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A Mollification Regularization Method for the Inverse Source Problem for a Time Fractional Diffusion Equation

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Abstract: We consider a time-fractional diffusion equation for an inverse problem to determine an unknown source term, whereby the input data is obtained at a certain time. In general, the inverse problems are ill-posed in the sense of Hadamard. Therefore, in this study, we propose a mollification regularization method to solve this problem. In the theoretical results, the error estimate between the exact and regularized solutions is given by a priori and a posteriori parameter choice rules. Besides, the proposed regularized methods have been verified by a numerical experiment.

Keywords: time-fractional diffusion equation; inverse problem; ill-posed problem; convergence estimates

MSC: 35K05; 35K99; 47J06; 47H10x

1. Introduction

In this work, we study an inverse source problem for the time-fractional diffusion equation in a infinite domain as follows:

$$\begin{cases} \dfrac{\partial^\beta u(x,t)}{\partial t^\beta} = u_{xx}(x,t) + \phi(t) f(x), \quad (x,t) \in \mathbb{R} \times (0,T], \\ u(x,0) = 0, x \in \mathbb{R}, \\ u(x,T) = g(x), \quad x \in \mathbb{R}, \end{cases} \tag{1}$$

where the fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo derivative of order β $(0 < \beta < 1)$ as defined by

$$\frac{d^\beta f(t)}{dt^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{df(s)}{ds} \frac{ds}{(t-s)^\beta}, \tag{2}$$

and $\Gamma(\cdot)$ denotes the standard Gamma function.

The biggest motivation for developing the problem (1) is the inverse problems for the heat equation; we recover the unknown source function under different assumptions on the smoothness of input data, which were proposed by Igor Malyshev in Reference [1]. The inverse problems of the restoration of a source function in the heat equation with the classical derivative are studied by many researchers, that is, Geng [2] and Shidfar [3].

The mathematical model (1) arising in control theory, physical, generalized voltage divider, elasticity and the model of neurons in biology is studied in References [4–6].

According to our search, the fractional inverse source problems (1) are the subject of very few works, for example, Sakamoto et al. [7] used the data $u(x_0,t)(x_0 \in \mathbb{R})$ to determine $\phi(t)$ once $f(x)$ was given, where the authors obtained a Lipschitz stability for $\phi(t)$. In Problem (1) for a one-dimensional problem with special coefficients, Wei et al. [8] used the Fourier truncation method to solve an inverse source problem with $\phi(t) = 1$. In Reference [9], using the mollification regularization method, Yang and Fu determined the inverse spatial-dependent heat source problem. In Reference [10], Wei and Wang considered a modified quasi boundary value regularization method for identifying this problem. In Reference [11], using the quasi-reversibility regularization method, Yang and his group identified the unknown source for a time fractional diffusion equation. In Reference [12], with the quasi-reversibility regularization method, Wei and her group investigated a space-dependent source for the time fractional diffusion equation. Actually, to our knowledge, in the case $\phi(t)$, dependent on time, the results of the inverse source problem for the time-fractional diffusion equation still has a limited achievement, if $\phi(t) \neq 0$, we know Huy and his group investigated this problem by way of the Tikhonov regularization method, see Reference [13]. In these regularization methods, the priori parameter choice rule depends on the noise level and the priori bound. But in practice, to know exactly this is very difficult. In the above research, by using Morozov's Discrepancy Principle for choosing the regularization parameter in Tikhonov regularization, the authors show error estimation for both the priori choice rule parameter and the posteriori choice rule parameter.

In this paper, we use the mollification method to solve the inverse source problem. Instead of receiving the correct input data, we only get the approximate input data. We assume that the measured data in functions couple $\big(g_\varepsilon(x) \in \mathscr{L}_2(\mathbb{R}), \phi_\varepsilon(t) \in C[0,T]\big)$ satisfies

$$\|g - g_\varepsilon\|_{\mathscr{L}^2(\mathbb{R})} \leq \varepsilon, \;\; \|\phi - \phi_\varepsilon\|_{C[0,T]} \leq \varepsilon, \tag{3}$$

where the constant $\varepsilon > 0$ represents a noise level. It is known that the inverse source problem mentioned above is ill-posed in the sense of Hadamard, that is, a solution of this problem (1) does not always exist, if the solution does exist, it is not dependent continuously on the given data, meaning that the error of the initial data is small, the error of the solution will be large. This makes trouble for the numerical solution; here a regularization is required. The Fourier transform of a function $\mathcal{F}$ is defined by

$$\widehat{\mathcal{F}}(\xi) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} e^{-i\xi x} \mathcal{F}(x) dx. \tag{4}$$

We imposed an a priori bound on the input data, that is,

$$\|\mathcal{F}\|_{\mathbb{H}^k(\mathbb{R})} \leq \mathcal{M}, \;\; k > 0, \tag{5}$$

where $\mathcal{M} \geq 0$ is a constant,$\| \cdot \|_{\mathbb{H}^k(\mathbb{R})}$ denotes the norm in Sobolev space $\mathbb{H}^k(\mathbb{R})$ is defined

$$\mathbb{H}^k(\mathbb{R}) = \Big\{ \mathcal{F} \in \mathscr{L}_2(\mathbb{R}), \|\mathcal{F}\|_k \leq \infty \Big\}, \;\; and \;\; \|\mathcal{F}\|_{\mathbb{H}^k(\mathbb{R})} = \left(\int\limits_{\mathbb{R}} \Big| (1+\xi^2)^{k/2} \widehat{\mathcal{F}}(\xi) \Big|^2 d\xi \right)^{\frac{1}{2}}. \tag{6}$$

The outline of this paper is divided into the following sections: Section 2 gives some auxiliary results. In Section 3, by the priori bound assumption of the exact solution and the priori parameter choice rule, we present the convergence rate. In Section 4, we show the convergence rate between the exact and regularized solutions under the posteriori parameter choice rule. Next, a numerical example is proposed to show the illustration of the results in theory in Section 5. Finally, a conclusion is given in Section 6.

2. Some Auxiliary Results

Before showing some lemmas, we recall the Mittag-Leffler function which is defined by

$$E_{\beta,\kappa}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \kappa)}, \quad z \in \mathbb{C}, \tag{7}$$

where $\beta > 0$ and $\kappa \in \mathbb{R}$ are arbitrary constant. In Reference [14], the properties of the Mittag-Leffler function are discussed. Hereby, we present the following Lemmas of the Mittag-Leffler function which can be found in Reference ([14], Chapter 1).

Lemma 1. *Let $0 < \beta_0 < \beta_1 < 1$. Then there exist the constants $\overline{B}_1, \overline{B}_2, \overline{B}_3$ depending only on β_0, β_1 such that for all $\beta \in [\beta_0, \beta_1]$,*

$$\frac{\overline{B}_1}{\Gamma(1-\beta)} \frac{1}{1-x} \leq E_{\beta,1}(x) \leq \frac{\overline{B}_2}{\Gamma(1-\beta)} \frac{1}{1-x}, \quad E_{\beta,\alpha}(x) \leq \frac{\overline{B}_3}{1-x}, \quad \forall x \leq 0, \forall \alpha \in \mathbb{R}. \tag{8}$$

These estimates are uniform for all $\beta \in [\beta_0, \beta_1]\cdot$

Lemma 2. *(see Reference [7]) For $0 < \beta < 1$, we have:*

$$E_{\beta,\beta}(-\zeta) \geq 0, \quad \zeta \geq 0.$$

Proof. As for the proof, see Miller and Samko [15]. □

Lemma 3. *(see Reference [7]) For $\xi > 0$, $\alpha > 0$ and a positive integer $n \in \mathbb{N}$, we have:*

$$\begin{aligned} &\frac{d^n}{dt^n} E_{\beta,1}(-\xi^2 t^\beta) = -\xi^2 t^{\beta-n} E_{\beta,\beta-n+1}(-\xi^2 t^\beta), \quad t > 0, \\ &\frac{d}{dt}(t E_{\beta,2}(-\xi^2 t^\beta)) = E_{\beta,1}(-\xi^2 t^\beta), \quad t \geq 0. \end{aligned} \tag{9}$$

Lemma 4. *(see Reference [7]) By Lemma 2 and Lemma 3, we have*

$$\begin{aligned} \int_0^{\varrho} \left| t^{\gamma-1} E_{\beta,\beta}(-\xi^2 t^\gamma) \right| dt &= \int_0^{\varrho} t^{\gamma-1} E_{\beta,\beta}(-\xi^2 t^\gamma) dt \\ &= -\frac{1}{\xi^2} \int_0^{\varrho} \frac{d}{dt} E_{\beta,1}(-\xi^2 t^\gamma) dt = \frac{1}{\xi^2} \Big(1 - E_{\alpha,1}(-\xi^2 \varrho^\alpha) \Big), \quad \varrho > 0. \end{aligned} \tag{10}$$

Lemma 5. *(see Reference [16]) For $0 < \alpha < 1$, $\xi \in \mathbb{R}$, the following inequalities hold:*

$$\sup_{\xi \in \mathbb{R}} \left| (1+\xi^2)^{-k} (1 - e^{-\frac{\alpha^2 \xi^2}{4}}) \right| \leq \max \left\{ \alpha^{2k}, \alpha^2 \right\}. \tag{11}$$

Proof. The proof can be found in Reference [9]. □

Lemma 6. *Let $\beta \in (0,1)$ and $\xi \in \mathbb{R}$, the following estimate holds*

$$\left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-\xi^2 s^\beta)ds\right)^{-1} = \frac{\xi^2}{1-E_{\beta,1}(-\xi^2T^\beta)} \le \begin{cases} \dfrac{\xi^2}{1-E_{\beta,1}(-T^\beta)}, & \text{if } |\xi| \ge 1, \\ \dfrac{1}{1-E_{\beta,1}(-T^\beta)}, & \text{if } |\xi| < 1. \end{cases} \tag{12}$$

Proof. If $|\xi| \ge 1$ then since $E_{\beta,1}(-y)$ for $0 < \beta < 1$ is a decreasing function for $y > 0$, we get $E_{\beta,1}(-\xi^2T^\beta) \le E_{\beta,1}(-T^\beta)$. Whereupon

$$\left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-\xi^2 s^\beta)ds\right)^{-1} = \frac{\xi^2}{1-E_{\beta,1}(-\xi^2T^\beta)} \le \frac{\xi^2}{1-E_{\beta,1}(-T^\beta)}, \text{ for } |\xi| \ge 1. \tag{13}$$

If $|\xi| \le 1$ then since $E_{\beta,\beta}(-y)$ with $0 < \beta < 1$ is a decreasing function for $y > 0$, we get $E_{\beta,\beta}(-\xi^2 s^\beta) \ge E_{\beta,\beta}(-s^\beta)$, so

$$\left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-\xi^2 s^\beta)ds\right)^{-1} \le \left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-s^\beta)ds\right)^{-1} = \frac{1}{1-E_{\beta,1}(-T^\beta)}, \text{ for } |\xi| \le 1. \tag{14}$$

□

Lemma 7. *For $\alpha \in (0,1)$ and $\xi \in \mathbb{R}$, from Lemma 6, one has:*

$$\frac{1}{\left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-\xi^2 s^\beta)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} = \frac{\xi^2}{\left(1-E_{\beta,1}(-\xi^2T^\beta)\right)e^{\frac{\alpha^2\xi^2}{4}}}$$
$$\le \begin{cases} \dfrac{\frac{\xi^2\alpha^2}{4}}{\left(1-E_{\beta,1}(-T^\beta)\right)e^{\frac{\alpha^2\xi^2}{4}}} \le \left(\dfrac{4}{\alpha^2}\right)\left(\dfrac{1}{(1-E_{\beta,1}(-T^\beta)}\right), & \text{if } |\xi| \ge 1, \\ \dfrac{1}{\left(1-E_{\beta,1}(-T^\beta)\right)e^{\frac{\alpha^2\xi^2}{4}}} \le \left(\dfrac{4}{\alpha^2}\right)\left(\dfrac{1}{1-E_{\beta,1}(-T^\beta)}\right), & \text{if } |\xi| < 1. \end{cases} \tag{15}$$

This gives

$$\frac{1}{\left(\int_0^T s^{\beta-1}E_{\beta,\beta}(-\xi^2 s^\beta)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} \le \left(\frac{4}{\alpha^2}\right)\left(\frac{1}{1-E_{\beta,1}(-T^\beta)}\right). \tag{16}$$

3. The Priori Parameter Choice

Next, the error estimate of the mollification regularization method will be derived under the priori parameter choice rule in this section. We consider the Gauss function

$$\rho_\alpha(x) := \frac{1}{\alpha\sqrt{\pi}}e^{-\frac{x^2}{\alpha^2}}, \tag{17}$$

as the mollifer kernel, where α is a positive constant.

We define an operator K_α as

$$K_\alpha f(x) := \rho_\alpha f(x) := \int_{\mathbb{R}} \rho_\alpha(t) f(x-t)dt = \int_{\mathbb{R}} \rho_\alpha(x-t) f(t)dt, \tag{18}$$

for $f(x) \in \mathscr{L}_2(\mathbb{R})$. The original ill-posed problem is replaced by a new problem of searching its approximation $f_{\varepsilon,\alpha}(x)$ which is defined by

$$f_{\varepsilon,\alpha}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \rho_\alpha \widehat{f_\varepsilon(\xi)} d\xi, \tag{19}$$

The Inverse Source Problem

By using the Fourier transform, the problem (1) is formulated in the following frequency space

$$\begin{cases} \dfrac{\partial^\beta \widehat{u}(\xi,t)}{\partial t^\beta} + \xi^2 \widehat{u}(\xi,t) = \phi(t)\widehat{f}(\xi), & (\xi,t) \in \mathbb{R} \times (0,T], \\ \widehat{u}(\xi,0) = 0, & \xi \in \mathbb{R}, \\ \widehat{u}(\xi,T) = \widehat{g}(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{20}$$

From the equation and the initial value in (20), we obtain

$$\widehat{u}(\xi,t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\xi^2(t-s)^\beta)\phi(s)\widehat{f}(\xi)ds. \tag{21}$$

Or equivalently,

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \Big(\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(-\xi^2(t-s)^\beta)\phi(s)ds \Big) \widehat{f}(\xi)d\xi. \tag{22}$$

Set

$$\mathcal{D}_\beta(\xi, t-s) = (t-s)^{\beta-1} E_{\beta,\beta}(-\xi^2(t-s)^\beta).$$

And $\widehat{u}(\xi,T) = \widehat{g}(\xi)$ in (20), one has

$$\widehat{f}(\xi) = \frac{\widehat{g}(\xi)}{\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds}. \tag{23}$$

Using the inverse Fourier transform, then we obtain the formula of the source function f

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \frac{\widehat{g}(\xi)}{\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds} d\xi. \tag{24}$$

On the other hand, if $\phi(t)$ is bounded by $\inf_{t\in[0,T]} |\phi(t)| \le \phi(t) \le \sup_{t\in[0,T]} |\phi(t)| = \|\phi\|_{C[0,T]}$, we have $\Big(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds \Big)^{-1}$ can be written then $\frac{1}{\inf_{t\in[0,T]} |\phi(t)|} \frac{\xi^2}{\big(1-E_{\beta,1}(-\xi^2 T^\beta)\big)}$. The unbounded function $\frac{\xi^2}{\left(1-E_{\beta,1}(-\xi^2T^\beta)\right)}$ can be seen as an amplification when $\xi \to \infty$. From now on, putting $\inf_{t\in[0,T]} |\phi(t)| = \mathcal{A}_0$, $\inf_{t\in[0,T]} |\phi_\varepsilon(t)| = \mathcal{A}_1$, $\sup_{t\in[0,T]} |\phi(t)| = \|\phi\|_{C[0,T]} = \Phi$. From (19) with α is a regularization parameter and α depends on ε, we found the regularized solution

$$\widehat{f}_{\varepsilon,\alpha}(\xi) = \frac{\widehat{g}_\varepsilon(\xi)}{\left(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}}. \tag{25}$$

Using inverse Fourier transform, we get

$$f_{\varepsilon,\alpha}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} \frac{\widehat{g}_\varepsilon(\xi)}{\left(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} e^{i\xi x}d\xi. \tag{26}$$

The main conclusion of this section are given below.

Theorem 1. *Let $f(x)$, given by (24), be the exact solution of (1) with exact data $g \in \mathscr{L}_2(\mathbb{R})$, and $f_{\varepsilon,\alpha}(x)$ is approximation solution of $f(x)$ with measured data $g_\varepsilon \in \mathscr{L}_2(\mathbb{R})$. Then we obtain*

a. If $0 < k < 1$, and choosing $\alpha(\varepsilon) = \left(\frac{\varepsilon}{\mathcal{M}}\right)^{\frac{1}{2(k+1)}}$, we have a convergence estimate

$$\left\|f(.) - f_{\varepsilon,\alpha}(.)\right\|_{\mathscr{L}_2(\mathbb{R})} \le \varepsilon^{\frac{k}{k+1}}\mathcal{M}^{\frac{1}{k+1}}\left(\max\left\{1, \left(\frac{\varepsilon}{\mathcal{M}}\right)^{\frac{1-k}{k+1}}\right\} + \mathcal{R}(\mathcal{A}_0, \mathcal{A}_1, \widehat{g})\right). \tag{27}$$

b. If $k > 1$, by choosing $\alpha(\varepsilon) = \left(\frac{\varepsilon}{\mathcal{M}}\right)^{\frac{1}{4}}$, we have a convergence estimate

$$\left\|f(.) - f_{\varepsilon,\alpha}(.)\right\|_{\mathscr{L}_2(\mathbb{R})} \le \varepsilon^{\frac{1}{2}}\mathcal{M}^{\frac{1}{2}}\left(1 + \mathcal{R}(\mathcal{A}_0, \mathcal{A}_1, \widehat{g})\right), \tag{28}$$

in which

$$\mathcal{R}(\mathcal{A}_0, \mathcal{A}_1, \widehat{g}) = \frac{4}{(1-E_{\beta,1}(-T^\beta))}\left(\frac{1}{\mathcal{A}_1} + \frac{\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}}{\mathcal{A}_1\mathcal{A}_0}\right). \tag{29}$$

Proof. From (24) and (26), by the Parseval formula, the triangle inequality, we obtain

$$\begin{aligned} &\left\|f(.) - f_{\varepsilon,\alpha}(.)\right\|_{\mathscr{L}_2(\mathbb{R})} = \left\|\widehat{f}(.) - \widehat{f_{\varepsilon,\alpha}}(.)\right\|_{\mathscr{L}_2(\mathbb{R})} \\ &= \left\|\frac{\widehat{g}(\xi)}{\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds} - \frac{\widehat{g}_\varepsilon(\xi)}{\left(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}}\right\|_{\mathscr{L}_2(\mathbb{R})} \\ &= \left\|\mathcal{I}_1\right\|_{\mathscr{L}_2(\mathbb{R})} + \left\|\mathcal{I}_2\right\|_{\mathscr{L}_2(\mathbb{R})} + \left\|\mathcal{I}_3\right\|_{\mathscr{L}_2(\mathbb{R})}, \end{aligned} \tag{30}$$

in which

$$\begin{aligned}
\mathcal{I}_1 &= \frac{\widehat{g}(\xi)}{\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds} - \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}}, \\
\mathcal{I}_2 &= \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g_\varepsilon}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}}, \\
\mathcal{I}_3 &= \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}}.
\end{aligned} \tag{31}$$

Next, we estimate the error by diving it into three steps as follows
Step 1: Estimate for $\|\mathcal{I}_1\|^2_{\mathscr{L}_2(\mathbb{R})}$, we have

$$\begin{aligned}
\|\mathcal{I}_1\|^2_{\mathscr{L}_2(\mathbb{R})} &= \left\| \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)} (1 - e^{-\frac{\alpha^2\xi^2}{4}}) \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&= \left\| (1+\xi^2)^{-k}(1 - e^{-\frac{\alpha^2\xi^2}{4}})(1+\xi^2)^k \widehat{f}(\xi) \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \sup_{\xi\in\mathbb{R}} \left| (1+\xi^2)^{-k}(1 - e^{-\frac{\alpha^2\xi^2}{4}}) \right|^2 \|f\|^2_{H^k(\mathbb{R})} \le \mathcal{M}^2 \max\left\{\alpha^{4k}, \alpha^4\right\}.
\end{aligned}$$

Hence,

$$\|\mathcal{I}_1\|_{\mathscr{L}_2(\mathbb{R})} \le \mathcal{M} \max\left\{\alpha^{2k}, \alpha^2\right\}. \tag{32}$$

Step 2: Estimate for $\|\mathcal{I}_2\|^2_{\mathscr{L}_2(\mathbb{R})}$, we get

$$\begin{aligned}
\|\mathcal{I}_2\|^2_{\mathscr{L}_2(\mathbb{R})} &= \left\| \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g_\varepsilon}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \mathcal{A}_1^{-2} \|\widehat{g}(\xi) - \widehat{g_\varepsilon}(\xi)\|^2_{\mathscr{L}_2(\mathbb{R})} \sup_{\xi\in\mathbb{R}} \left| \Big(\int\limits_0^T (T-s)^{\beta-1} E_{\beta,\beta}(-\xi^2(T-s)^\beta)ds\Big) e^{\frac{\alpha^2\xi^2}{4}} \right|^{-2} \\
&\le \mathcal{A}_1^{-2} \|\widehat{g}(\xi) - \widehat{g_\varepsilon}(\xi)\|^2_{\mathscr{L}_2(\mathbb{R})} \sup_{\xi\in\mathbb{R}} \left| \frac{\xi^2}{(1 - E_{\beta,1}(-\xi^2 T^\beta)e^{\frac{\alpha^2\xi^2}{4}}} \right|^2 \\
&\le \left(\frac{16\varepsilon^2}{\alpha^4}\right) \Big(\mathcal{A}_1 (1 - E_{\beta,1}(-T^\beta))\Big)^{-2}.
\end{aligned} \tag{33}$$

Hence, we conclude that

$$\|\mathcal{I}_2\|_{\mathscr{L}_2(\mathbb{R})} \le \left(\frac{4\varepsilon}{\alpha^2}\right) \Big(\mathcal{A}_1 (1 - E_{\beta,1}(-T^\beta))\Big)^{-1}. \tag{34}$$

Step 3: Estimate for $\left\|\mathcal{I}_3\right\|^2_{\mathscr{L}_2(\mathbb{R})}$, we have

$$\begin{aligned}
\|\mathcal{I}_3\|^2_{\mathscr{L}_2(\mathbb{R})} &= \left\| \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&= \left\| \frac{\widehat{g}(\xi)}{e^{\frac{\alpha^2\xi^2}{4}}} \frac{\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\Big(\phi_\varepsilon(s)-\phi(s)\Big)ds}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\Big)\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)} \right\|^2_{\mathscr{L}_2(\mathbb{R})}. \qquad (35)
\end{aligned}$$

From (35), we get

$$\begin{aligned}
\|\mathcal{I}_3\|^2_{\mathscr{L}_2(\mathbb{R})} &= \mathcal{A}_1^{-2}\|\phi_\varepsilon - \phi\|^2_{C[0,T]} \left\| \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \mathcal{A}_1^{-2}\|\phi_\varepsilon - \phi\|^2_{C[0,T]} \left\| \frac{\widehat{g}(\xi)}{\Big(\int\limits_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)e^{\frac{\alpha^2\xi^2}{4}}} \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \big(\mathcal{A}_0\mathcal{A}_1\big)^{-2}\|\phi_\varepsilon - \phi\|^2_{C[0,T]} \left| \frac{\xi^2}{(1-E_{\beta,1}(-\xi^2T^\beta))\, e^{\frac{\alpha^2\xi^2}{4}}} \right|^2 \int\limits_{\mathbb{R}} |\widehat{g}(\xi)|^2 d\xi \\
&\le \left(\frac{16}{\alpha^4}\right)\Big(\mathcal{A}_0\mathcal{A}_1\big(1-E_{\beta,1}(-T^\beta)\big)\Big)^{-2}\|\phi_\varepsilon - \phi\|^2_{C[0,T]} \int\limits_{\mathbb{R}} |\widehat{g}(\xi)|^2 d\xi. \qquad (36)
\end{aligned}$$

Hence,

$$\|\mathcal{I}_3\|_{\mathscr{L}_2(\mathbb{R})} \le \left(\frac{4\varepsilon}{\alpha^2}\right)\Big(\mathcal{A}_0\mathcal{A}_1\big(1-E_{\beta,1}(-T^\beta)\big)\Big)^{-1}\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}. \qquad (37)$$

Combining (32), (34) and (37), we received

(a) If $0 \le k \le 1$ by choosing $\alpha(\varepsilon) = \left(\frac{\varepsilon}{\mathcal{M}}\right)^{\frac{1}{2(k+1)}}$, we have a convergent estimation:

$$\big\|f(.) - f_{\varepsilon,\alpha}(.)\big\|_{\mathscr{L}_2(\mathbb{R})} \text{ is of order } \varepsilon^{\frac{k}{k+1}}. \qquad (38)$$

(b) If $k > 1$, by choosing $\alpha(\varepsilon) = \left(\frac{\varepsilon}{\mathcal{M}}\right)^{\frac{1}{4}}$, we have a convergent estimation:

$$\big\|f(.) - f_{\varepsilon,\alpha}(.)\big\|_{\mathscr{L}_2(\mathbb{R})} \text{ is of order } \varepsilon^{\frac{1}{2}}. \qquad (39)$$

□

4. The Discrepancy Principle

Now, we present the posteriori regularization parameter choice rule. The most general of the posteriori rules is the Morozov discrepancy principle [17]. Choosing the regularization parameter α as the solution of the equation

$$l(\alpha) = \left\| e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}_\varepsilon(\xi) \right\|_{\mathscr{L}_2(\mathbb{R})} = \varepsilon + \eta \left(\log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right)^{-1}, \tag{40}$$

where $\eta > 1$ is a constant.

Remark 1. *To ensure the existence and uniqueness, we can choose η such that*

$$0 < \varepsilon + \eta \left(\log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right)^{-1} < \|\widehat{g}_\varepsilon\|_{\mathscr{L}_2(\mathbb{R})}.$$

To establish the existence and uniqueness of the solution of Equation (40), we consider the following lemmas

Lemma 8. *If $\varepsilon > 0$ then there holds:*

(a) $l(\alpha)$ *is a continous function.*
(b) $\lim_{\alpha \to 0^+} l(\alpha) = 0$.
(c) $\lim_{\alpha \to +\infty} l(\alpha) = \|\widehat{g}_\varepsilon\|_{\mathscr{L}_2(\mathbb{R})}$.
(d) $l(\alpha)$ *is a strictly increasing function.*

The proof is very easy and we omit it here.

Lemma 9. *The following inequality holds:*

$$\|e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\|_{\mathscr{L}_2(\mathbb{R})} \le 2\varepsilon + \eta \left(\log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right)^{-1}. \tag{41}$$

Proof. Applying the triangle inequality and (40), we have

$$\begin{aligned} \|e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\|_{\mathscr{L}_2(\mathbb{R})} &\le \|e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}_\varepsilon(\xi)\|_{\mathscr{L}_2(\mathbb{R})} + \|\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\|_{\mathscr{L}_2(\mathbb{R})} \\ &\le \|e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}_\varepsilon(\xi)\|_{\mathscr{L}_2(\mathbb{R})} + \|\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\|_{\mathscr{L}_2(\mathbb{R})} \\ &\le 2\varepsilon + \eta \left(\log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right)^{-1}. \end{aligned} \tag{42}$$

□

Lemma 10. *For any $0 \ne \xi \in \mathbb{R}$, let $s, t \in [0, T]$ such that $0 \le s \le t \le T$, making the substitution ξ^2 and using the inequality: $\frac{\overline{B}_3 s^{\beta-1}}{1+\xi^2 s^\beta} \le \overline{B}_3 s^{\beta-1}$, we have the following estimate*

$$\int_0^T \mathcal{D}_\beta(\xi, T-s)ds = \int_0^T (T-s)^{\beta-1} E_{\beta,\beta}(-\xi^2(T-s)^\beta)ds \le \frac{\overline{B}_3 T^\beta}{\beta}. \tag{43}$$

Lemma 11. *If α is the solution of Equation (40), then the following inequality also holds:*

$$\frac{4}{\alpha^2} \le \frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\eta} \left(\log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right). \tag{44}$$

whereby $\mathcal{M} \geq \|f\|_{\mathbb{H}^k(\mathbb{R})}$.

Proof. Due to (40), we receive

$$\varepsilon + \eta\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{-1} = \Big\|(1-e^{-\frac{\alpha^2\xi^2}{4}})\widehat{g}_\varepsilon(\xi)\Big\|_{\mathscr{L}_2(\mathbb{R})}$$
$$\leq \Big\|(1-e^{-\frac{\alpha^2\xi^2}{4}})\widehat{g}_\varepsilon(\xi) - (1-e^{-\frac{\alpha^2\xi^2}{4}})\widehat{g}(\xi) + (1-e^{-\frac{\alpha^2\xi^2}{4}})\widehat{g}(\xi)\Big\|_{\mathscr{L}_2(\mathbb{R})}$$
$$\leq \varepsilon + \Big\|(1-e^{-\frac{\alpha^2\xi^2}{4}})(1+\xi^2)^{-k}\Big(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)(1+\xi^2)^k\widehat{f}(\xi)\Big\|_{\mathscr{L}_2(\mathbb{R})}$$
$$\leq \varepsilon + \sup_{\xi\in\mathbb{R}}\Big|(1-e^{-\frac{\alpha^2\xi^2}{4}})(1+\xi^2)^{-k}\Big(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\Big)\Big|\mathcal{M}$$
$$\leq \varepsilon + \frac{\alpha^2}{4}\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M}), \tag{45}$$

whereby

$$\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M}) = (\beta)^{-1}\Phi\overline{B}_3 T^\beta\mathcal{M}. \tag{46}$$

So

$$\frac{4}{\alpha^2} \leq \frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\eta}\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big). \tag{47}$$

□

Lemma 12. *For* $0 < \alpha < 1$, *using the Lemma 7, the following inequality holds:*

$$\sup_{\xi\in\mathbb{R}}\Big|\Big(\frac{\xi^2}{1-E_{\beta,1}(-\xi^2T^\beta)}\Big)^{k+1}e^{-\frac{\alpha^2\xi^2}{4}}\Big| \leq \Big(\frac{k+1}{(1-E_{\beta,1}(-T^\beta))}\Big)^{k+1}\Big(\frac{4}{\alpha^2}\Big)^{k+1}. \tag{48}$$

The proof is similar to Lemma 7 and we omit it here.

Next, the main results of this section are shown under Theorem.

Theorem 2. *Assume the condition* $\|g_\varepsilon - g\| \leq \varepsilon$ *where* $\|.\|$ *denotes the* $\mathscr{L}_2(\mathbb{R})$*-norm with* $\varepsilon > 0$ *is a noise level and the condition (5) holds, then there holds the following error estimate*

$$\|f(.) - f_{\varepsilon,\alpha}(.)\|_{\mathscr{L}_2(\mathbb{R})} = \|\widehat{f}(.) - \widehat{f_{\varepsilon,\alpha}}(.)\|_{\mathscr{L}_2(\mathbb{R})}$$
$$\leq \Bigg(\varepsilon\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{k+1}\Big(\frac{\mathcal{L}_\beta(k,T)\ \mathcal{H}_\beta(\overline{B}_3, T, \Phi)}{\mathcal{A}_0\eta}\Big)^{k+1}\mathcal{M}^k$$
$$+ \frac{\Phi}{\mathcal{A}_0^{k+1}}\Big(\frac{1}{1-E_{\beta,1}(-T^\beta)}\Big)^k\Bigg)^{\frac{1}{k+1}}\mathcal{M}^{\frac{1}{k+1}}.\Big(2\varepsilon + \eta\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{-1}\Big)^{\frac{k}{k+1}}$$
$$+ \Big(\varepsilon\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)\Bigg(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\eta\mathcal{A}_0(1-E_{\beta,1}(-T^\beta))} + \Big(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}}{\eta(1-E_{\beta,1}(-T^\beta))\mathcal{A}_0\mathcal{A}_1}\Big)\Bigg). \tag{49}$$

Proof. By the Parseval formula, we get

$$\begin{aligned}
\big\|f(.)-f_{\varepsilon,\alpha}(.)\big\|_{\mathscr{L}_2(\mathbb{R})} &= \big\|\widehat{f}(.)-\widehat{f_{\varepsilon,\alpha}}(.)\big\|_{\mathscr{L}_2(\mathbb{R})} \\
&\le \left\| \frac{e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)}{\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi(s)ds} \right\|_{\mathscr{L}_2(\mathbb{R})} \\
&+ \left\| \frac{\widehat{g}_\varepsilon(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g}(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} \right\|_{\mathscr{L}_2(\mathbb{R})} \\
&+ \left\| \frac{\widehat{g}(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g}(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi_\varepsilon(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} \right\|_{\mathscr{L}_2(\mathbb{R})} \\
&\le \|\mathcal{J}_1\|_{\mathscr{L}_2(\mathbb{R})} + \|\mathcal{J}_2\|_{\mathscr{L}_2(\mathbb{R})} + \|\mathcal{J}_3\|_{\mathscr{L}_2(\mathbb{R})}, \qquad (50)
\end{aligned}$$

We can divide the proof into three steps as follows:
Step 1: Estimate for $\|\mathcal{J}_1\|^2_{\mathscr{L}_2(\mathbb{R})}$, using the Hölder inequality, we obtain

$$\begin{aligned}
\|\mathcal{J}_1\|^2_{\mathscr{L}_2(\mathbb{R})} &= \frac{1}{\left(\int_0^T \mathcal{D}_\beta(\xi,T-s)\phi(s)ds\right)^2} \left\| e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \left\| \frac{\xi^2}{\mathcal{A}_0(1-E_{\beta,1}(-\xi^2T^\beta))}\left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right)\right\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le (\mathcal{C}_1^2)^{\frac{1}{k+1}} \times (\mathcal{C}_2^2)^{\frac{k}{k+1}}, \qquad (51)
\end{aligned}$$

whereby

$$\begin{aligned}
\mathcal{C}_1^2 &= \left(\int_{\mathbb{R}} \left(\left(\frac{\xi^2}{\mathcal{A}_0(1-E_{\beta,1}(-\xi^2T^\beta))}\right)^2\left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right)^{\frac{2}{k+1}}\right)^{k+1}d\xi\right), \\
\mathcal{C}_2^2 &= \left(\int_{\mathbb{R}} \left(\left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right)^{\frac{2k}{k+1}}\right)^{k+1}d\xi\right). \qquad (52)
\end{aligned}$$

From (52), we can check that $\left(\mathcal{C}_2^2\right)^{\frac{k}{k+1}}$ as follows

$$\begin{aligned}
(\mathcal{C}_2^2)^{\frac{k}{k+1}} &= \left(\int_{\mathbb{R}} \left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right)^2 d\xi\right)^{\frac{k}{k+1}} \\
&= \left\|e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}_\varepsilon(\xi)-\widehat{g}(\xi)\right\|^{\frac{2k}{k+1}}_{\mathscr{L}_2(\mathbb{R})} \le \left(2\varepsilon+\eta\left(\log\left(\log\left(\frac{T}{\varepsilon}\right)\right)\right)^{-1}\right)^{\frac{2k}{k+1}}. \qquad (53)
\end{aligned}$$

On the other hand, we deduce

$$(\mathcal{C}_1^2)^{\frac{1}{k+1}} \leq \bigg(\int\limits_{-\infty}^{+\infty} \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{2(k+1)} \Big(e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big)^2 d\xi \bigg)^{\frac{1}{k+1}}$$

$$= \Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} \Big(e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})}^{\frac{2}{k+1}}$$

$$\leq \bigg(\Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} e^{-\frac{\alpha^2\xi^2}{4}} \Big(\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})}$$

$$+ \Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} \Big(e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})} \bigg)^{\frac{2}{k+1}}. \tag{54}$$

To estimate $\mathcal{C}_1$, we give two Lemmas as follows:

Lemma 13. *Assume that the condition* $\|\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\|_{\mathscr{L}_2(\mathbb{R})} \leq \varepsilon$ *holds. Then we have the following estimate*

$$\Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} e^{-\frac{\alpha^2\xi^2}{4}} \Big(\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})}$$

$$\leq \varepsilon \Bigg(\log \bigg(\log \Big(\frac{T}{\varepsilon} \Big) \bigg) \Bigg)^{k+1} \bigg(\frac{\mathcal{L}_\beta(k,T)\, \mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\mathcal{A}_0 \eta} \bigg)^{k+1}. \tag{55}$$

Proof. Using the Lemma 12 and setting $\mathcal{L}_\beta(k,T) = \Big(\frac{k+1}{(1 - E_{\beta,1}(-T^\beta))} \Big)$, we get

$$\Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} e^{-\frac{\alpha^2\xi^2}{4}} \Big(\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})}$$

$$\leq \varepsilon \Big(\frac{4}{\alpha^2} \Big)^{k+1} \bigg(\frac{k+1}{\mathcal{A}_0\big(1 - E_{\beta,1}(-T^\beta)\big)} \bigg)^{k+1}$$

$$\leq \varepsilon \Bigg(\log \bigg(\log \Big(\frac{T}{\varepsilon} \Big) \bigg) \Bigg)^{k+1} \bigg(\frac{\mathcal{L}_\beta(k,T)\, \mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\mathcal{A}_0 \eta} \bigg)^{k+1} \tag{56}$$

in which $\mathcal{H}_\beta(\overline{B}_3, T, \Phi, M)$ is defined in Lemma 11. □

Lemma 14. *Let* $\xi \in \mathbb{R}$ *and exist* $\mathcal{M}$ *is a positive constant such that* $\mathcal{M} \geq \|f\|_{H^k(\mathbb{R})}$, *we get*

$$\Big\| \Big(\frac{\xi^2}{\mathcal{A}_0\big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} \Big(e^{-\frac{\alpha^2\xi^2}{4}} \widehat{g}(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})} \leq \frac{\Phi}{\mathcal{A}_0^{k+1}} \bigg(\frac{1}{1 - E_{\beta,1}(-T^\beta)} \bigg)^k \mathcal{M}. \tag{57}$$

Proof. Applying the Lemma 4, we receive

$$\begin{aligned}
&\Big\| \Big(\frac{\xi^2}{\mathcal{A}_0 \big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} \Big(e^{-\frac{\alpha^2 \xi^2}{4}} \widehat{g}(\xi) - \widehat{g}(\xi) \Big) \Big\|_{\mathscr{L}_2(\mathbb{R})} \\
&= \Bigg\| \Big(\frac{\xi^2 (1 - e^{-\frac{\alpha^2 \xi^2}{4}})^{\frac{1}{k+1}}}{\mathcal{A}_0 \big(1 - E_{\beta,1}(-\xi^2 T^\beta)\big)} \Big)^{k+1} (1+\xi^2)^{-k} (1+\xi^2)^k \widehat{f}(\xi) \int_0^T \mathcal{D}_\beta(\xi, T-s) \phi(s) ds \Bigg\|_{\mathscr{L}_2(\mathbb{R})} \\
&\le \frac{\Phi}{\mathcal{A}_0} \sup_{\xi \in \mathbb{R}} \Big| \Big(\frac{\xi^2}{\mathcal{A}_0 (1 - E_{\beta,1}(-\xi^2 T^\beta))} \Big)^k \frac{(1 - e^{-\frac{\alpha^2 \xi^2}{4}})}{(1+\xi^2)^k} \Big| \mathcal{M} \\
&\le \frac{\Phi}{\mathcal{A}_0^{k+1}} \sup_{\xi \in \mathbb{R}} \Big| \Big(\frac{\xi^2}{(1+\xi^2)(1 - E_{\beta,1}(-\xi^2 T^\beta))} \Big)^k (1 - e^{-\frac{\alpha^2 \xi^2}{4}}) \Big| \mathcal{M} \\
&\le \frac{\Phi}{\mathcal{A}_0^{k+1}} \left(\frac{1}{1 - E_{\beta,1}(-T^\beta)} \right)^k \mathcal{M}. \qquad (58)
\end{aligned}$$

□

Combining (54), (56) and (58), we have estimate $(\mathcal{C}_1^2)^{\frac{1}{k+1}}$ as follows

$$\begin{aligned}
(\mathcal{C}_1^2)^{\frac{1}{k+1}} \le \Bigg(&\varepsilon \Big(\log \Big(\log \Big(\frac{T}{\varepsilon} \Big) \Big) \Big)^{k+1} \left(\frac{\mathcal{L}_\beta(k,T)\, \mathcal{H}_\beta(\overline{B}_3, T, \Phi)}{\mathcal{A}_0 \eta} \right)^{k+1} \mathcal{M}^k \\
&+ \frac{\Phi}{\mathcal{A}_0^{k+1}} \left(\frac{1}{1 - E_{\beta,1}(-T^\beta)} \right)^k \Bigg)^{\frac{2}{k+1}} \mathcal{M}^{\frac{2}{k+1}}. \qquad (59)
\end{aligned}$$

From (51) to (59), so

$$\begin{aligned}
\|\mathcal{J}_1\|_{\mathscr{L}_2(\mathbb{R})} \le \Bigg(&\varepsilon \Big(\log \Big(\log \Big(\frac{T}{\varepsilon} \Big) \Big) \Big)^{k+1} \left(\frac{\mathcal{L}_\beta(k,T)\, \mathcal{H}_\beta(\overline{B}_3, T, \Phi)}{\mathcal{A}_0 \eta} \right)^{k+1} \mathcal{M}^k \\
&+ \frac{\Phi}{\mathcal{A}_0^{k+1}} \left(\frac{1}{1 - E_{\beta,1}(-T^\beta)} \right)^k \Bigg)^{\frac{1}{k+1}} \mathcal{M}^{\frac{1}{k+1}} . \Bigg(2\varepsilon + \eta \Big(\log \Big(\log \Big(\frac{T}{\varepsilon} \Big) \Big) \Big)^{-1} \Bigg)^{\frac{k}{k+1}} . \qquad (60)
\end{aligned}$$

Step 2: Estimate for $\|\mathcal{J}_2\|^2_{\mathscr{L}_2(\mathbb{R})}$, we have

$$\begin{aligned}
\|\mathcal{J}_2\|^2_{\mathscr{L}_2(\mathbb{R})} &\le \Bigg\| \frac{\big(\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi)\big)}{\Big(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s) ds \Big) e^{\frac{\alpha^2 \xi^2}{4}}} \Bigg\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&= \Bigg\| \frac{\xi^2}{\mathcal{A}_0 \Big(1 - E_{\beta,1}(-\xi^2 T^\beta)\Big)} e^{-\frac{\alpha^2 \xi^2}{4}} \Big(\widehat{g}_\varepsilon(\xi) - \widehat{g}(\xi) \Big) \Bigg\|^2_{\mathscr{L}_2(\mathbb{R})} \\
&\le \Big(\frac{16}{\alpha^4} \Big) \frac{\|\widehat{g}_\varepsilon - \widehat{g}\|^2_{\mathscr{L}_2(\mathbb{R})}}{\mathcal{A}_0^2} \Big(\frac{1}{1 - E_{\beta,1}(-T^\beta)} \Big)^2 . \qquad (61)
\end{aligned}$$

Applying the Lemmas 11 and 12 in case $k = 0$, we know that

$$\|\mathcal{J}_2\|^2_{\mathscr{L}_2(\mathbb{R})} \le \left(\varepsilon \log \left(\log \left(\frac{T}{\varepsilon} \right) \right) \right)^2 \left(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\eta \mathcal{A}_0 (1 - E_{\beta,1}(-T^\beta))} \right)^2 . \qquad (62)$$

Hence, we conclude that

$$\|\mathcal{J}_2\|_{\mathscr{L}_2(\mathbb{R})} \leq \left(\varepsilon \log\left(\log\left(\frac{T}{\varepsilon}\right)\right)\right)\left(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})}{\eta \mathcal{A}_0(1 - E_{\beta,1}(-T^\beta))}\right). \tag{63}$$

Step 3: Estimate for $\|\mathcal{J}_3\|^2_{\mathscr{L}_2(\mathbb{R})}$, we have :

$$\begin{aligned}\|\mathcal{J}_3\|^2_{\mathscr{L}_2(\mathbb{R})} &\leq \left\| \frac{\widehat{g}(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} - \frac{\widehat{g}(\xi)}{\left(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\right)e^{\frac{\alpha^2\xi^2}{4}}} \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\ &= \left\| \frac{\left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}(\xi)\right)\int_0^T \mathcal{D}_\beta(\xi, T-s)(\phi_\varepsilon(s) - \phi(s))ds}{\left(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi(s)ds\right)\left(\int_0^T \mathcal{D}_\beta(\xi, T-s)\phi_\varepsilon(s)ds\right)} \right\|^2_{\mathscr{L}_2(\mathbb{R})}. \end{aligned} \tag{64}$$

From (64), it gives

$$\begin{aligned}\|\mathcal{J}_3\|^2_{\mathscr{L}_2(\mathbb{R})} &\leq \left\| \frac{\|\phi_\varepsilon - \phi\|_{C[0,T]} \int_0^T \mathcal{D}_\beta(\xi, T-s)ds}{\mathcal{A}_0\mathcal{A}_1\left(\int_0^T \mathcal{D}_\beta(\xi, T-s)ds\right)^2}\left(e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}(\xi)\right) \right\|^2_{\mathscr{L}_2(\mathbb{R})} \\ &\leq \frac{\|\phi - \phi_\varepsilon\|^2_{C[0,T]}}{\mathcal{A}_0^2\mathcal{A}_1^2}\left\| \frac{\xi^2}{(1 - E_{\beta,1}(-\xi^2 T^\beta))} e^{-\frac{\alpha^2\xi^2}{4}}\widehat{g}(\xi) \right\|^2_{\mathscr{L}_2(\mathbb{R})}. \end{aligned} \tag{65}$$

Applying Lemma 12 with $k = 0$ and Lemma 11 , we know that

$$\begin{aligned}\|\mathcal{J}_3\|^2_{\mathscr{L}_2(\mathbb{R})} &\leq \left(\frac{16}{\alpha^4}\right)\left(\frac{1}{1 - E_{\beta,1}(-T^\beta)}\right)^2 \frac{\|\phi - \phi_\varepsilon\|^2_{C[0,T]}}{\mathcal{A}_0^2\mathcal{A}_1^2}\|\widehat{g}\|^2_{\mathscr{L}_2(\mathbb{R})} \\ &\leq \left(\varepsilon \log\left(\log\left(\frac{T}{\varepsilon}\right)\right)\right)^2\left(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}}{\eta(1 - E_{\beta,1}(-T^\beta))\mathcal{A}_0\mathcal{A}_1}\right)^2. \end{aligned} \tag{66}$$

Therefore,

$$\|\mathcal{J}_3\|_{\mathscr{L}_2(\mathbb{R})} \leq \left(\varepsilon \log\left(\log\left(\frac{T}{\varepsilon}\right)\right)\right)\left(\frac{\mathcal{H}_\beta(\overline{B}_3, T, \Phi, \mathcal{M})\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}}{\eta(1 - E_{\beta,1}(-T^\beta))\mathcal{A}_0\mathcal{A}_1}\right). \tag{67}$$

Combining (60), (63) and (67), we get:

$$\begin{aligned}&\|f(.)-f_{\varepsilon,\alpha}(.)\|_{\mathscr{L}_2(\mathbb{R})}=\|\widehat{f}(.)-\widehat{f_{\varepsilon,\alpha}}(.)\|_{\mathscr{L}_2(\mathbb{R})}\\&\le\Bigg(\varepsilon\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{k+1}\Big(\frac{\mathcal{L}_\beta(k,T)\,\mathcal{H}_\beta(\overline{B}_3,T,\Phi)}{\mathcal{A}_0\eta}\Big)^{k+1}\mathcal{M}^k\\&+\frac{\Phi}{\mathcal{A}_0^{k+1}}\Big(\frac{1}{1-E_{\beta,1}(-T^\beta)}\Big)^k\Bigg)^{\frac{1}{k+1}}\mathcal{M}^{\frac{1}{k+1}}.\Big(2\varepsilon+\eta\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{-1}\Big)^{\frac{k}{k+1}}\\&+\Big(\varepsilon\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)\Bigg(\frac{\mathcal{H}_\beta(\overline{B}_3,T,\Phi,\mathcal{M})}{\eta\mathcal{A}_0(1-E_{\beta,1}(-T^\beta))}+\Big(\frac{\mathcal{H}_\beta(\overline{B}_3,T,\Phi,\mathcal{M})\|\widehat{g}\|_{\mathscr{L}_2(\mathbb{R})}}{\eta(1-E_{\beta,1}(-T^\beta))\mathcal{A}_0\mathcal{A}_1}\Big)\Bigg).\end{aligned}\tag{68}$$

Nohing that

$$\lim_{\varepsilon\to 0}\varepsilon\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)=0,\ \ \lim_{\varepsilon\to 0}\varepsilon\Big(\log\Big(\log\Big(\frac{T}{\varepsilon}\Big)\Big)\Big)^{k+1}=0.\tag{69}$$

Combining (68) and (69), we conclude that

$$\|f(.)-f_{\varepsilon,\alpha}(.)\|_{\mathscr{L}_2(\mathbb{R})}=\|\widehat{f}(.)-\widehat{f_{\varepsilon,\alpha}}(.)\|_{\mathscr{L}_2(\mathbb{R})}\ \to\ 0,\text{as}\ \ \varepsilon\to 0.\tag{70}$$

The proof of Theorem 2 is completed. □

5. Numerical Experiments

In this section, in order to illustrate the usefulness of the proposed methods, we consider the numerical examples intended. We carry out numerically above regularization methods to verify our proposed methods. The numerical examples with $T=1$, and $\beta=0.4$, $\beta=0.95$ are shown in this section, respectively. In the following, we give an example which had the exact expression of the solutions $(u(x,t),f(x))$. We use the computations in Matlab codes which are given by Podlubny [18] for computing the generalized Mittag-Leffler function and the accuracy control in computing is 10^{-10}. We will do the numerical tests with $x\in[-7,7]$ and $\eta=1.1$. The couple of $(\phi_\varepsilon,g_\varepsilon)$, which are determined below, play as measured data with a random noise as follows:

$$\phi_\varepsilon(\cdot)=\phi(\cdot)+\varepsilon\left(2\text{rand}(.)-1\right),\ g_\varepsilon(\cdot)=g(\cdot)+\varepsilon\left(2\text{rand}(.)-1\right).\tag{71}$$

Following Reference [9], we know the function $rand(\cdot)$ generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2=1$ and standard deviation $\sigma=1$, and it gives rand(size(.)) and rand(size(.)) returns an array of random entries that is the same size as g and ϕ, respectively. We can easily verify the validity of the inequality:

$$\|\phi_\varepsilon-\phi\|_{C[0,T]}\le\varepsilon,\ \ \|g_\varepsilon-g\|_{\mathscr{L}_2(\mathbb{R})}\le\varepsilon.\tag{72}$$

In this example, we consider particularly a one-dimensional case of the problem (1) for f is an exact data function.

$$\begin{cases}\dfrac{\partial^\beta u(x,t)}{\partial t^\beta}=u_{xx}(x,t)+\phi(t)f(x),\ \ (x,t)\in\mathbb{R}\times(0,T],\\u(x,0)=0,x\in\mathbb{R},\\u(x,1)=g(x),\ \ x\in\mathbb{R}\cdot\end{cases}\tag{73}$$

In this example, we choose the following solution

$$u(x,t)=\Big(E_{\beta,1}(t^\beta)-E_{\beta,1}(-t^\beta)\Big)\sin\Big(\frac{x}{2}\Big).\tag{74}$$

Then a simple computation yields

$$\phi(t) = \frac{5}{4}\, E_{\beta,1}(t^{\beta}) + \frac{3}{4} E_{\beta,1}(-t^{\beta}). \tag{75}$$

and $f(x) = \sin\left(\frac{x}{2}\right)$. Moreover, we have $u(x,0) = u_0(x) = 0$ and

$$u(x,1) = u_1(x) = g(x) = \Big(E_{\beta,1}(1) - E_{\beta,1}(-1)\Big) \sin\left(\frac{x}{2}\right). \tag{76}$$

Next, for computing the integral in the latter equality, see Reference [19], we use the fact that

$$\int_0^x u^{\kappa-1} E_{\beta,\kappa}(yu^{\beta})(x-u)^{\beta-1} E_{\beta,\beta}(z(x-u)^{\beta}) du = \frac{yE_{\beta,\kappa+\beta}(yx^{\beta}) - zE_{\beta,\beta+\kappa}(zx^{\beta})}{y-z} x^{\beta+\kappa+1}. \tag{77}$$

From $\phi_{\varepsilon}(.) = \phi(.) + \varepsilon\, \Big(2\text{rand}(.) - 1\Big)$, we have

$$\begin{aligned} \int_0^1 s^{\beta-1} E_{\beta,\beta}(-\xi^2 s^{\beta}) \phi_{\varepsilon}(1-s) ds &= \int_0^1 s^{\beta-1} E_{\beta,\beta}(-\xi^2 s^{\beta}) \phi(1-s) ds \\ &\quad + \varepsilon\, \big(2\text{rand}(.) - 1\big) \int_0^1 s^{\beta-1} E_{\beta,\beta}(-\xi^2 s^{\beta}) ds\cdot \end{aligned} \tag{78}$$

Combining (72), (75) and (78), we have

$$\begin{aligned} \int_0^1 s^{\beta-1} E_{\beta,\beta}(-\xi^2 s^{\beta-1}) \phi_{\varepsilon}(1-s) ds &= \frac{5}{4}\Big(\frac{E_{\beta,\beta+1}(1) + \xi^2 E_{\beta,\beta+1}(-\xi^2)}{1+\xi^2}\Big) \\ &\quad - \frac{3}{4}\Big(\frac{E_{\beta,\beta+1}(-1) - \xi^2 E_{\beta,\beta+1}(-\xi^2)}{-1+\xi^2}\Big) \\ &\quad + \frac{\varepsilon\Big(2\text{rand}(.) - 1\Big)}{\xi^2} \Big(1 - E_{\beta,1}(-\xi^2)\Big). \end{aligned} \tag{79}$$

In general, the numerical methods referenced by References [20,21] are summarized in three steps as follows.

Step 1: Choose N to generate the spatial and temporal discretization in such a manner as:

$$x_i = i\Delta x,\ \Delta x = \frac{\pi}{N},\ i = \overline{0, N}. \tag{80}$$

Obviously, the higher value of N will provide numerical results that are more accurate and stable. Here we choose $N = 100$ is satisfied.

Step 2: Setting $f_{\varepsilon,\alpha}(x_i) = f^i_{\varepsilon,\alpha}$ and $f(x_i) = f^i$, constructing two vectors contained all discrete values of $f_{\varepsilon,\alpha}$ and f denoted by $\Lambda_{\varepsilon,\alpha}$ and Ψ, respectively.

$$\begin{aligned} &\Lambda_{\varepsilon,\alpha} = [f^0_{\varepsilon,\alpha}\ f^1_{\varepsilon,\alpha} \ldots f^Q_{\varepsilon,\alpha}] \in \mathbb{R}^{Q+1}, \\ &\Psi = [f^0\ f^1 \ldots f^{Q-1}\ f^Q] \in \mathbb{R}^{Q+1}. \end{aligned} \tag{81}$$

Step 3: The estimation is defined:

Relative error estimation:

$$E_1 = \frac{\sqrt{\sum_{i=1}^{N} |f_{\varepsilon,\alpha}(x_i) - f(x_i)|^2_{\mathscr{L}_2(-7,7)}}}{\sqrt{\sum_{i=1}^{N} |f(x_i)|^2_{\mathscr{L}_2(-7,7)}}}. \tag{82}$$

Absolute error estimation:

$$E_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |f_{\varepsilon,\alpha}(x_i) - f(x_i)|^2_{\mathscr{L}_2(-7,7)}}. \tag{83}$$

Figure 1 shows a 2D figures between the sought and its regularized solutions for $N = 100$ and $\beta = 0.95$. All figures are presented with $\varepsilon = 0.1$, $\varepsilon = 0.01$ and $\varepsilon = 0.001$, respectively.

In Tables 1 and 2 of this example, we show the error estimation both the priori and the posteriori within case $N = 100$, that is, in Table 1 we show the error estimation for both the priori and the posteriori at $\beta = 0.95$ with $\varepsilon \in \{0.1, 0.01, 0.001\}$, respectively. In Table 2, we show the relative error estimation and absolute error estimation both the priori and the posteriori with $\varepsilon = 0.01$ with the different values of $\beta \in \{0.2, 0.4, 0.6, 0.8\}$, when ε is fixed and the mesh resolutions are increased, the regularized solution convergence is better than that of the exact solution. From observing the results from the tables and figures above, we conclude that when ε tends to zero, the regularized solution approaches the exact solution.

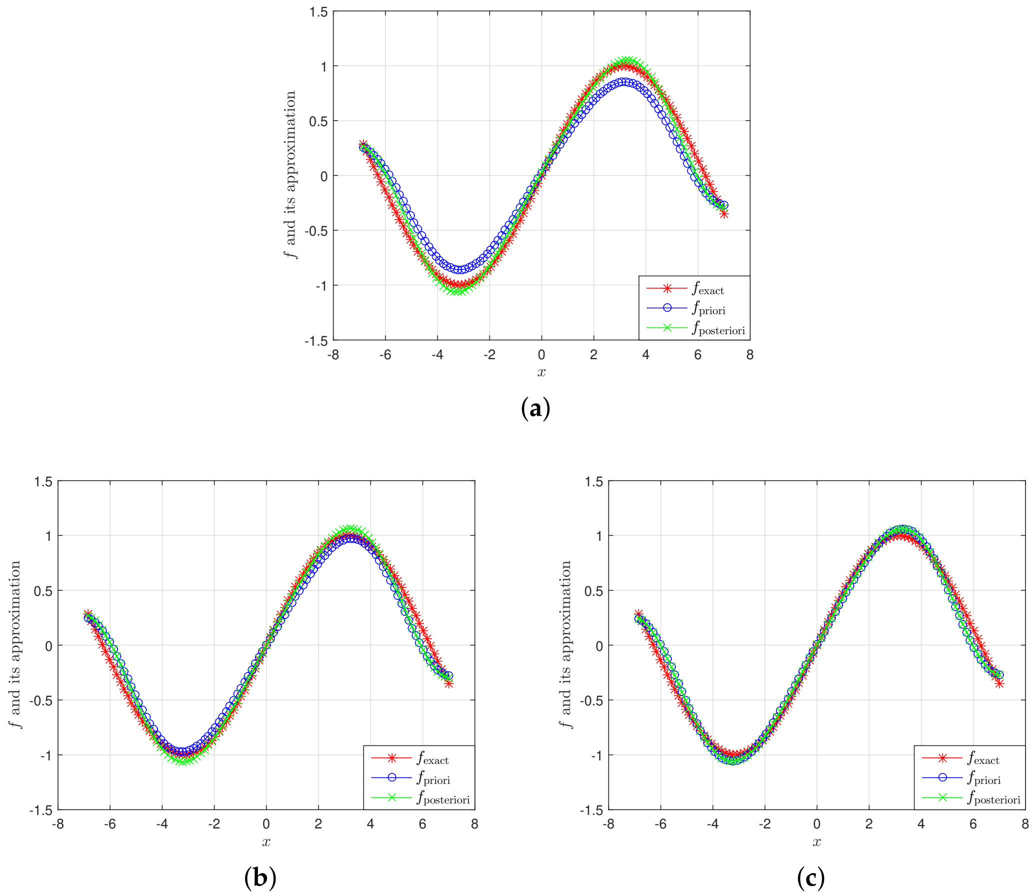

Figure 1. A comparison between the exact and regularized solutions for $k = 1$, $\beta = 0.95$ with $N = 100$. (**a**) $\varepsilon = 0.1$. (**b**) $\varepsilon = 0.01$. (**c**) $\varepsilon = 0.001$.

Table 1. The error estimation between the exact and regularized solutions of this example at $\beta = 0.95$ with $N = 100$.

ε	$E_1^{\beta_{pri}}$	$E_1^{\beta_{pos}}$	$E_2^{\beta_{pri}}$	$E_2^{\beta_{pos}}$
0.1	0.279660141830880	0.163452531664322	0.188256991900635	0.110030273632189
0.01	0.167130513450332	0.146077554813055	0.112506156619184	0.098334073898654
0.001	0.144054212078375	0.144599158066180	0.096972033479447	0.097338871212350

Table 2. The error estimation between the exact and regularized solutions with the different values of β, $\varepsilon = 0.01$ and $N = 100$.

β	$E_1^{\beta_{pri}}$	$E_1^{\beta_{pos}}$	$E_2^{\beta_{pri}}$	$E_2^{\beta_{pos}}$
0.2	0.156401672575436	0.176079016470940	0.078962919638416	0.092189970426402
0.4	0.146364358305196	0.165153671589525	0.073895354649786	0.086469770247512
0.6	0.136338164832119	0.153413164488168	0.068833404246973	0.080322774289912
0.8	0.124692172130227	0.140316883268202	0.062953661590221	0.073465933522836

6. Conclusions

In this study, by using the mollification regularization method, we solved the inverse problem and recovered the source term for time fractional diffusion equation with the time dependent coefficient. In the theoretical results, which we have shown, we obtained the error estimates of both a priori and a posteriori parameter choice rule methods based on a priori condition. In addition, in the numerical results, it shows that the regularized solutions are converged to the exact solution. Furthermore, it also shows that the smaller error of the input data, the better the convergence results.

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