


Article

Generalized Integral Inequalities for Hermite–Hadamard-Type Inequalities via s -Convexity on Fractal Sets

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Abstract: In this article, we establish new Hermite–Hadamard-type inequalities via Riemann–Liouville integrals of a function ψ taking its value in a fractal subset of \mathbb{R} and possessing an appropriate generalized s -convexity property. It is shown that these fractal inequalities give rise to a generalized s -convexity property of ψ . We also prove certain inequalities involving Riemann–Liouville integrals of a function ψ provided that the absolute value of the first or second order derivative of ψ possesses an appropriate fractal s -convexity property.

Keywords: s -convex function; Hermite–Hadamard inequalities; Riemann–Liouville fractional integrals; fractal space

1. Introduction

Convexity is considered to be an important property in mathematical analysis. The applications of convex functions can be found in many fields of studies including economics, engineering and optimization (see for example [1,2]). A well-known result which was identified as Hermite–Hadamard inequalities is the reformulation through convexity. These inequalities, widely reported in the literature, can be defined as follows:

Theorem 1. Suppose that $\psi : [u, v] \subset \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on $[u, v]$ with $u < v$, then

$$\psi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \psi(x) dx \leq \frac{\psi(u) + \psi(v)}{2}. \quad (1)$$

These two inequalities, which are refinement of convexity, can be held in reverse order as concave. Following this, many refinements of convex functions using Hermite–Hadamard inequalities have been continuously studied [3–6]. Given the variation of Hermite–Hadamard inequalities, Dragomir and Fitzpatrick [7] established a new generalization of s -convex functions in the second sense.

Theorem 2. Suppose that $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a s -convex function in the second sense, where $0 < s \leq 1$, $u, v \in \mathbb{R}_+$ and $u < v$. If $\psi \in L^1([u, v])$, then

$$2^{s-1} \psi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \psi(x) dx \leq \frac{\psi(u) + \psi(v)}{s+1}. \quad (2)$$

Though the Hermite–Hadamard inequalities were established for classical integrals [8], the inequalities can also hold for fractional calculus, such as Riemann–Liouville [9–11], Katugampola [12] and local fractional integrals [13]. Some of these were studied through Mittag–Leffler function [14,15]. Other important generalizations include the work of Sarikaya et al. [16], who proved the Hermite–Hadamard inequalities through fractional integrals as follows:

Theorem 3. Suppose that $\psi : [u, v] \rightarrow \mathbb{R}$ is a non-negative function with $0 \leq u < v$ and $\psi \in L_1[u, v]$. If ψ is convex function on $[u, v]$, we have:

$$\psi\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \leq \frac{\psi(u) + \psi(v)}{2},$$

where $0 < \alpha \leq 1$.

The s -convexity mentioned in Hudzik and Maligranda [2] was also given as the generalization on fractal sets.

Definition 1 ([17]). A function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ is called generalized s -convex in the second sense if

$$\psi(\gamma_1 u + \gamma_2 v) \leq \gamma_1^{\alpha s} \psi(u) + \gamma_2^{\alpha s} \psi(v), \tag{3}$$

holds for all $u, v \in \mathbb{R}_+, \gamma_1, \gamma_2 \geq 0$, with $\gamma_1 + \gamma_2 = 1$ and for some fixed $s \in (0, 1]$. The symbol GK_s^2 denotes the class of this functions.

The Riemann–Liouville fractional integral is introduced here due to its importance.

Definition 2 ([18]). Suppose that $\psi \in L_1[u, v]$. The Riemann–Liouville integrals $J_{u^+}^\alpha \psi$ and $J_{v^-}^\alpha \psi$ of order $\alpha \in \mathbb{R}_+$ are defined by

$$J_{u^+}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_u^x (x-\gamma)^{\alpha-1} \psi(\gamma) d\gamma, \quad x > u,$$

and

$$J_{v^-}^\alpha \psi(x) = \frac{1}{\Gamma(\alpha)} \int_x^v (\gamma-x)^{\alpha-1} \psi(\gamma) d\gamma, \quad x < v,$$

respectively.

The following lemma for differentiable function is given by Sarikaya et al. [16].

Lemma 1. Let $\psi : [u, v] \rightarrow \mathbb{R}$ be a differentiable function on (u, v) with $u < v$. If $\psi' \in L^1[u, v]$, then we have:

$$\frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] = \frac{v-u}{2} \int_0^1 [(1-\gamma)^\alpha - \gamma^\alpha] \psi'(\gamma u + (1-\gamma)v) d\gamma.$$

Wang et al. [9] extended Lemma 1 to include two cases, one of which involves the second derivative of Riemann–Liouville fractional integrals.

Lemma 2. Let $\psi : [u, v] \rightarrow \mathbb{R}$ be a twice-differentiable function on (u, v) with $u < v$. If $\psi'' \in L^1[u, v]$, then

$$\begin{aligned} & \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \\ &= \frac{(v - u)^2}{2} \int_0^1 \frac{1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1}}{\alpha + 1} \psi''(\gamma u + (1 - \gamma)v) d\gamma, \end{aligned}$$

holds.

Even though studies were conducted on generalized Hermite–Hadamard inequality via Riemann–Liouville fractional integrals for s -convexity [16,19–21], inequalities of this type for generalized s -convexity are lacking. Therefore, this paper is aimed at establishing some new integral inequalities via generalized s -convexity on fractal sets. We show that the newly established inequalities are generalizations of Theorem 2. The new Hermite–Hadamard-type inequalities in the class of functions with derivatives in absolute values are shown to be s -convex function on fractal sets. This was achieved using Riemann–Liouville fractional integrals inequalities.

2. Main Results

Our first main result is obtained in the following theorem.

Theorem 4. Suppose that $\psi : [u, v] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+^\alpha$ is a generalized s -convex on $[u, v]$, where $0 < s < 1$, $u, v \in \mathbb{R}_+$ and $u \leq v$. If $\psi \in L^1[u, v]$, then we obtain

$$2^{\alpha(s-1)} \psi\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \leq \left[\frac{1}{s+1} + \frac{\Gamma(\alpha s+1)\Gamma(\alpha+1)}{\Gamma(\alpha(s+1)+1)} \right] \frac{[\psi(u)+\psi(v)]}{2}. \tag{4}$$

Proof. Since $\psi \in GK_s^2$, we get

$$\psi\left(\frac{x+y}{2}\right) \leq \frac{\psi(x) + \psi(y)}{2^{\alpha s}}. \tag{5}$$

Substituting $x = \gamma u + (1 - \gamma)v$ and $y = (1 - \gamma)u + \gamma v$ with $\gamma \in [0, 1]$ in inequality (5), we obtain

$$2^{\alpha s} \psi\left(\frac{x+y}{2}\right) \leq \psi(\gamma u + (1 - \gamma)v) + \psi(\gamma v + (1 - \gamma)u). \tag{6}$$

Multiplying both sides of (6) by $\gamma^{\alpha-1}$ and integrating the resulting inequality with respect to γ over $[0, 1]$ yields

$$\begin{aligned} \frac{2^{\alpha s}}{\alpha} \psi\left(\frac{x+y}{2}\right) &\leq \int_0^1 \gamma^{\alpha-1} \psi(\gamma u + (1 - \gamma)v) + \int_0^1 \gamma^{\alpha-1} \psi(\gamma v + (1 - \gamma)u) \\ &= \frac{\Gamma(\alpha)}{(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)]. \end{aligned} \tag{7}$$

Then the first inequality in (4) is proved.

To prove the second inequality in (4), since $\psi \in GK_s^2$, we get

$$\psi(\gamma u + (1 - \gamma)v) \leq \gamma^{\alpha s} \psi(u) + (1 - \gamma)^{\alpha s} \psi(v), \tag{8}$$

and

$$\psi(\gamma v + (1 - \gamma)u) \leq \gamma^{\alpha s} \psi(v) + (1 - \gamma)^{\alpha s} \psi(u). \tag{9}$$

Combining the inequalities (8) and (9), we obtain

$$\begin{aligned} \psi(\gamma u + (1 - \gamma)v) + \psi((1 - \gamma)u + \gamma v) &\leq \gamma^{\alpha s} \psi(u) + (1 - \gamma)^{\alpha s} \psi(v) + \gamma^{\alpha s} \psi(u) + (1 - \gamma)^{\alpha s} \psi(u) \\ &= [\gamma^{\alpha s} + (1 - \gamma)^{\alpha s}] [\psi(u) + \psi(v)]. \end{aligned} \tag{10}$$

A similar technique used in (6) is applied to inequality (10) to get the following:

$$\begin{aligned} \frac{\Gamma(\alpha)}{(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] &\leq \int_0^1 \gamma^{\alpha-1} [\gamma^{\alpha s} + (1 - \gamma)^{\alpha s}] [\psi(u) + \psi(v)] d\gamma \\ &\leq \left[\frac{1}{\alpha(s + 1)} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha)}{\Gamma(\alpha(s + 1) + 1)} \right] [\psi(u) + \psi(v)], \end{aligned} \tag{11}$$

where

$$\int_0^1 \gamma^{\alpha s + s - 1} d\gamma = \frac{1}{\alpha s + \alpha},$$

and

$$\int_0^1 \gamma^{\alpha-1} (1 - \gamma)^{\alpha s} d\gamma = \frac{\Gamma(\alpha s + 1)\Gamma(\alpha)}{\Gamma(\alpha s + \alpha + 1)}.$$

Using inequalities (7) and (11), we prove Theorem 4. \square

Remark 1. In the second inequality of Theorem 4, the expression $\frac{1}{2} \left[\frac{1}{s+1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right]$ for $0 < s \leq 1$ is the best possible. The map $\psi : [0, 1] \rightarrow [0^\alpha, 1^\alpha]$ given by $\psi(z) = z^{s\alpha}$ is generalized s -convex in the second sense, and it satisfies the following equalities:

$$\begin{aligned} \frac{\Gamma(\alpha + 1)}{2} [J_{u^+}^\alpha \psi(1) + J_{v^-}^\alpha \psi(0)] &= \frac{\Gamma(\alpha + 1)}{2} \left[\frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha(s + 1)} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha)}{\Gamma(\alpha s + \alpha + 1)} \right) \right] \\ &= \frac{\Gamma(\alpha + 1)}{2} \left[\frac{1}{\alpha\Gamma(\alpha)(s + 1)} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\alpha s + \alpha + 1)} \right] \\ &= \frac{1}{2} \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right], \end{aligned}$$

and

$$\left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \frac{[\psi(0) + \psi(1)]}{2} = \frac{1}{2} \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right].$$

Corollary 1. By taking $\alpha = 1$ in Theorem 4, the inequalities in (2) of Theorem 2 are recovered.

This result is the same as Theorem 2.1 in Dragomir and Fitzpatrick [7].

Remark 2. The equality

$$\beta(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}$$

implies

$$\begin{aligned}
 2^{\alpha s-1} \psi\left(\frac{u+v}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \leq \left[\frac{1}{s+1} + \frac{\Gamma(\alpha s+1)\Gamma(\alpha+1)}{\Gamma(\alpha(s+1)+1)} \right] \frac{[\psi(u) + \psi(v)]}{2} \\
 &\leq \left[\frac{1}{s+1} + \beta(\alpha s+1, \alpha+1)(\alpha(s+1)+1) \right] \\
 &\quad \times \frac{[\psi(u) + \psi(v)]}{2}.
 \end{aligned}$$

Theorem 5. Suppose that $M : [0, 1] \rightarrow \mathbb{R}^\alpha$ is the mapping given by

$$M(\gamma) = \frac{\Gamma(\alpha+1)}{(v-u)^\alpha} \left[J_{u^+}^\alpha \psi\left(\gamma v + (1-\gamma)\frac{u+v}{2}\right) + J_{v^-}^\alpha \psi\left(\gamma u + (1-\gamma)\frac{u+v}{2}\right) \right], \gamma \in [0, 1],$$

where $\psi : [u, v] \rightarrow \mathbb{R}^\alpha$ belongs to GK_s^2 , $s \in (0, 1]$, $u, v \in \mathbb{R}_+$, $u < v$ and $\psi \in L^1([u, v])$. Then

- (i) $M \in GK_s^2$ on $[0, 1]$.
- (ii) We have the following inequality:

$$M(\gamma) \geq 2^{\alpha s} \psi\left(\frac{u+v}{2}\right). \tag{12}$$

- (iii) We have the following inequality:

$$M(\gamma) \leq \min\{M_1(\gamma), M_2(\gamma)\}, \gamma \in [0, 1], \tag{13}$$

where

$$M_1(\gamma) = \gamma^{\alpha s} \frac{\Gamma(\alpha+1)}{(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] + (1-\gamma)^{\alpha s} \psi\left(\frac{u+v}{2}\right),$$

and

$$M_2(\gamma) = \left[\frac{1}{s+1} + \frac{\Gamma(\alpha s+1)\Gamma(\alpha+1)}{\Gamma(\alpha(s+1)+2)} \right] \left[\psi\left(\gamma u + (1-\gamma)\frac{u+v}{2}\right) + \psi\left(\gamma v + (1-\gamma)\frac{u+v}{2}\right) \right].$$

- (iv) If $\tilde{M} = \max\{M_1(\gamma), M_2(\gamma)\}, \gamma \in [0, 1]$, then we have

$$\tilde{M} \leq \left[\frac{1}{s+1} + \frac{\Gamma(\alpha s+1)\Gamma(\alpha+1)}{\Gamma(\alpha(s+1)+1)} \right] \left[\gamma^{\alpha s} [\psi(u) + \psi(v)] + 2^{\alpha s} (1-\gamma)^{\alpha s} \psi\left(\frac{u+v}{2}\right) \right].$$

Proof.

- (i) Let $\gamma_1, \gamma_2 \in [0, 1]$ and $\mu_1, \mu_2 \geq 0$ with $\mu_1 + \mu_2 = 1$, then

$$\begin{aligned}
 M(\mu_1 \gamma_1 + \mu_2 \gamma_2) &= \frac{\Gamma(\alpha+1)}{(v-u)^\alpha} \left[J_{u^+}^\alpha \psi\left((\mu_1 \gamma_1 + \mu_2 \gamma_2)\frac{u+v}{2} + (1 - (\mu_1 \gamma_1 + \mu_2 \gamma_2))v\right) \right. \\
 &\quad \left. + J_{v^-}^\alpha \psi\left((\mu_1 \gamma_1 + \mu_2 \gamma_2)u + (1 - (\mu_1 \gamma_1 + \mu_2 \gamma_2))\frac{u+v}{2}\right) \right] \\
 &\leq \frac{\Gamma(\alpha+1)}{(v-u)^\alpha} \left[\mu_1^{\alpha s} \left(J_{u^+}^\alpha \psi\left(\gamma_1 \frac{u+v}{2} + (1-\gamma_1)v\right) + J_{v^-}^\alpha \psi\left(\gamma_1 u + (1-\gamma_1)\frac{u+v}{2}\right) \right) \right. \\
 &\quad \left. + \mu_2^{\alpha s} \left(J_{u^+}^\alpha \psi\left(\gamma_2 \frac{u+v}{2} + (1-\gamma_2)v\right) + J_{v^-}^\alpha \psi\left(\gamma_2 u + (1-\gamma_2)\frac{u+v}{2}\right) \right) \right] \\
 &= \mu_1^{\alpha s} M(\gamma_1) + \mu_2^{\alpha s} M(\gamma_2).
 \end{aligned}$$

(ii) Assume that $\gamma \in (0, 1]$. Then by the change of variables $q = \gamma v + (1 - \gamma)\frac{u+v}{2}$ and $p = \gamma u + (1 - \gamma)\frac{u+v}{2}$, we have

$$\begin{aligned} M(\gamma) &= \frac{\Gamma(\alpha + 1)}{\gamma^\alpha(v - u)^\alpha} \left[J_{(\gamma v + (1 - \gamma)\frac{u+v}{2})^+}^\alpha \psi \left(\gamma u + (1 - \gamma)\frac{u+v}{2} \right) \right. \\ &\quad \left. + J_{(\gamma u + (1 - \gamma)\frac{u+v}{2})^-}^\alpha \psi \left(\gamma v + (1 - \gamma)\frac{u+v}{2} \right) \right] \\ &= \frac{\Gamma(\alpha + 1)}{(p - q)^\alpha} [J_{q^+}^\alpha \psi(p) + J_{p^-}^\alpha \psi(q)]. \end{aligned}$$

Applying the first generalized Hermite–Hadamard inequality, we obtain

$$\frac{\Gamma(\alpha + 1)}{(p - q)^\alpha} [J_{q^+}^\alpha \psi(p) + J_{p^-}^\alpha \psi(q)] \geq 2^{\alpha s} \psi \left(\frac{q + p}{2} \right) = 2^{\alpha s} \psi \left(\frac{u + v}{2} \right),$$

and inequality (12) is obtained.

If $\gamma = 0$, the inequality

$$\psi \left(\frac{u + v}{2} \right) \geq 2^{\alpha s - 1} \psi \left(\frac{u + v}{2} \right),$$

also holds.

(iii) Applying the second generalized Hermite–Hadamard inequality, we obtain

$$\begin{aligned} \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] &\leq \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] [\psi(u) + \psi(v)] \\ &= \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \left[\psi \left(\gamma u + (1 - \gamma)\frac{u + v}{2} \right) \right. \\ &\quad \left. + \psi \left(\gamma v + (1 - \gamma)\frac{u + v}{2} \right) \right] \\ &= A_2(\gamma), \forall \gamma \in [0, 1]. \end{aligned}$$

Please note that if $\gamma = 0$, then the inequality

$$\psi \left(\frac{u + v}{2} \right) = M(0) \leq M_2(0) = 2^\alpha \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \psi \left(\frac{u + v}{2} \right),$$

holds as it is equivalent to

$$\left(\left[\frac{\Gamma(\alpha s + \alpha + 1)(s + 1)}{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)} \right] - 2^\alpha \right) \psi \left(\frac{u + v}{2} \right) \leq 0^\alpha,$$

which is known to hold for $s \in (0, 1]$.

Since for all $\gamma \in [0, 1]$ and $x \in [u, v]$ the inequalities

$$\psi \left(\gamma u + (1 - \gamma)\frac{u + v}{2} \right) \leq \gamma^{\alpha s} \psi(u) + (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right),$$

and

$$\psi \left(\gamma v + (1 - \gamma)\frac{u + v}{2} \right) \leq \gamma^{\alpha s} \psi(v) + (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right).$$

are true, we obtain

$$\begin{aligned} M(\gamma) &= \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} \left[J_{u^+}^\alpha \psi \left(\gamma v + (1 - \gamma) \frac{u + v}{2} \right) \right. \\ &\quad \left. + J_{v^-}^\alpha \psi \left(\gamma u + (1 - \gamma) \frac{u + v}{2} \right) \right] \\ &\leq \gamma^{\alpha s} \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] + (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right) \\ &= M_1(\gamma) \end{aligned}$$

and the inequality (13) is proved.

(iv) We have

$$\begin{aligned} M_2(\gamma) &= \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 2)} \right] \left[\psi \left(\gamma u + (1 - \gamma) \frac{u + v}{2} \right) \right. \\ &\quad \left. + \psi \left(\gamma v + (1 - \gamma) \frac{u + v}{2} \right) \right] \\ &\leq \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \left[\gamma^{\alpha s} \psi(u) + (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right) + \gamma^{\alpha s} \psi(v) \right. \\ &\quad \left. + (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right) \right] \\ &= \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \left[\gamma^{\alpha s} [\psi(u) + \psi(v)] + 2^{\alpha s} (1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right) \right]. \end{aligned}$$

Since

$$\frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \leq \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \frac{[\psi(u) + \psi(v)]}{2},$$

and

$$(1 - \gamma)^{\alpha s} \psi \left(\frac{u + v}{2} \right) \leq 2^\alpha (1 - \gamma)^{\alpha s} \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \psi \left(\frac{u + v}{2} \right),$$

then

$$\begin{aligned} M_1(\gamma) &\leq \gamma^{\alpha s} \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] (\psi(u) + \psi(v)) \\ &\quad + 2^\alpha (1 - \gamma)^{\alpha s} \left[\frac{1}{s + 1} + \frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 1)} \right] \psi \left(\frac{u + v}{2} \right), \end{aligned}$$

and the proof of Theorem 5 is complete.

□

Corollary 2. *Choosing $s = 1$ in Theorem 5, we have*

(i)

$$\begin{aligned} & \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} \left[J_{u^+}^\alpha \psi \left(\gamma v + (1 - \gamma) \frac{u + v}{2} \right) + J_{v^-}^\alpha \psi \left(\gamma u + (1 - \gamma) \frac{u + v}{2} \right) \right] \\ & \leq \min \left\{ \gamma^\alpha \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right. \\ & \quad \left. + (1 - \gamma)^\alpha \psi \left(\frac{u + v}{2} \right), \left[\frac{1}{2} + \frac{\Gamma(\alpha + 1)^2}{\Gamma(2(\alpha + 1))} \right] \right. \\ & \quad \left. \times \left[\psi \left(\gamma u + (1 - \gamma) \frac{u + v}{2} \right) + \psi \left(\gamma v + (1 - \gamma) \frac{u + v}{2} \right) \right] \right\}. \end{aligned}$$

(ii) Since

$$\begin{aligned} \tilde{M} = \max & \left\{ \gamma^\alpha \frac{\Gamma(\alpha + 1)}{(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] + (1 - \gamma)^\alpha \psi \left(\frac{u + v}{2} \right), \right. \\ & \left. \left[\frac{1}{2} + \frac{\Gamma(\alpha + 1)^2}{\Gamma(2(\alpha + 1))} \right] \left[\psi \left(\gamma u + (1 - \gamma) \frac{u + v}{2} \right) + \psi \left(\gamma v + (1 - \gamma) \frac{u + v}{2} \right) \right] \right\}, \end{aligned}$$

we get

$$\tilde{M} = \left[\frac{1}{2} + \frac{\Gamma(\alpha + 1)\Gamma(\alpha + 1)}{\Gamma(2(\alpha + 1))} \right] \left(\gamma^\alpha (\psi(u) + \psi(v)) + 2^\alpha (1 - \gamma)^\alpha \psi \left(\frac{u + v}{2} \right) \right).$$

Theorem 6. Let $\psi : [u, v] \subset \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ be a differentiable function on (u, v) where $0 \leq u < v$. For some fixed $q \geq 1$, if $|\psi'|^q$ is generalized s -convex on (u, v) , we obtain

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| & \leq \frac{v - u}{2} \left(\frac{2}{\alpha + 1} \right)^{1 - \frac{1}{q}} \\ & \quad \times \left(\left[\beta(\alpha + 1, \alpha s + 1) + \frac{1}{\alpha(s + 1) + 1} \right] \right. \\ & \quad \left. \times [|\psi'(u)|^q + |\psi'(v)|^q] \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Applying Lemma 1, we obtain

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| & = \frac{v - u}{2} \left| \int_0^1 [(1 - \gamma)^\alpha - \gamma^\alpha] \right. \\ & \quad \left. \times \psi'(\gamma u + (1 - \gamma)v) d\gamma \right|. \end{aligned} \tag{14}$$

First, suppose $q = 1$. Since the function $|\psi'|$ is generalized s -convex on (u, v) , we obtain

$$|\psi'(\gamma u + (1 - \gamma)v)| \leq \gamma^{\alpha s} |\psi'(u)| + (1 - \gamma)^{\alpha s} |\psi'(v)|. \tag{15}$$

Therefore,

$$\begin{aligned} \left| \int_0^1 [(1-\gamma)^\alpha - \gamma^\alpha] \psi'(\gamma u + (1-\gamma)v) d\gamma \right| &\leq |\psi'(u)| \int_0^1 [(1-\gamma)^\alpha \gamma^{\alpha s} + \gamma^{\alpha s + \alpha}] d\gamma \\ &\quad + |\psi'(v)| \int_0^1 [(1-\gamma)^{\alpha s + \alpha} + (1-\gamma^{\alpha s}) \gamma^\alpha] d\gamma \\ &= \left(\beta(\alpha + 1, \alpha s + 1) + \frac{1}{\alpha s + \alpha + 1} \right) [\psi'(u) + \psi'(v)]. \end{aligned} \tag{16}$$

Next suppose that $q > 1$. From the power mean inequality and the generalized s -convexity of the function $|\psi'|^q$ we obtain

$$\begin{aligned} \left| \int_0^1 [(1-\gamma)^\alpha - \gamma^\alpha] \psi'(\gamma u + (1-\gamma)v) d\gamma \right| &\leq \left| \int_0^1 [(1-\gamma)^\alpha - \gamma^\alpha]^{1-\frac{1}{q}} [(1-\gamma)^\alpha - \gamma^\alpha]^{\frac{1}{q}} \right. \\ &\quad \left. \times \psi'(\gamma u + (1-\gamma)v) d\gamma \right| \\ &\leq \left(\int_0^1 |(1-\gamma)^\alpha - \gamma^\alpha| d\gamma \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 |(1-\gamma)^\alpha - \gamma^\alpha| |\psi'(\gamma u + (1-\gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^1 (1-\gamma)^\alpha + \gamma^\alpha d\gamma \right)^{1-\frac{1}{q}} \\ &\quad \times \left(\int_0^1 [(1-\gamma)^\alpha + \gamma^\alpha] [\gamma^{\alpha s} |\psi'(u)|^q + (1-\gamma)^{\alpha s} |\psi'(v)|^q] d\gamma \right)^{\frac{1}{q}} \\ &= \left(\frac{2}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\left[\beta(\alpha + 1, \alpha s + 1) + \frac{1}{\alpha s + \alpha + 1} \right] \right. \\ &\quad \left. \times [|\psi'(u)|^q + |\psi'(v)|^q] \right)^{\frac{1}{q}}. \end{aligned} \tag{17}$$

In view of inequalities (14), (16) and (17) the proof of Theorem 6 is complete now. \square

Corollary 3. Under the conditions of Theorem 6, we get

(i) If $q = s = 1$, then

$$\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \leq \frac{v-u}{2} \left[\frac{1}{2\alpha+1} + \beta(\alpha + 1, \alpha + 1) \right] (|\psi'(u)| + |\psi'(v)|).$$

(ii) If $q = \alpha = s = 1$, then

$$\left| \frac{\psi(u) + \psi(v)}{2} - \frac{1}{v-u} \int_u^v \psi(x) dx \right| \leq \frac{v-u}{4} (|\psi'(u)| + |\psi'(v)|).$$

(iii) If $q > 1$ and $s = 1$

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| &\leq \frac{v-u}{2} \left(\frac{2}{\alpha+1} \right)^{1-\frac{1}{q}} \left(\left[\beta(\alpha + 1, \alpha s + 1) + \frac{1}{\alpha s + \alpha + 1} \right] \right. \\ &\quad \left. \times [|\psi'(u)|^q + |\psi'(v)|^q] \right)^{\frac{1}{q}}. \end{aligned}$$

(iv) If $q > 1$ and $\alpha = s = 1$ then

$$\left| \frac{\psi(u) + \psi(v)}{2} - \frac{1}{v-u} \int_u^v \psi(x) dx \right| \leq \frac{v-u}{2} \left(\frac{|\psi'(u)|^q + |\psi'(v)|^q}{2} \right)^{\frac{1}{q}}.$$

Theorem 7. Let $\psi : [u, v] \subset \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ be a differentiable function on (u, v) where $0 \leq u < v$. If $|\psi'|^q$ is generalized s -convex on (u, v) for $q > 1$, we get

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| &\leq \frac{v - u}{2} \left(\frac{2}{\alpha + 1} \left[1 - \frac{1}{2^\alpha} \right] \right)^{1 - \frac{1}{q}} \\ &\times \left(\left[\frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha s + \alpha + 2)} + \frac{1}{\alpha s + \alpha + 1} \right] \right)^{\frac{1}{q}} \\ &\times [|\psi'(u)|^q + |\psi'(v)|^q]^{\frac{1}{q}}. \end{aligned}$$

Proof. Since $|\psi'|$ is generalized s -convex on (u, v) , we obtain

$$|\psi'(\gamma u + (1 - \gamma)v)| \leq \gamma^{\alpha s} |\psi'(u)| + (1 - \gamma)^{\alpha s} |\psi'(v)|.$$

From this fact and applying the Hölder’s inequality, we have

$$\begin{aligned} \left| \int_0^1 [(1 - \gamma)^\alpha - \gamma^\alpha] \psi'(\gamma u + (1 - \gamma)v) d\gamma \right| &\leq \left| \int_0^1 [(1 - \gamma)^\alpha - \gamma^\alpha]^{1 - \frac{1}{q}} [(1 - \gamma)^\alpha - \gamma^\alpha]^{\frac{1}{q}} \psi'(\gamma u + (1 - \gamma)v) d\gamma \right| \\ &\leq \left(\int_0^{\frac{1}{2}} [(1 - \gamma)^\alpha - \gamma^\alpha] d\gamma + \int_{\frac{1}{2}}^1 [\gamma^\alpha - (1 - \gamma)^\alpha] d\gamma \right)^{1 - \frac{1}{q}} \\ &\times \left(\int_0^1 |(1 - \gamma)^\alpha - \gamma^\alpha| |\psi'(\gamma u + (1 - \gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^{\frac{1}{2}} [(1 - \gamma)^\alpha - \gamma^\alpha] d\gamma + \int_{\frac{1}{2}}^1 [\gamma^\alpha - (1 - \gamma)^\alpha] d\gamma \right)^{1 - \frac{1}{q}} \tag{18} \\ &\times \left(\int_0^1 [(1 - \gamma)^\alpha + \gamma^\alpha] [\gamma^{\alpha s} |\psi'(u)|^q + (1 - \gamma)^{\alpha s} |\psi'(v)|^q] d\gamma \right)^{\frac{1}{q}} \\ &= \left(\frac{2}{\alpha + 1} \left[1 - \frac{1}{2^\alpha} \right] \right)^{1 - \frac{1}{q}} \left(\left[\frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha s + \alpha + 2)} + \frac{1}{\alpha s + \alpha + 1} \right] \right)^{\frac{1}{q}} \\ &\times [|\psi'(u)|^q + |\psi'(v)|^q]^{\frac{1}{q}}. \end{aligned}$$

Thus, the inequalities (14) and (18) complete the proof of Theorem 7. \square

Theorem 8. Let $\psi : [u, v] \subset \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ be a differentiable mapping on (u, v) with $0 \leq u < v$. For some fixed $q > 1$, if $|\psi'|^q$ is generalized s -convex on $[u, v]$, then we have:

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^+}^\alpha \psi(u)] \right| &\leq \frac{v - u}{2} \left(\frac{2}{\alpha p + 1} \left[1 - \frac{1}{2^{\alpha p}} \right] \right)^{\frac{1}{p}} \left(\frac{1}{\alpha s + 1} \right)^{\frac{1}{q}} \\ &\times \left(|\psi'(u)|^q + |\psi'(v)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. By applying Hölder’s inequality and (15), we obtain

$$\begin{aligned} \left| \int_0^1 [(1 - \gamma)^\alpha - \gamma^\alpha] \psi'(\gamma u + (1 - \gamma)v) d\gamma \right| &\leq \left(\int_0^1 |(1 - \gamma)^\alpha - \gamma^\alpha|^p d\gamma \right)^{\frac{1}{p}} \\ &\times \left(\int_0^1 |\psi'(\gamma u + (1 - \gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ &\leq \left(\int_0^{\frac{1}{2}} [(1 - \gamma)^\alpha - \gamma^\alpha]^p d\gamma + \int_{\frac{1}{2}}^1 [\gamma^\alpha - (1 - \gamma)^\alpha]^p d\gamma \right)^{\frac{1}{p}} \tag{19} \\ &\times \left(\int_0^1 [\gamma^{\alpha s} |\psi'(u)|^q + (1 - \gamma)^{\alpha s} |\psi'(v)|^q] d\gamma \right)^{\frac{1}{q}} \\ &\leq \left(\frac{2}{\alpha p + 1} \left[1 - \frac{1}{2^{\alpha p}} \right] \right)^{\frac{1}{p}} \left(\frac{|\psi'(u)|^q + |\psi'(v)|^q}{\alpha s + 1} \right)^{\frac{1}{q}}. \end{aligned}$$

Finally, from (14) and (19) we get the desired result. \square

Remark 3. From Theorems 6–8, we obtain the following inequality for $q > 1$

$$\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \leq \min\{S_1, S_2, S_3\} \frac{(v - u)}{2} [|\psi'(u)|^q + |\psi'(v)|^q]^{\frac{1}{q}},$$

where

$$S_1 = \left(\frac{2}{\alpha + 1} \right)^{1 - \frac{1}{q}} \left(\left[\beta(\alpha + 1, \alpha s + 1) + \frac{1}{\alpha(s + 1) + 1} \right] \right)^{\frac{1}{q}},$$

$$S_2 = \left(\frac{2}{\alpha + 1} \left[1 - \frac{1}{2^\alpha} \right] \right)^{1 - \frac{1}{q}} \left(\left[\frac{\Gamma(\alpha s + 1)\Gamma(\alpha + 1)}{\Gamma(\alpha(s + 1) + 2)} + \frac{1}{\alpha(s + 1) + 1} \right] \right)^{\frac{1}{q}},$$

$$S_3 = \left(\frac{2}{\alpha p + 1} \left[1 - \frac{1}{2^{\alpha p}} \right] \right)^{\frac{1}{p}} \left(\frac{1}{\alpha s + 1} \right)^{\frac{1}{q}}.$$

Theorem 9. Let $\psi : [u, v] \subset \mathbb{R}_+ \rightarrow \mathbb{R}^\alpha$ be a twice-differentiable function on (u, v) with $0 \leq u < v$. If, for some fixed $q \geq 1$, the function $|\psi''|^q$ is generalized s -convex on the interval $[u, v]$, then we have

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| &\leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\frac{\alpha}{\alpha + 2} \right)^{1 - \frac{1}{q}} \\ &\times \left(\frac{1}{\alpha s + 1} - \beta(\alpha s + 1, \alpha + 2) - \frac{1}{\alpha s + \alpha + 2} \right)^{\frac{1}{q}} \\ &\times \left(|\psi''(u)|^q + |\psi''(v)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. Applying Lemma 2, we have

$$\begin{aligned} &\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ &= \frac{(v - u)^2}{2} \int_0^1 \frac{1 - (1 - \gamma)^{\alpha + 1} - \gamma^{\alpha + 1}}{\alpha + 1} |\psi''(\gamma u + (1 - \gamma)v)| d\gamma. \end{aligned} \tag{20}$$

First, suppose that $q = 1$. Since the mapping $|\psi''|$ is generalized s -convex on (u, v) , we obtain

$$|\psi''(\gamma u + (1 - \gamma)v)| \leq \gamma^{\alpha s} |\psi''(u)| + (1 - \gamma)^{\alpha s} |\psi''(v)|. \tag{21}$$

Therefore,

$$\begin{aligned} &\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ &\leq \frac{(v - u)^2}{2} \int_0^1 \frac{1 - (1 - \gamma)^{\alpha + 1} - \gamma^{\alpha + 1}}{\alpha + 1} \left(\gamma^{\alpha s} |\psi''(u)| + (1 - \gamma)^{\alpha s} |\psi''(v)| \right) d\gamma \\ &= \frac{(v - u)^2}{2(\alpha + 1)} \left(\frac{1}{\alpha s + 1} - \beta(\alpha s + 1, \alpha + 2) - \frac{1}{\alpha s + \alpha + 2} \right) (|\psi''(u)| + |\psi''(v)|), \end{aligned}$$

where

$$\beta(\alpha s + 1, \alpha + 2) = \beta(\alpha + 2, \alpha s + 1).$$

Secondly, for $q > 1$. From Lemma 2 and the power mean inequality, we have

$$\begin{aligned} & \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 (1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1}) d\gamma \right)^{1 - \frac{1}{q}} \\ & \quad \times \left(\int_0^1 (1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1} |\psi''(\gamma u + (1 - \gamma)v)|^q) d\gamma \right)^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Hence, from inequalities (21) and (22), we obtain

$$\begin{aligned} & \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1} \right)^{1 - \frac{1}{q}} \left(\int_0^1 1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1} |\psi''(\gamma u + (1 - \gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1} \right)^{1 - \frac{1}{q}} \\ & \quad \times \left(\int_0^1 1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1} [\gamma^{\alpha s} |\psi''(u)|^q + (1 - \gamma)^{\alpha s} |\psi''(v)|^q] d\gamma \right)^{\frac{1}{q}} \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\frac{\alpha}{\alpha + 2} \right)^{1 - \frac{1}{q}} \left(\frac{1}{\alpha s + 1} - \beta(\alpha s + 1, \alpha + 2) - \frac{1}{\alpha s + \alpha + 2} \right)^{\frac{1}{q}} \left(|\psi''(u)|^q + |\psi''(v)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of Theorem 9. \square

Theorem 10. Let $0 \leq u < v < \infty$ and let the function $\psi : [u, v] \rightarrow \mathbb{R}^\alpha$ be twice-differentiable on the open interval (u, v) , and fix $s \in (0, 1]$ and fix $q > 1$. If, in addition, the function $|\psi''|^q$ is generalized s -convex on $[u, v]$, then

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(1 - \frac{2}{p(\alpha + 1)} \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|\psi''(u)|^q + |\psi''(v)|^q}{\alpha s + 1} \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From (20), (21) and the Hölder’s inequality, we have

$$\begin{aligned} & \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 (1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1})^p d\gamma \right)^{\frac{1}{p}} \left(\int_0^1 |\psi''(\gamma u + (1 - \gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 (1 - (1 - \gamma)^{p(\alpha+1)} - \gamma^{p(\alpha+1)}) d\gamma \right)^{\frac{1}{p}} \left(|\psi''(u)|^q \int_0^1 \gamma^{\alpha s} d\gamma + |\psi''(v)|^q \int_0^1 (1 - \gamma)^{\alpha s} d\gamma \right)^{\frac{1}{q}} \\ & \leq \frac{(v - u)^2}{2(\alpha + 1)} \left(1 - \frac{2}{p(\alpha + 1)} \right)^{\frac{1}{p}} \left(\frac{|\psi''(u)|^q + |\psi''(v)|^q}{\alpha s + 1} \right)^{\frac{1}{q}}. \end{aligned}$$

We use

$$\left(1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1}\right)^q \leq 1 - (1 - \gamma)^{q(\alpha+1)} - t^{q(\alpha+1)},$$

for any $\gamma \in [0, 1]$, which follows from

$$(V - N)^q \leq V^q - N^q,$$

where

$$V > N \geq 0 \text{ and } q \geq 1.$$

The proof of Theorem 10 is complete now. \square

The following result exhibits another Hermite–Hadamard type inequality in terms of the second derivative of a function.

Theorem 11. *Under the same assumptions of Theorem 10, we have*

$$\begin{aligned} \left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| &\leq \frac{(v - u)^2}{2(\alpha + 1)} \\ &\times \left(\frac{1}{\alpha s + 1} - \beta(\alpha s + 1, q(\alpha + s) + 1) \right. \\ &\left. - \frac{1}{(\alpha + 1)q + \alpha s + 1} \right)^{\frac{1}{q}} \\ &\times \left(|\psi''(u)|^q + |\psi''(v)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. By applying Lemma 2 and the Hölder’s inequality, we obtain

$$\begin{aligned} &\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - b)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \\ &\leq \frac{(v - u)^2}{2(\alpha + 1)} \left(\int_0^1 1 d\gamma \right)^{\frac{1}{p}} \left(\int_0^1 (1 - (1 - \gamma)^{\alpha+1} - \gamma^{\alpha+1})^q |\psi''(\gamma u + (1 - \gamma)v)|^q d\gamma \right)^{\frac{1}{q}} \\ &\leq \frac{(v - u)^2}{2(\alpha + 1)} \left(|\psi''(u)|^q \int_0^1 (\gamma^{\alpha s} - (1 - \gamma)^{q(\alpha+s)} \gamma^{\alpha s} - \gamma^{q(\alpha+1)+\alpha s}) d\gamma \right. \\ &\quad \left. + |\psi''(v)|^q \int_0^1 ((1 - \gamma)^{\alpha s} - (1 - \gamma)^{q(\alpha+1)+\alpha s} - \gamma^{q(\alpha+1)} (1 - \gamma)^{\alpha s}) d\gamma \right)^{\frac{1}{q}} \\ &\leq \frac{(v - u)^2}{2(\alpha + 1)} \times \left(\frac{1}{\alpha s + 1} - \beta(\alpha s + 1, 1 + q(\alpha + s)) - \frac{1}{(\alpha + 1)q + \alpha s + 1} \right)^{\frac{1}{q}} \\ &\quad \times \left(|\psi''(u)|^q + |\psi''(v)|^q \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof of Theorem 11. \square

Remark 4. *From Theorems 9, 10 and 11, we have*

$$\left| \frac{\psi(u) + \psi(v)}{2} - \frac{\Gamma(\alpha + 1)}{2(v - u)^\alpha} [J_{u^+}^\alpha \psi(v) + J_{v^-}^\alpha \psi(u)] \right| \leq \min\{K_1, K_2, K_3\},$$

where

$$K_1 = \frac{(v-u)^2}{2(\alpha+1)} \left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}} \left(\frac{1}{\alpha s+1} - \beta(\alpha s+1, \alpha+2) - \frac{1}{\alpha s+\alpha+2}\right)^{\frac{1}{q}} \left(|\psi''(u)|^q + |\psi''(v)|^q\right)^{\frac{1}{q}},$$

$$K_2 = \frac{(v-u)^2}{2(\alpha+1)} \left(1 - \frac{2}{p(\alpha+1)+}\right)^{\frac{1}{p}} \left(\frac{|\psi''(u)|^q + |\psi''(v)|^q}{\alpha s+1}\right)^{\frac{1}{q}},$$

$$K_3 = \frac{(v-u)^2}{2(\alpha+1)} \left(\frac{1}{\alpha s+1} - \beta(\alpha s+1, q(\alpha+s)+1) - \frac{1}{q(\alpha+1)+\alpha s+1}\right)^{\frac{1}{q}} \left(|\psi''(u)|^q + |\psi''(v)|^q\right)^{\frac{1}{q}}.$$

3. Applications to Special Means

Using the obtained results, we examine some applications to special means of non-negative numbers u and v .

1. The arithmetic mean:
 $A = A(u, v) = \frac{u+v}{2}; u, v \in \mathbb{R}, \text{ with } u, v > 0.$
2. The logarithmic mean:
 $L(u, v) = \frac{v-u}{\log v - \log u}; u, v \in \mathbb{R}, \text{ with } u, v > 0.$
3. The generalized logarithmic mean:
 $L_r(u, v) = \left[\frac{v^{r+1} - u^{r+1}}{(v-u)(r+1)}\right]^{\frac{1}{r}}; r \in \mathbb{Z} \setminus \{-1, 0\} u, v \in \mathbb{R}, \text{ with } u, v > 0.$

Using the results obtained in Section 2, and the above applications of means, we get the following proposition.

Proposition 1. Suppose that $r \in \mathbb{Z}, |r| \geq 2$ and $u, v \in \mathbb{R}$ such that $0 < u < v$. Then we get the following inequality:

$$\left|A(u^r, v^r) - L_r^r(u, v)\right| \leq \frac{(v-u)|r|}{2} A(|u|^{r-1}, |v|^{r-1}).$$

Proof. This result follows Corollary 3 (ii) applied to the function $\psi(x) = x^r$. □

Proposition 2. Suppose that $n \in \mathbb{Z}, |r| \geq 2$ and $u, v \in \mathbb{R}$ such that $0 < u < v$. Then for $q \geq 1$, we get the following:

$$\left|A(u^r, v^r) - L_r^r(u, v)\right| \leq \frac{(v-u)|r|}{2} A^{\frac{1}{q}}(|u|^{q(r-1)}, |v|^{q(r-1)}).$$

Proof. This result follows from Corollary 3 (iv) applied to the function $\psi(x) = x^r$. □

Proposition 3. Suppose that $u, v \in \mathbb{R}$ such that $0 < u < v$, then

$$\left|A(u^{-1}, v^{-1}) - L(u, v)\right| \leq \frac{(v-u)}{2} A(|u|^{-2}, |v|^{-2}).$$

Proof. This result follows from Corollary 3 (ii) applied to the function $\psi(x) = x^{-1}$. □

Proposition 4. Suppose that $u, v \in \mathbb{R}$ such that $0 < u < v$, then

$$\left| A(u^{-1}, v^{-1}) - L(u, v) \right| \leq \frac{(v-u)}{2} A^{\frac{1}{q}}(|u|^{-2q}, |v|^{-2q}).$$

Proof. This result follows from Corollary 3 (iv) applied to the function $\psi(x) = x^{-1}$. \square

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References

1. Yang, X.M. On E-convex sets, E-convex functions, and E-convex programming. *J. Optim. Theory Appl.* **2001**, *109*, 699–704. [[CrossRef](#)]
2. Hudzik, H.; Maligranda, L. Some remarks on s-convex functions. *Aequ. Math.* **1994**, *48*, 100–111. [[CrossRef](#)]
3. Almutairi, O.; Kılıçman, A. New refinements of the Hadamard inequality on coordinated convex function. *J. Inequal. Appl.* **2019**, *1*, 192. [[CrossRef](#)]
4. Dragomir, S.S.; Pearce, C. Selected topics on Hermite-Hadamard inequalities and applications. *Math. Prepr. Arch.* **2003**, *3*, 463–817.
5. Agarwal, P.; Jleli, M.; Tomar, M. Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals. *J. Inequal. Appl.* **2017**, *1*, 55. [[CrossRef](#)] [[PubMed](#)]
6. Jhathanam, S.; Tariboon, J.; Ntouyas, S.K.; Nonlaopon, K. On q-Hermite-Hadamard Inequalities for Differentiable Convex Functions. *Mathematics* **2019**, *7*, 632. [[CrossRef](#)]
7. Dragomir, S.S.; Fitzpatrick, S. The Hadamard inequalities for s-convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696. [[CrossRef](#)]
8. Almutairi, O.; Kılıçman, A. New fractional inequalities of midpoint type via s-convexity and their application. *J. Inequal. Appl.* **2019**, *1*, 1–19. [[CrossRef](#)]
9. Chen, F. On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity. *Chin. J. Math.* **2014**, *92*, 2241–2253. [[CrossRef](#)]
10. Sarikaya, M.Z. On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals. *Integr. Transform. Spec. Funct.* **2014**, *25*, 134–147. [[CrossRef](#)]
11. Qaisar, S.; Nasir, J.; Butt, S.I.; Hussain, S. On Some Fractional Integral Inequalities of Hermite-Hadamard's Type through Convexity. *Symmetry* **2019**, *11*, 137. [[CrossRef](#)]
12. Set, E.; Karaoglan, A. Hermite-Hadamard and Hermite-Hadamard-Fejer Type Inequalities for (k, h) -Convex Function via Katugampola Fractional Integrals. *Konuralp J. Math.* **2017**, *5*, 181–191.
13. Sun, W. On generalization of some inequalities for generalized harmonically convex functions via local fractional integrals. *Quaest. Math.* **2018**, 1–25. [[CrossRef](#)]
14. Abbas, G.; Farid, G. Some integral inequalities for m -convex functions via generalized fractional integral operator containing generalized Mittag-Leffler function. *Cogent Math. Stat.* **2016**, *3*, 1269589. [[CrossRef](#)]
15. Kılınc, S.; Akkurt, A.; Yildirim, H. Generalization Mittag-Leffler Function Associated with of the Hadamard and Fejer Hadamard Inequalities for $(h-m)$ -Strongly convex Functions via Fractional Integrals. *J. Univ. Math.* **2019**, *2*, 8–15. [[CrossRef](#)]
16. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [[CrossRef](#)]
17. Mo, H.; Sui, X. Generalized-convex functions on fractal sets. *Abstr. Appl. Anal.* **2014**, *2014*, 254737. [[CrossRef](#)]
18. Kilbas, A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2014; Volume 204.
19. Wang, J.; Li, X.; Zhou, Y. Hermite-Hadamard Inequalities Involving Riemann-Liouville Fractional Integrals via s-convex Functions and Applications to Special Means. *Filomat* **2016**, *30*, 1143–1150. [[CrossRef](#)]

20. Özdemir, M.E.; Merve, A.A.; Kavurmaci-Önalın, H. Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals. *Turk. J. Sci.* **2016**, *1*, 28–40.
21. İşcan, I. Generalization of different type integral inequalities for s-convex functions via fractional integrals. *Appl. Anal.* **2014**, *93*, 1846–1862. [[CrossRef](#)]



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