_q$ is the well-known q-derivative of the function $f(x)$ defined by:

$$\mathcal{D}_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Definition 6 ([23])**.** *Let $f : I \to \mathbb{R}$ be a continuous function. Then the second-order q-derivative of f on I, which is denoted as $_{\ell_1}\mathcal{D}^2_q f$, provided $_{\ell_1}\mathcal{D}_q f$ is q-differentiable on I with $_{\ell_1}\mathcal{D}^2_q f = {}_{\ell_1}\mathcal{D}_q \left({}_{\ell_1}\mathcal{D}_q f \right) : I \to \mathbb{R}$.*

Definition 7 ([23]). *Let $f : I \to \mathbb{R}$ be a continuous function. Then the q-integral on I is defined by:*

$$\int_{\ell_1}^{x} f(\imath)\,{}_{\ell_1}d_q\imath = (1-q)(x-\ell_1)\sum_{n=0}^{+\infty} q^n f\left(q^n x + (1-q^n)\ell_1\right).$$

for $x \in I$. Note that if $\ell_1 = 0$, then we have the classical q-integral, which is defined by:

$$\int_{0}^{x} f(\imath)\,{}_{0}d_q\imath = (1-q)x\sum_{n=0}^{+\infty} q^n f\left(q^n x\right)$$

for $x \in [0, +\infty)$.

Theorem 2 ([23]). *Assume that $f, g : I \to \mathbb{R}$ are continuous functions, $c \in \mathbb{R}$. Then, for $x \in I$, we have:*

$$\int_{\ell_1}^{x} \left[f(\imath) + g(\imath)\right]\,{}_{\ell_1}d_q\imath = \int_{\ell_1}^{x} f(\imath)\,{}_{\ell_1}d_q\imath + \int_{\ell_1}^{x} g(\imath)\,{}_{\ell_1}d_q\imath;$$

$$\int_{\ell_1}^{x} (cf)(\imath)\,{}_{\ell_1}d_q\imath = c\int_{\ell_1}^{x} f(\imath)\,{}_{\ell_1}d_q\imath.$$

Definition 8 ([17]). *For any real number ℓ_1 :*

$$[\ell_1]_q = \frac{q^{\ell_1} - 1}{q - 1}$$

is called the q-analogue of ℓ_1. If $n \in \mathbb{Z}$, we denote:

$$[n] = \frac{q^n - 1}{q - 1} = q^{n-1} + \cdots + q + 1.$$

Definition 9 ([17]). *If $n \in \mathbb{Z}$, the q-analogue of $(x - \ell_1)^n$ is the polynomial:*

$$(x-\ell_1)_q^n = \begin{cases} 1, & n = 0; \\ (x-\ell_1)(x-q\ell_1)\cdots(x-q^{n-1}\ell_1), & n \geq 1. \end{cases}$$

Definition 10 ([17]). *For any $t, s > 0$:*

$$\beta_q(t,s) = \int_0^1 \imath^{t-1}(1-q\imath)_q^{s-1}\,{}_0d_q\imath$$

is called the q-Beta function. Note that:

$$\beta_q(t,1) = \int_0^1 \imath^{t-1}\,{}_0d_q\imath = \frac{1}{[t]},$$

where $[t]$ is the q-analogue of t.

Finally, from [24], four simple lemmas will be used in this paper.

Lemma 1. *Let $f(\imath) = 1$, then we have:*

$$\int_0^1 {}_0d_q\imath = 1.$$

Lemma 2. *Let $f(\imath) = \imath$, then we have:*

$$\int_0^1 \imath \, {}_0d_q\imath = \frac{1}{1+q}.$$

Lemma 3. *Let $f(\imath) = 1 - q\imath$, where $0 < q < 1$ is a constant, then we have:*

$$\int_0^1 (1 - q\imath) \, {}_0d_q\imath = \frac{1}{1+q}.$$

Lemma 4. *Let $f(\imath) = \imath(1 - q\imath)$, where $0 < q < 1$ is a constant, then we have:*

$$\int_0^1 \imath(1 - q\imath) \, {}_0d_q\imath = \frac{1}{(1+q)(1+q+q^2)}.$$

Liu et al. in [20] established the following q-integral identity.

Lemma 5. *Let $f : [\ell_1, \ell_2] \subseteq \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable function on (ℓ_1, ℓ_2) with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on $[\ell_1, \ell_2]$, where $0 < q < 1$. Then the following identity holds:*

$$\frac{qf(\ell_1) + f(\ell_2)}{1+q} - \frac{1}{\ell_2 - \ell_1} \int_{\ell_1}^{\ell_2} f(\imath) \, {}_{\ell_1}d_q\imath = \frac{q^2(\ell_2 - \ell_1)^2}{1+q} \int_0^1 \imath(1 - q\imath) \, {}_{\ell_1}\mathcal{D}_q^2 f((1-\imath)\ell_1 + \imath\ell_2) \, {}_0d_q\imath.$$

Motivated by the aforementioned literature, this paper is organized as follows: In Section 3, a quantum trapezium-type inequality using a new class of functions, the so-called generalized ϕ-convex, will be represented. A quantum trapezium-type inequality for the product of two generalized ϕ-convex functions will also be provided. In Section 4, an identity for a twice q-differentiable functions involving Raina's function will be established. Applying these identities, we develop some quantum estimate inequalities for the above class of functions. Various special cases will be obtained. In Section 5, a brief conclusion is given.

3. Quantum Trapezium-Type Inequalities

Throughout this paper the following notations are used:

$$O = \left[\ell_1, \ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right], \quad \text{where} \quad \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1) > 0,$$

where ρ, $\lambda > 0$ and $\sigma = (\sigma(0), \ldots, \sigma(k), \ldots)$ are bounded sequence of positive real numbers. Moreover, for convenience we denote $d_q\imath$ for ${}_0d_q\imath$, where $0 < q < 1$.

In this section, we first establish quantum estimates for trapezium-type inequalities via generalized ϕ-convexity.

Theorem 3. *Let $f : O \longrightarrow \mathbb{R}$ be a generalized ϕ-convex function on O° (the interior of O) such that $f \in L(O)$. Then the following double inequality holds:*

$$f\left(\frac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) \leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)} \int_{\ell_1}^{\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)} f(\imath) \, {}_{\ell_1}d_q\imath \leq \frac{f(\ell_1) + f(\ell_2)}{2}. \tag{6}$$

Proof. Let $x = \ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)$ and $y = \ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)$. Then:

$$f\left(\frac{x+y}{2}\right) = f\left(\frac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right).$$

Since f is a generalized ϕ-convex function, we have:

$$f(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \le (1-\imath)f(\ell_1) + \imath f(\ell_2)$$

and:

$$f(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) \le \imath f(\ell_1) + (1-\imath)f(\ell_2).$$

So:

$$2f\left(\frac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) \le f(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)) + f(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)). \tag{7}$$

Taking the q-integral of both sides in Equation (7) with respect to $\imath$ on $[0,1]$, we get:

$$2f\left(\frac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) \le \int_0^1 f(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))d_q\imath \tag{8}$$

$$+ \int_0^1 f(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))d_q\imath.$$

Changing the variables of integration in Equation (8), we obtain the left-side inequality of Equation (6). To prove the right-side inequality of Equation (6), from generalized ϕ-convexity of f, we have:

$$\int_0^1 f(\ell_1 + (1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))d_q\imath + \int_0^1 f(\ell_1 + \imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))d_q\imath \le \int_0^1 \left[f(\ell_1) + f(\ell_2)\right]d_q\imath. \tag{9}$$

Changing the variables of integration, we obtain the right-side inequality of Equation (6). The proof of Theorem 3 is completed. □

Corollary 1. *In Theorem 3, taking $q \to 1^-$, we get the following new double inequality:*

$$f\left(\frac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) \le \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)} \int_{\ell_1}^{\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)} f(\imath)d\imath \le \frac{f(\ell_1) + f(\ell_2)}{2}. \tag{10}$$

Remark 2. *In Corollary 1, taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1) = \ell_2 - \ell_1$, we recapture Theorem 1.*

We are now in a position to derive a new quantum trapezium-type inequality for the product of two generalized ϕ-convex functions.

Theorem 4. *Let $f, g : O \longrightarrow \mathbb{R}$ be two generalized ϕ-convex functions on O° (the interior of O) such that $f, g \in L(O)$. Then the following double inequality holds:*

$$4f\left(\tfrac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) g\left(\tfrac{2\ell_1 + \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)}{2}\right) - C(q)\left[2q^2 M(\ell_1, \ell_2) + (q^3 + 2q + 1)N(\ell_1, \ell_2)\right]$$

$$\leq \frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} f(\imath)g(\imath)\ {}_{\ell_1}d_q\imath \tag{11}$$
$$\leq \frac{2C(q)}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[(q^3-1)f(\ell_1)g(\ell_1)+(q+1)M(\ell_1,\ell_2)+q^2N(\ell_1,\ell_2)\right],$$

where:

$$M(\ell_1,\ell_2)=f(\ell_1)g(\ell_1)+f(\ell_2)g(\ell_2),\quad N(\ell_1,\ell_2)=f(\ell_1)g(\ell_2)+f(\ell_2)g(\ell_1)$$

and:

$$C(q)=\frac{1}{(1+q)(1+q+q^2)}.$$

Proof. Let $x=\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)$ and $y=\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)$. Then:

$$f\left(\frac{x+y}{2}\right)=f\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)$$

and:

$$g\left(\frac{x+y}{2}\right)=g\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right).$$

Since f and g are generalized ϕ-convex functions, we have:

$$f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq(1-\imath)f(\ell_1)+\imath f(\ell_2),$$
$$g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq(1-\imath)g(\ell_1)+\imath g(\ell_2)$$

and:

$$f(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq\imath f(\ell_1)+(1-\imath)f(\ell_2),$$
$$g(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\leq\imath g(\ell_1)+(1-\imath)g(\ell_2).$$

Multiplying the above inequalities, we have:

$$\begin{aligned}
4f&\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)g\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)\\
&\leq\left[f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+f(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\right]\\
&\qquad\times\left[g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))+g(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\right]\\
&=f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\quad+f(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\quad+f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\quad+f(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\leq f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\quad+f(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+(1-\imath)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))\\
&\quad+\left[(1-\imath)f(\ell_1)+\imath f(\ell_2)\right]\left[\imath g(\ell_1)+(1-\imath)g(\ell_2)\right]
\end{aligned}\tag{12}$$

$$+\left[\imath f(\ell_1)+(1-\imath)f(\ell_2)\right]\left[(1-\imath)g(\ell_1)+\imath g(\ell_2)\right].$$

Taking the q-integral of both sides in Equation (12) with respect to $\imath$ on $[0,1]$ and changing the variables of integration, we get:

$$\begin{aligned}
&4f\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)g\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)\\
&\quad\leq\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}f(\imath)g(\imath)\,{}_{\ell_1}d_q\imath\\
&\qquad+2M(\ell_1,\ell_2)\int_0^1\imath(1-\imath)d_q\imath+N(\ell_1,\ell_2)\int_0^1\left[\imath^2+(1-\imath)^2\right]d_q\imath.
\end{aligned}$$

By using Lemmas 1–4, the left-side inequality of Equation (11) is proved. To prove the right-side inequality of Equation (11), from the generalized ϕ-convexity of f and g, we have:

$$\begin{aligned}
&\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}f(\imath)g(\imath)\,{}_{\ell_1}d_q\imath\\
&=\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_0^1 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))g(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))d_q\imath\\
&\leq\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_0^1\left[(1-\imath)f(\ell_1)+\imath f(\ell_2)\right]\left[(1-\imath)g(\ell_1)+\imath g(\ell_2)\right]d_q\imath\\
&=\frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[f(\ell_1)g(\ell_1)\int_0^1(1-\imath)^2d_q\imath+f(\ell_2)g(\ell_2)\int_0^1\imath^2d_q\imath+N(\ell_1,\ell_2)\int_0^1\imath(1-\imath)d_q\imath\right].
\end{aligned}$$

The right-side inequality of Equation (11) is thus proved. The proof of Theorem 4 is completed. □

Corollary 2. *In Theorem 4, taking $q\to 1^-$, we get the following new double inequality:*

$$\begin{aligned}
&f\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)g\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)-\frac{\left[M(\ell_1,\ell_2)+2N(\ell_1,\ell_2)\right]}{12}\\
&\quad\leq\frac{1}{2\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}f(\imath)g(\imath)d\imath\leq\frac{\left[2M(\ell_1,\ell_2)+N(\ell_1,\ell_2)\right]}{12\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}.
\end{aligned}\tag{13}$$

Corollary 3. *In Corollary 2, taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)=\ell_2-\ell_1$, we get:*

$$\begin{aligned}
&f\left(\frac{\ell_1+\ell_2}{2}\right)g\left(\frac{\ell_1+\ell_2}{2}\right)-\frac{\left[M(\ell_1,\ell_2)+2N(\ell_1,\ell_2)\right]}{12}\\
&\quad\leq\frac{1}{2(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2}f(\imath)g(\imath)d\imath\leq\frac{\left[2M(\ell_1,\ell_2)+N(\ell_1,\ell_2)\right]}{12(\ell_2-\ell_1)}.
\end{aligned}\tag{14}$$

Corollary 4. *In Theorem 4, taking $g\equiv 1$, we get:*

$$4f\left(\frac{2\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}{2}\right)-(1+2q+2q^2+q^3)C(q)\left[f(\ell_1)+f(\ell_2)\right]$$

$$\leq \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} f(\imath)d\imath \leq \frac{2}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\left[\frac{qf(\ell_1)+f(\ell_2)}{1+q}\right]. \tag{15}$$

4. Other Quantum Inequalities

In this section, we first derive a new quantum integral identity for twice q-differentiable functions involving Raina's function.

Lemma 6. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}^2_q f$ being continuous and integrable on O. Then the following identity holds:*

$$T_f(\ell_1,\ell_2;q) = \frac{q^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}{1+q}\int_0^1 \imath(1-q\imath)\,{}_{\ell_1}\mathcal{D}^2_q f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))d_q\imath,$$

where:

$$T_f(\ell_1,\ell_2;q) = \frac{qf(\ell_1)+f(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{1+q} - \frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} f(\imath)\,{}_{\ell_1}d_q\imath.$$

Proof. Using Definitions 5–7, we have:

$$\int_0^1 \imath(1-q\imath)\,{}_{\ell_1}\mathcal{D}^2_q f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))d_q\imath$$

$$= \int_0^1 \imath(1-q\imath)\frac{qf(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) - (1+q)f(\ell_1+q\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + f(\ell_1+q^2\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q(1-q)^2\imath^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}d_q\imath$$

$$= \frac{q\sum_{n=0}^{+\infty} f(\ell_1+q^n\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) - (1+q)\sum_{n=0}^{+\infty} f(\ell_1+q^{n+1}\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q(1-q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}$$

$$+\frac{\sum_{n=0}^{+\infty} f(\ell_1+q^{n+2}\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q(1-q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}$$

$$-q\left[\frac{q(1-q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\sum_{n=0}^{+\infty} q^n f(\ell_1+q^n\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q(1-q)^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\right.$$

$$-\frac{(1+q)(1-q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\sum_{n=0}^{+\infty} q^{n+1} f(\ell_1+q^{n+1}\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q^2(1-q)^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}$$

$$\left.+\frac{(1-q)\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\sum_{n=0}^{+\infty} q^{n+2} f(\ell_1+q^{n+2}\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q^3(1-q)^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\right]$$

$$= \frac{q\Big(f(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) - f(\ell_1)\Big) - f(\ell_1+q\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)) + f(\ell_1)}{q(1-q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}$$

$$-\left[\frac{(1+q)}{q^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)} f(\imath)\,{}_{\ell_1}d_q\imath\right.$$

$$+\frac{(q^2+q-1)}{q^2(1-q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}f(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))$$

$$\left.-\frac{f(\ell_1+q\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q(1-q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}\right]$$

$$=\frac{qf(\ell_1)+f(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{q^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}-\frac{(1+q)}{q^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^3}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}f(\imath)\,{}_{\ell_1}d_q\imath.$$

Multiplying both sides of the above equality: by $\frac{q^2\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}{1+q}$, we get the desired result. The proof of Lemma 6 is completed. □

Remark 3. *Taking $q \to 1^-$ in Lemma 6, we obtain the following new identity:*

$$T_f(\ell_1,\ell_2)=\frac{\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2}{2}\int_0^1 \imath(1-\imath)f''(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))d\imath,$$

where:

$$T_f(\ell_1,\ell_2)=\frac{f(\ell_1)+f(\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))}{2}-\frac{1}{\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_1+\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)}f(\imath)\,d\imath.$$

Remark 4. *Taking $\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)=\ell_2-\ell_1$ in Lemma 6, we get Lemma 5.*

Now, applying Lemma 6, we establish some quantum estimate inequalities for the generalized ϕ-convex function.

Theorem 5. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}^2_q f$ being continuous and integrable on O. If $|{}_{\ell_1}\mathcal{D}^2_q f|$ is generalized ϕ-convex on O, then the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right|\le\frac{q^2}{1+q}A(q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left[q^2|{}_{\ell_1}\mathcal{D}^2_q f(\ell_1)|+|{}_{\ell_1}\mathcal{D}^2_q f(\ell_2)|\right], \tag{16}$$

where:

$$A(q)=\frac{1}{(1+q+q^2)(1+q+q^2+q^3)}.$$

Proof. Using Lemmas 1–4 and Lemma 6 and the fact that $|{}_{\ell_1}\mathcal{D}^2_q f|$ is a generalized ϕ-convex function, we have:

$$\left|T_f(\ell_1,\ell_2;q)\right|\le\frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1\imath(1-q\imath)|{}_{\ell_1}\mathcal{D}^2_q f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath$$

$$\le\frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1\imath(1-q\imath)\left[(1-\imath)\,|{}_{\ell_1}\mathcal{D}^2_q f(\ell_1)|+\imath\,|{}_{\ell_1}\mathcal{D}^2_q f(\ell_2)|\right]d_q\imath$$

$$=\frac{q^2}{1+q}A(q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left[q^2|{}_{\ell_1}\mathcal{D}^2_q f(\ell_1)|+|{}_{\ell_1}\mathcal{D}^2_q f(\ell_2)|\right].$$

The proof of Theorem 5 is completed. □

Corollary 5. *Taking $q \to 1^-$ in Theorem 5, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \leq \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left[\frac{|f''(\ell_1)|+|f''(\ell_2)|}{24}\right]. \tag{17}$$

Corollary 6. *Taking $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 5, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2(q^2+1)}{1+q}A(q)\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{18}$$

Theorem 6. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex on O, for $r > 1$ and $\frac{1}{p}+\frac{1}{r}=1$, then the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq \frac{q^2}{1+q}\sqrt[p]{B(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{\frac{(q+1)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}{1+q}}, \tag{19}$$

where:

$$B(p;q) = (1-q)\sum_{n=0}^{+\infty}(q^n)^{p+1}\left(1-q^{n+1}\right)^p.$$

Proof. Using Lemmas 1–4 and Lemma 6, Hölder's inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1\left[\imath(1-q\imath)\right]^p d_q\imath\right)^{\frac{1}{p}}\left(\int_0^1 |_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1\left[\imath(1-q\imath)\right]^p d_q\imath\right)^{\frac{1}{p}}\left(\int_0^1\left[(1-\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q}\sqrt[p]{B(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{\frac{(q+1)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}{1+q}}.
\end{aligned}$$

The proof of Theorem 6 is completed. □

Corollary 7. *Taking $q \to 1^-$ in Theorem 6, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \leq \frac{\sqrt[p]{\beta(p+1,p+1)}}{2}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{\frac{2|f''(\ell_1)|^r+|f''(\ell_2)|^r}{2}}. \tag{20}$$

Corollary 8. *Taking $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 6, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{1+q}\sqrt[r]{\frac{2+q}{1+q}}\sqrt[p]{B(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{21}$$

Theorem 7. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|{}_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, then for $r \geq 1$, the following inequality holds:*

$$\left|T_f(\ell_1, \ell_2; q)\right| \leq \frac{q^2}{1+q} C(q) \sqrt[r]{\frac{A(q)}{C(q)}} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \sqrt[r]{q^2 |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}, \tag{22}$$

where $C(q)$ is defined from Theorem 4 and $A(q)$ is defined from Theorem 5.

Proof. Using Lemmas 1–4 and Lemma 6, the well-known power mean inequality and the fact that $|{}_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$\begin{aligned}
&\left|T_f(\ell_1, \ell_2; q)\right| \\
&\leq \frac{q^2}{1+q} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \left(\int_0^1 \left[\imath(1 - q\imath)\right] d_q\imath\right)^{1-\frac{1}{r}} \left(\int_0^1 \imath(1 - q\imath) |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_1 + \imath \mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q^2}{1+q} C^{1-\frac{1}{r}}(q) \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \left(\int_0^1 \imath(1 - q\imath) \left[(1 - \imath) |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + \imath |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right] d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q} C(q) \sqrt[r]{\frac{A(q)}{C(q)}} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \sqrt[r]{q^2 |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}$$

The proof of Theorem 7 is completed. □

Corollary 9. *Taking $r = 1$ in Theorem 7, we get Theorem 5.*

Corollary 10. *Taking $q \to 1^-$ in Theorem 7, we get:*

$$\left|T_f(\ell_1, \ell_2)\right| \leq \frac{1}{12} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \sqrt[r]{\frac{|f''(\ell_1)|^r + |f''(\ell_2)|^r}{2}}. \tag{23}$$

Corollary 11. *Taking $|{}_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 7, we get:*

$$\left|T_f(\ell_1, \ell_2; q)\right| \leq K \frac{q^2}{1+q} \sqrt[r]{q^2 + 1} C(q) \sqrt[r]{\frac{A(q)}{C(q)}} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2. \tag{24}$$

Theorem 8. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|{}_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, for $r > 1$ and $\frac{1}{p} + \frac{1}{r} = 1$, then the following inequality holds:*

$$\left|T_f(\ell_1, \ell_2; q)\right| \leq \frac{q^2}{1+q} \sqrt[r]{C(q)} \sqrt[p]{D(p; q)} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2 - \ell_1)\right]^2 \sqrt[r]{q^2 |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + (q+1) |{}_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r} \tag{25}$$

where:

$$D(p; q) = (1 - q) \sum_{n=0}^{+\infty} q^{2n} \left(1 - q^{n+1}\right)^p$$

and $C(q)$ is defined from Theorem 4.

Proof. Using Lemmas 1–4 and Lemma 6, Hölder's inequality and the fact that $|{}_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$
\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 \imath(1-q\imath)^p d_q\imath\right)^{\frac{1}{p}}\left(\int_0^1 \imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 \imath(1-q\imath)^p d_q\imath\right)^{\frac{1}{p}}\left(\int_0^1 \imath\left[(1-\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q}\sqrt[r]{C(q)}\sqrt[p]{D(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{q^2|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+(q+1)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}
$$

The proof of Theorem 8 is completed. □

Corollary 12. *Taking $q \to 1^-$ in Theorem 8, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \leq \frac{\sqrt[p]{\beta(p+1,2)}}{2\sqrt[r]{6}}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{|f''(\ell_1)|^r+2|f''(\ell_2)|^r}. \tag{26}$$

Corollary 13. *Taking $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 8, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{(1+q)\sqrt[r]{1+q}}\sqrt[p]{D(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{27}$$

Theorem 9. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, then for $r \geq 1$, the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq \frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{E(r;q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+F(r;q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}, \tag{28}$$

where:

$$E(r;q) = \frac{[r+2]-[r+1]}{[r+1][r+2]} - \frac{q([r+3]-[r+2])}{[r+2][r+3]}, \quad F(r;q) = \frac{[r+3]-q[r+2]}{[r+2][r+3]}$$

and $[r]$ is the q-analogue of r.

Proof. Using Lemmas 1–4 and Lemma 6, the well-known power mean inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$
\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 (1-q\imath)d_q\imath\right)^{1-\frac{1}{r}}\left(\int_0^1 \imath^r(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \tfrac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 (1-q\imath)d_q\imath\right)^{1-\frac{1}{r}}\left(\int_0^1 \imath^r(1-q\imath)\left[(1-\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\imath\right)^{\frac{1}{r}}
\end{aligned}
$$

$$= \frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{E(r;q)|_{\ell_1}\mathcal{D}^2_q f(\ell_1)|^r + F(r;q)|_{\ell_1}\mathcal{D}^2_q f(\ell_2)|^r}.$$

The proof of Theorem 9 is completed. □

Corollary 14. *Taking $q \to 1^-$ in Theorem 9, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \le \frac{\sqrt[r]{E(r)}}{4}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{2|f''(\ell_1)|^r + (r+1)|f''(\ell_2)|^r}, \tag{29}$$

where:

$$E(r) = \frac{2}{(r+1)(r+2)(r+3)}.$$

Corollary 15. *Taking $|_{\ell_1}\mathcal{D}^2_q f| \le K$ in Theorem 9, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \le K\frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{E(r;q)+F(r;q)}. \tag{30}$$

Theorem 10. *Let $f: O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}^2_q f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}^2_q f|^r$ is generalized ϕ-convex on O, for $r > 1$ and $\frac{1}{p}+\frac{1}{r} = 1$, then the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \le \tfrac{q^2}{1+q}\sqrt[r]{C(q)}\sqrt[p]{\beta_q(p+1,2)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{(q^2+q)|_{\ell_1}\mathcal{D}^2_q f(\ell_1)|^r + |_{\ell_1}\mathcal{D}^2_q f(\ell_2)|^r}, \tag{31}$$

where $C(q)$ is defined from Theorem 4 and $\beta_q(\cdot,\cdot)$ is a q-Beta function.

Proof. Using Lemmas 1–4 and Lemma 6, Hölder's inequality and the fact that $|_{\ell_1}\mathcal{D}^2_q f|^r$ is a generalized ϕ-convex function, we have:

$$\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\le \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}^2_q f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\le \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left(\int_0^1 \imath^p(1-q\imath)d_q\imath\right)^{\frac{1}{p}} \left(\int_0^1 (1-q\imath)|_{\ell_1}\mathcal{D}^2_q f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\le \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left(\int_0^1 \imath^p(1-q\imath)d_q\imath\right)^{\frac{1}{p}} \left(\int_0^1 (1-q\imath)\left[(1-\imath)|_{\ell_1}\mathcal{D}^2_q f(\ell_1)|^r + \imath|_{\ell_1}\mathcal{D}^2_q f(\ell_2)|^r\right]d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q}\sqrt[r]{C(q)}\sqrt[p]{\beta_q(p+1,2)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{(q^2+q)|_{\ell_1}\mathcal{D}^2_q f(\ell_1)|^r + |_{\ell_1}\mathcal{D}^2_q f(\ell_2)|^r}.
\end{aligned}$$

The proof of Theorem 10 is completed. □

Corollary 16. *Taking $q \to 1^-$ in Theorem 10, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \le \frac{\sqrt[p]{\beta(p+1,2)}}{2}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{\frac{2|f''(\ell_1)|^r + |f''(\ell_2)|^r}{6}}. \tag{32}$$

Corollary 17. *Taking* $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ *in Theorem 10, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{(1+q)\sqrt[r]{1+q}}\sqrt[p]{\beta_q(p+1,2)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{33}$$

Theorem 11. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with $_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, then for $r \geq 1$, the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq \frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\sqrt[r]{D(r;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{(1-q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + q|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}, \tag{34}$$

where $D(r;q)$ is defined from Theorem 8, for $p = r$.

Proof. Using Lemmas 1–4 and Lemma 6, the well-known power mean inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 \imath d_q\imath\right)^{1-\frac{1}{r}}\left(\int_0^1 \imath(1-q\imath)^r|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\left(\int_0^1 \imath d_q\imath\right)^{1-\frac{1}{r}}\left(\int_0^1 \imath(1-q\imath)^r\left[(1-\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + \imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\sqrt[r]{D(r;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{(1-q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + q|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}$$

The proof of Theorem 11 is completed. □

Corollary 18. *Taking $q \to 1^-$ in Theorem 11, we get:*

$$\left|T_f(\ell_1,\ell_2)\right| \leq \frac{\sqrt[r]{2\beta(r+1,2)}}{4}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2|f''(\ell_2)|. \tag{35}$$

Corollary 19. *Taking $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 11, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\sqrt[r]{D(r;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{36}$$

Theorem 12. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with $_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, for $r > 1$ and $\frac{1}{p}+\frac{1}{r}=1$, then the following inequality holds:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq \frac{q^2}{1+q}\frac{1}{\sqrt[p]{[p+1]}}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2\sqrt[r]{[L(r;q)-D(r;q)]|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + D(r;q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}, \tag{37}$$

where $[p]$ is the q-analogue of p :

$$L(r;q) = (1-q)\sum_{n=0}^{+\infty} q^n(1-q^{n+1})^r$$

and $D(r;q)$ is defined from Theorem 8, for $p = r$.

Proof. Using Lemmas 1–4 and Lemma 6, Hölder's inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$
\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right| \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left(\int_0^1 \imath^p d_q\imath\right)^{\frac{1}{p}} \left(\int_0^1 (1-q\imath)^r|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|^r d_q\imath\right)^{\frac{1}{r}} \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left(\int_0^1 \imath^p d_q\imath\right)^{\frac{1}{p}} \left(\int_0^1 (1-q\imath)^r\left[(1-\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\imath|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right] d_q\imath\right)^{\frac{1}{r}} \\
&= \frac{q^2}{1+q}\frac{1}{\sqrt[p]{[p+1]}}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{\left[L(r;q)-D(r;q)\right]|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + D(r;q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}
$$

The proof of Theorem 12 is completed. □

Corollary 20. *Taking $q \to 1^-$ in Theorem 12, we get:*

$$
\left|T_f(\ell_1,\ell_2)\right| \leq \frac{1}{2\sqrt[p]{p+1}}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \sqrt[r]{\frac{|f''(\ell_1)|^r}{r+2} + \beta(r+1,2)|f''(\ell_2)|^r}. \tag{38}
$$

Corollary 21. *Taking $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ in Theorem 12, we get:*

$$
\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{1+q}\frac{1}{\sqrt[p]{[p+1]}}\sqrt[r]{L(r;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{39}
$$

Theorem 13. *Let $f : O \to \mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, for $r > 1$ and $\frac{1}{p}+\frac{1}{r} = 1$, then the following inequality holds:*

$$
\begin{aligned}
\left|T_f(\ell_1,\ell_2;q)\right| \leq \frac{q^2}{1+q}\frac{1}{\sqrt[r]{[r+1][r+2]}}\sqrt[p]{L(p;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \qquad (40)\\
\times\sqrt[r]{([r+2]-[r+1])\,|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r + [r+1]|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r},
\end{aligned}
$$

where $[r]$ is the q-analogue of r and $L(p,q)$ is defined from Theorem 12, for $r = p$.

Proof. Using Lemmas 1–4 and Lemma 6, Hölder's inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$
\begin{aligned}
\left|T_f(\ell_1,\ell_2;q)\right| &\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \int_0^1 \imath(1-q\imath)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\imath\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1))|d_q\imath \\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2 \left(\int_0^1 (1-q\imath)^p d_q\imath\right)^{\frac{1}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_0^1 \iota^r|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\iota\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1))|^r d_q\iota\right)^{\frac{1}{r}}\\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\left(\int_0^1(1-q\iota)^p d_q\iota\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1 \iota^r\left[(1-\iota)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\iota|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\iota\right)^{\frac{1}{r}}\\
&=\frac{q^2}{1+q}\frac{1}{\sqrt[r]{[r+1][r+2]}}\sqrt[p]{L(p;q)}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\\
&\times\sqrt[r]{([r+2]-[r+1])\,|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+[r+1]|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}
$$

The proof of Theorem 13 is completed. □

Corollary 22. *Taking $q\to 1^-$ in Theorem 13, we get:*

$$
\left|T_f(\ell_1,\ell_2)\right|\leq \frac{1}{2\sqrt[p]{p+1}}\frac{1}{\sqrt[r]{(r+1)(r+2)}}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\sqrt[r]{|f''(\ell_1)|^r+(r+1)|f''(\ell_2)|^r}. \tag{41}
$$

Corollary 23. *Taking $|_{\ell_1}\mathcal{D}_q^2 f|\leq K$ in Theorem 13, we get:*

$$
\left|T_f(\ell_1,\ell_2;q)\right|\leq K\frac{q^2}{1+q}\frac{\sqrt[p]{L(p;q)}}{\sqrt[r]{[r+1]}}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2. \tag{42}
$$

Theorem 14. *Let $f: O\to\mathbb{R}$ be a twice q-differentiable function on O° with ${}_{\ell_1}\mathcal{D}_q^2 f$ being continuous and integrable on O. If $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is generalized ϕ-convex on O, then for $r\geq 1$, the following inequality holds:*

$$
\left|T_f(\ell_1,\ell_2;q)\right|\leq \tfrac{q^2}{1+q}\sqrt[r]{U(r;q)}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\sqrt[r]{(1-q(1-q))|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+q(1-q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}, \tag{43}
$$

where:

$$
U(r;q)=(1-q)\sum_{n=0}^{+\infty}(q^n)^{r+1}(1-q^{n+1})^r.
$$

Proof. Using Lemmas 1–4 and Lemma 6, the well-known power mean inequality and the fact that $|_{\ell_1}\mathcal{D}_q^2 f|^r$ is a generalized ϕ-convex function, we have:

$$
\begin{aligned}
&\left|T_f(\ell_1,\ell_2;q)\right|\\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\int_0^1 \iota(1-q\iota)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\iota\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1))|d_q\iota\\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\left(\int_0^1 d_q\iota\right)^{1-\frac{1}{r}}\left(\int_0^1\left[\iota(1-q\iota)\right]^r|_{\ell_1}\mathcal{D}_q^2 f(\ell_1+\iota\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1))|^r d_q\iota\right)^{\frac{1}{r}}\\
&\leq \frac{q^2}{1+q}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\left(\int_0^1 d_q\iota\right)^{1-\frac{1}{r}}\left(\int_0^1\left[\iota(1-q\iota)\right]^r\left[(1-\iota)|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+\iota|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r\right]d_q\iota\right)^{\frac{1}{r}}\\
&=\frac{q^2}{1+q}\sqrt[r]{U(r;q)}\left[\mathcal{F}_{\rho,\lambda}^{\sigma}(\ell_2-\ell_1)\right]^2\sqrt[r]{(1-q(1-q))|_{\ell_1}\mathcal{D}_q^2 f(\ell_1)|^r+q(1-q)|_{\ell_1}\mathcal{D}_q^2 f(\ell_2)|^r}.
\end{aligned}
$$

The proof of Theorem 14 is completed. □

Corollary 24. *Taking* $|_{\ell_1}\mathcal{D}_q^2 f| \leq K$ *in Theorem 14, we get:*

$$\left|T_f(\ell_1,\ell_2;q)\right| \leq K\frac{q^2}{1+q}\sqrt[r]{U(r;q)}\left[\mathcal{F}^{\sigma}_{\rho,\lambda}(\ell_2-\ell_1)\right]^2. \tag{44}$$

Remark 5. *For different choices of* ρ, $\lambda > 0$, *where* $\sigma = (\sigma(0),\ldots,\sigma(k),\ldots)$ *are bounded sequence of positive real numbers in Raina's function, we can arrive at new fascinating inequalities. Moreover, our results can be applied to derive some inequalities using special means. For example, from Corollary 5 taking* $f(x) = x^m, \sigma = (0,1,0,\cdots), \rho = 1, \lambda = 0$, *and recalling the well known means:*

1. *Arithmetic:*
$$A(u,v) = \frac{u+v}{2}$$
2. *Generalized Logarithmic:*
$$L_m(u,v) = \left(\frac{v^{m+1}-u^{m+1}}{(m+1)(v-u)}\right)^{1/m}$$

we have the following inequality:

$$|A(\ell_1^m,\ell_2^m) - L_m^m(\ell_1,\ell_2)| \leq \frac{m(m-1)(\ell_2-\ell_1)^2}{12}A(\ell_1^{m-1},\ell_2^{m-1})$$

The details are left to the interested reader.

5. Conclusions

Since quantum calculus is broadly applicable in many mathematical areas, this new class of functions, the so-called generalized ϕ-convex, can be applied to obtain various results in convex analyses, special functions, quantum mechanics, related optimization theories and mathematical inequalities and it may stimulate further research in different areas of pure and applied sciences.

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