

Article

Optimal Uncertain Controls for Cash Holding Problems

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Abstract: Determining whether an enterprise has target holdings and figuring out how to reasonably determine these cash holdings are common problems faced by all enterprises. This paper first establishes an uncertain optimal cash holdings model with a security area constraint, and then proves that the model is a typical bang–bang control model. The control variables in the model can be expressed as a symbolic function. Then, under the specific objective function, the optimal cash holdings are discussed for two cases including whether to consider transaction costs or not. Finally, the applicability of the model is verified by specific examples, and the influences of factors, including risky asset returns and the transaction cost of unit securities on decision-making results are discussed.

Keywords: uncertainty theory; optimal control; bang–bang control; cash holding; safe area; transaction cost

1. Introduction

Cash is not only the most liquid asset but also the most profitable. A lack of cash affects the production and operation of an enterprise, and excess cash reduces the profit level of the enterprise. How much cash should a company keep? This is what the best cash holding determines. This problem has long been a topic of research for current assets management in enterprises. The earliest solution was cost optimization analysis. Its main idea is that when the sum of the costs of holding cash is the smallest, the cash holding amount of the enterprise is the best. At present, the control models that use this method include the cash inventory model, the Baumol–Tobin model, and the Miller–Orr model.

The cash inventory model proposed by Baumol [1] was the first theoretical model based on the idea of cost optimization. The contribution of this model is that it uses the basic principle of the best inventory level for reference, regards cash as a kind of special inventory, and its cost includes the expenses and opportunity cost of each capital raise. At this time, the total cost of cash management is the sum of the opportunity cost and the conversion cost, and the lowest total cost is the best cash holding level. The Baumol–Tobin (B–T) model was proposed by Tobin [2]. Its value lies in the comprehensive consideration of the cash inventory model and interest rate factors. The model has been continuously improved. Beckham and Foreman [3] conducted empirical research on the B–T model and suggested that the model also needs to consider the variables of decision-making cost, otherwise it cannot be widely used. On the basis of their work, scholars have done further research. For example, Chang [4] included the properties of money demand in an inventory model. The research also showed that an increase in transaction costs increases the elasticity of cash flow, the elasticity of absolute value interest rate, and the elasticity of uncertainty, but reduces the elasticity of transaction cost. Melo and

Bilich [5] proposed the expected equilibrium model, which minimizes the total cost of maintaining and converting other forms of assets (the sum of holding costs and shortage costs). A stochastic model based on the cash inventory model was proposed by Miller and Orr [6]. The stochastic model is a method to control cash holdings when cash demand is unpredictable. For enterprises, cash demand often fluctuates greatly and is unpredictable. However, according to historical experience and practical needs, enterprises can calculate a control range for cash holdings, that is, establish the upper and lower limits of cash holdings. In a recent extension of this model, Bar-Ilan et al. [7] presented a general cash management model that was considered to be an impulse control problem in the process of stochastic money flow. Moraes and Nagano [8] proposed the application of computational evolutionary models to minimize the total cost of cash balance maintenance, and obtain the parameters for a cash management policy. Song et al. [9] presented a model using a dynamic programming method that does not need to satisfy the condition that cash flow has a normal distribution. Wang et al. [10] also presented a recent discussion on cash holding using the dynamic programming method.

Scholarly research on the cash holding problem has mainly been carried out in a deterministic environment. All the decision-making models have been established under the assumption that the return of investors is a random variable. The stochastic analysis of cash holding is mainly based on an objective probability. However, when the amount of cash management data available to the enterprise is small, it must rely on expert experience to solve its problem, which means the problem is one of subjective probability. That is, when the return of investment is an uncertain variable, how does one make a decision on the best cash holdings?

Liu [11] established the theory of uncertainty to characterize and process uncertain variables. Liu [12] later perfected uncertainty theory and established a mathematical system based on the normative axiom, product measure axiom, sub-additive axiom, and self-duality axiom. On the basis of uncertain differential equations, for uncertain continuous systems, Zhu [13] proposed an optimal control problem and established the expected value model of the problem. The optimality equation of the model was given using the Bellman [14] dynamic programming method, and its action was equivalent to the HJB equation in the stochastic control problem. The results were successfully applied to the portfolio selection problem. Yao and Qin [15] gave an analytical solution to the linear quadratic uncertain optimal control problem. Xu and Zhu [16] studied the continuous and discrete time models of uncertain bang–bang control. These kinds of bang–bang control problems occupy an important position in optimal control theory. For example, Wang and Wang [17], Zheng and Ma [18], and Kunisch and Wang [19] all derived the bang–bang property for a certain time optimal control problem by using the pontryagin maximum principle. The latter used the bang–bang property to establish a certain relationship between the time and norm optimal control problem of heat conduction equation and obtained the sufficient and necessary conditions for the optimal time and optimal control. Phung, Wang and Zhang [20] discussed the bang–bang property of time optimal control for a semi linear heat equation in a bounded domain. Yan and Zhu [21] studied the uncertain switched systems bang–bang control model. First, they gave the optimal equation of the model, and then found the optimal solution to the problem using a two-stage algorithm. Kunisch and Wang [22] established the bang–bang property of time optimal control for a class of semi-linear parabolic equations. Pogodaev [23] developed suboptimal solutions using bang–bang controls to form a coupled control system.

The main objective of this paper is to establish an optimal cash holding model in an uncertain environment. Unlike general uncertain optimization models, the model must meet the constraint of cash holding in a safe area (Wang et al. [10]). In this paper, we take the terminal wealth maximization as the ultimate goal of cash management under the constraint that cash holding is always in a safe area. The results show that our model is a bang–bang control model, and managers only need to compare the future value of unit assets to make decisions about cash holding. The method in this paper can also be used in some prediction models in the future [24,25]. Our study is an extension of the stochastic cash holding model and also makes use of uncertainty theory in finance.

After analyzing and summarizing the existing research results, the article is arranged as follows. In Section 2, we introduce the cash holding problem with a safe area constraint. In Section 3, we establish the models without considering transaction costs first, and then establish the models considering transaction costs. In Section 4, we give an illustrative example. The last section presents our conclusions and the limitations of the paper.

2. The Uncertain Cash Holding Problem with a Safe Area Constraint

2.1. Modeling

In the stochastic model, enterprises can budget for a controlled range of cash holdings and set the upper and lower limits of cash holdings according to their historical operating experience. Suppose that the highest value is H and the lowest value is L . Wang et al. [10] called $[L, H]$ a safe area. The lower limit, L , is affected by the minimum daily cash requirement of the enterprise and the risk tolerance tendency of the managers. Within the safe range of cash holdings, the problem with cash holdings is that when the cash amount reaches the control limit H , purchasing securities with cash reduces the cash holdings. Conversely, when the cash holdings fall to the control limit L , securities are sold for cash, which increases the cash holdings. If the amount of cash is within the upper and lower limits of control, there is no need to convert cash into securities to maintain the respective existing stocks.

Now, we address how to determine the conversion ratio when cash assets and risky assets need to be converted to each other. On the basis of the above multidimensional uncertain optimal control model, the following optimal cash decision model can be established:

$$\begin{cases} J(0, x_0) = \sup_{\alpha(t)} E[\int_0^T F(\alpha(t), X(t), t) dt + h(X_T, T)] \\ \text{s.t.} \begin{cases} dX(t) = \mu(\alpha(t), X(t), t) dt + \sigma(\alpha(t), X(t), t) dC_t \\ X(0) = x_0 \\ \alpha(t) \in \Lambda \end{cases} \end{cases} \quad (1)$$

where $X(t) = (X_1(t), X_2(t), \dots, X_n(t))$ is an n -dimensional state vector with the initial state $X(0) = x_0$ at time t , and $X_i(t)$ is the investment of asset $i, i = 1, 2, 3, \dots, n$. $\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_r(t))^T$ is an n -dimensional control variable of $X(t)$ at time t , and $\alpha_i(t)$ represents the adjustment of assets $i, i = 1, 2, 3, \dots, n$. C_t is a canonical process. $F(\alpha(t), X(t), t)$ is the objective function of $R^n \times R^r \times [0, T] \rightarrow R$, and $h(X_T, T)$ is the terminal reward function of $R^n \times [0, T] \rightarrow R$. $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_n)^T$, $\Lambda_j = [\underline{\alpha}_j, \bar{\alpha}_j], j = 1, 2, \dots, n$.

For a given $\alpha(t)$, $dX(t)$ is defined by the equation

$$dX(t) = \mu(\alpha(t), X(t), t) dt + \sigma(\alpha(t), X(t), t) dC_t,$$

where $\mu(\alpha(t), X(t))$ is a vector function of $R^n \times R^n \times [0, T] \rightarrow R^n$ and $\sigma(\alpha(t), X(t), t)$ is a matrix function of $R^n \times R^n \times [0, T] \rightarrow R^n \times R^n$.

It is assumed that enterprises will only invest surplus assets in riskless assets and risky assets. In (1), we let $X_1(t)$ be the investment of riskless assets and $X_i(t), i = 2, 3, \dots, n$ be the investment of risk asset i . If the cash holdings exceed the upper limit, H , of the safety area, then enterprises will convert some riskless assets into risky assets, so at this moment we have $\alpha_1(t) > 0, \alpha_i(t) = 0, i = 2, 3, \dots, n$. If the cash holding is below L , enterprises will convert some risky assets into riskless assets, and then we have $\alpha_1(t) = 0, \alpha_i(t) > 0, i = 2, 3, \dots, n$. Therefore, (1) can be further rewritten as

$$\left\{ \begin{array}{l} J(0, x_0) = \sup_{\alpha(t)} E[\int_0^T F(\alpha(t), X(t), t) dt + h(X_T, T)] \\ s.t. \left\{ \begin{array}{l} dX(t) = \mu(\alpha(t), X(t), t) dt + \sigma(\alpha(t), X(t), t) dC_t \\ X(0) = x_0 \\ \alpha_1(t)\alpha_i(t) = 0, \alpha_1(t) + \alpha_i(t) \neq 0 \quad i = 2, \dots, n \\ \alpha(t) \in \Lambda \end{array} \right. \end{array} \right. \quad (2)$$

This model is an uncertain optimal control problem with control constraints. The difficulty encountered when solving the model depends on the form and nature of the value function. When (2) is applied to decide the optimal cash holding, it can be further extended to the following form:

$$\left\{ \begin{array}{l} J(0, x_0) = \max_{\alpha(t)} E[\int_0^T f(t)^T X(t) dt + H(T)^T X(T)] \\ s.t. \left\{ \begin{array}{l} dX(t) = [P(t)X(t) + Q(t)\alpha(t)] dt + R(t)X(t)\sigma dC_t \\ X(0) = x_0 \\ \alpha_1(t)\alpha_i(t) = 0, \alpha_1(t) + \alpha_i(t) \neq 0 \quad i = 2, \dots, n \\ \alpha(t) \in \Lambda \end{array} \right. \end{array} \right. , \quad (3)$$

where $f: [0, T] \rightarrow R^n$ is the twice differentiable objective function. $H_T \in R^n$ is terminal revenue function. $P: [0, T] \rightarrow R^{n \times n}$ and $Q: [0, T] \rightarrow R^{n \times n}$ are both quadratic differentiable continuous functions. For the convenience of the following derivation, we define $Q(t) = (q_{ij}(t))_{n \times n}$.

Then, the following two definitions are given:

Definition 1. If the control variable of the optimal control problem is valued on the boundary, or if the solution of the optimal control problem can be expressed by a symbolic function, then the optimal control is called a bang–bang control.

Definition 2. A control $\alpha(t), t \in [0, T]$ is said to be an admissible control if it satisfies the constraints $\alpha_1(t)\alpha_i(t) = 0, \alpha_1(t) + \alpha_i(t) \neq 0$ and $\alpha(t) \in \Lambda$. An admissible control $\alpha^*(t), t \in [0, T]$ is called an optimal control if $J(0, x_0)_{|\alpha^*(t)} = \max J(0, x_0)_{|\alpha(t)}$.

2.2. The Solution of the Model

$\alpha_1(t)\alpha_i(t) = 0, \alpha_1(t) + \alpha_i(t) \neq 0$ means that $\alpha_1(t)$ and $\alpha_i(t)$ are not equal to 0 at the same time. Thus, the solution to (3) is either $\alpha^*(t) = (\alpha_1^*(t), 0, \dots, 0)^T$ or $\alpha^*(t) = (0, \alpha_2^*(t), \dots, \alpha_n^*(t))^T$.

Theorem 1. The optimal control variable $\alpha^*(t) = (\alpha_1^*(t), \alpha_2^*(t), \dots, \alpha_n^*(t))^T$ of (3) is a bang–bang control with

$$\alpha_j^*(t) = \frac{\text{sgn}\{(q_{1j}(t), q_{1j}(t), \dots, q_{nj}(t))h(t)\}(\bar{\alpha}_j - \underline{\alpha}_j) + (\bar{\alpha}_j + \underline{\alpha}_j)}{2}, \quad (4)$$

where $j = 1, 2, 3, \dots, n$, and $h(t) \in R^n$ satisfies

$$\frac{dh(t)}{dt} = -f(t) - P(t)^T h(t), h(T) = H(T). \quad (5)$$

The optimal value of (3) is

$$J(0, x(0)) = h(t)^T x(0) + \int_0^T h(t)^T Q(t) \alpha^*(t) dt. \quad (6)$$

Proof. This can be obtained from Zhu's equation of optimality that

$$-J_t(t, x) = \max_{\alpha(t) \in \Lambda} \left\{ f(t)^T x + [P(t)x + Q(t)\alpha(t)] \nabla_x J(t, x)^T \right\}. \quad (7)$$

If there is $\alpha^*(t)$ to maximize the right side of Equation (7), then it is equivalent to solve

$$\max_{\alpha(t) \in \Lambda} \left\{ Q(t)\alpha(t) \nabla_x J(t, x)^T \right\} = Q(t)\alpha^*(t) \nabla_x J(t, x)^T. \quad (8)$$

We define $\alpha^*(t) = (\alpha_1^*(t), \alpha_2^*(t), \dots, \alpha_n^*(t))^T$ and $\nabla_x J(t, x)^T Q(t) = (g_1(t, x), g_2(t, x), \dots, g_n(t, x))$.

In particular, according to the analysis at the beginning of Section 2.2, the solution to model (3) is either $\alpha^*(t) = (\alpha_1^*(t), 0, \dots, 0)^T$ or $\alpha^*(t) = (0, \alpha_2^*(t), \dots, \alpha_n^*(t))^T$. We can have $\alpha_j^*(t) = 0$ when $g_j(t, x) = 0$. Then, we can obtain the following conclusion by comparing the two ends of (8):

$$\alpha_j^*(t) = \begin{cases} \bar{\alpha}_j & \text{if } g_j(t, x) > 0 \\ \alpha_j & \text{if } g_j(t, x) < 0 \\ 0 & \text{if } g_j(t, x) = 0 \end{cases}. \quad (9)$$

From Equation (9), we know that the optimal control $\alpha^*(t)$ is a bang–bang control according to Definition 1.

According to $J(T, X(T)) = H(T)^T X(T)$, we can guess $J(t, x) = h(t)^T x + k(t)$ and $h(T) = H(T)$, $K(T) = 0$. Then,

$$\nabla_x J(t, x) = h(t), J_t(t, x) = \frac{dh(t)^T}{dt} x + \frac{dk(t)}{dt}. \quad (10)$$

Taking (10) into (7) yields

$$-\frac{dh(t)^T}{dt} x - \frac{dk(t)}{dt} = f(t)^T x + [P(t)x + Q(t)\alpha^*(t)] h(t)^T.$$

Therefore,

$$-\frac{dh(t)^T}{dt} = f(t)^T + h(t)^T P(t), -\frac{dk(t)}{dt} = h(t)^T Q(t) \alpha^*(t). \quad (11)$$

Using (9), we can get

$$\alpha_j^*(t) = \frac{\text{sgn}\{g_j(t, x)\}(\bar{\alpha}_j - \underline{\alpha}_j) + (\bar{\alpha}_j + \underline{\alpha}_j)}{2}.$$

and using (11), we have

$$k(t) = \int_0^T h(t)^T Q(t) \alpha^*(t) dt.$$

Hence,

$$J(0, x(0)) = h(0)^T x(0) + \int_0^T h(t)^T Q(t) \alpha^*(t) dt.$$

So far, the theorem has been proved. $\square$

3. The Optimal Cash Holding Models with Specific Objective Function

We suppose that at time t , the cash assets and risky assets are $X_1(t)$ and $X_2(t)$. Managers are interested in maximizing the terminal value $X_1(T) + X_2(T)$ over an infinite time horizon $[0, T]$. We will discuss the optimal cash holding problem for the following two cases.

3.1. Optimal Models without Transaction Costs

Case 1: For $X_1(t) > H$, we suppose that the transaction amount is $e(t) = (e_1(t) \quad e_2(t))^T$. $e_1(t)$ represents the portion of cash assets converted to risky assets at time t , and $e_2(t)$ represents the portion of risky assets converted to cash assets at time t . The two cases do not happen at the same time; that is, when $X_1(t) > H$, we need to convert part of $X_1(t)$ into $X_2(t)$ and $e_2(t) = 0$, $e(t) = (e_1(t) \quad 0)^T$. Now, we only need to know the value of $e_1(t)$. Based on the above assumptions, we get the following two equations:

$$\begin{aligned} dX_1(t) &= (r_1 X_1(t) - e_1(t))dt, \\ dX_2(t) &= (\mu_1 X_2(t) + e_1(t))dt + \sigma_1 X_2(t) dC_t, \end{aligned}$$

where C_t is a canonical process. Here, we need to be aware that the remaining cash holding after conversion still needs to be in the safe area Λ , that is, $L \leq X_1(t) - e_1(t) \leq H$, $X_1(t) - H \leq e_1(t) \leq X_1(t) - L$. We define the set of constraints at this time as $E = [\underline{e_1(t)}, \bar{e_1(t)}] = [X_1(t) - H, X_1(t) - L]$.

Now we can construct the following model:

$$\begin{cases} \max_{e_1(t)} [X_1(T) + X_2(T)] \\ \text{s.t.} \begin{cases} dX_1(t) = (r_1(t)X_1(t) - e_1(t))dt \\ dX_2(t) = (\mu_1(t)X_2(t) + e_1(t))dt + \sigma_1(t)X_2(t)dC_t \\ e_1(t) \in E \end{cases} \end{cases} \quad (12)$$

By comparing (12) with (3), we have $P(t) = \begin{pmatrix} r_1(t) & 0 \\ 0 & \mu_1(t) \end{pmatrix}$, $Q(t) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$, $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $H(T) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f(t) = 0$. Using (11) we have

$$\frac{dh(t)^T}{dt} = - \begin{pmatrix} r_1(t) & 0 \\ 0 & \mu_1(t) \end{pmatrix} h(t), \quad h(T) = H(T) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (13)$$

and by simple calculation, we can obtain the solution to (13), which is $h(t) = \begin{pmatrix} e^{\int_t^T r_1(s)ds} & e^{\int_t^T \mu_1(s)ds} \end{pmatrix}^T$.

Then,

$$\begin{aligned} h(t)^T Q(t) &= \begin{pmatrix} e^{\int_t^T r_1(s)ds} & e^{\int_t^T \mu_1(s)ds} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -e^{\int_t^T r_1(s)ds} + e^{\int_t^T \mu_1(s)ds} & 0 \end{pmatrix}, \\ g(t, x) &= -e^{\int_t^T r_1(s)ds} + e^{\int_t^T \mu_1(s)ds}. \end{aligned}$$

Hence, by using (4), we can get

$$e_1^*(t) = \frac{\text{sgn}\{-e^{\int_t^T r_1(s)ds} + e^{\int_t^T \mu_1(s)ds}\}(\bar{e_1} - \underline{e_1}) + (\bar{e_1} + \underline{e_1})}{2} = \begin{cases} \bar{e_1} & \text{if } e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \\ \underline{e_1} & \text{if } e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \end{cases}$$

Furthermore, the optimal cash holding can be obtained as follows:

$$X_1(t) - e_1^*(t) = \begin{cases} L & \text{if } e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \\ H & \text{if } e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \end{cases} \quad (14)$$

$e^{\int_t^T r_1(s)ds}$ represents the future value of cash assets per unit, and $e^{\int_t^T \mu_1(s)ds}$ represents the future value of risk assets per unit at time t . Equation (14) means that if the future value of cash assets per unit is less than the future value of risk assets per unit, the company will maximize the purchase of risky assets. On the contrary, if the future value of cash assets per unit is more than the future value of

risk assets per unit, the company will maximize the holding of cash assets. Here, we do not consider the case of the two being equal, because when the two are equal, the conversion is meaningless.

Case 2: For $X_1(t) < L$. In this case, in order not to be confused with the case of $X_1(t) > H$, we assume that the transaction amount is $m(t) = (m_1(t) \ m_2(t))^T$, where $m_1(t)$ represents the portion of cash assets converted to risky assets at time t , and $m_2(t)$ represents the portion of risky assets converted to cash assets at time t , and the two do not occur at the same time.

When $X_1(t) < L$, we need to convert part of $X_2(t)$ into $X_1(t)$; then we have $m_1(t) = 0$ and $m(t) = (m_1(t) \ 0)^T$. Now we only need to get the value of $m_1(t)$ and we have the following two equations:

$$\begin{aligned} dX_1(t) &= (r_1 X_1(t) + m_2(t))dt, \\ dX_2(t) &= (\mu_1 X_2(t) - m_2(t))dt + \sigma_1 X_2(t)dC_t. \end{aligned}$$

Here, we still need to be aware that the remaining cash holding after conversion needs to be in the safe area Λ , that is, $L \leq X_1(t) + m_2(t) \leq H$. Therefore, $L - X_1(t) \leq m_2(t) \leq H - X_1(t)$. We define the set of constraints at this time as $M = [m_2(t), \overline{m_2(t)}] = [L - X_1(t), H - X_1(t)]$ and assume that managers are still interested in maximizing the terminal value $X_1(T) + X_2(T)$. Now, the problem can be represented by

$$\begin{cases} \max_{m_2(t)} [X_1(T) + X_2(T)] \\ s.t. \begin{cases} dX_1(t) = (r_1(t)X_1(t) + m_2(t))dt \\ dX_2(t) = (\mu_1(t)X_2(t) - m_2(t))dt + \sigma_1(t)X_2(t)dC_t \\ m_2(t) \in M \end{cases} \end{cases} \quad (15)$$

By comparing (15) with (3), we can see that $P(t) = \begin{pmatrix} r_1(t) & 0 \\ 0 & \mu_1(t) \end{pmatrix}$, $Q(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, $H(T) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $f(t) = 0$. Using (11) we can obtain the same $h(t) = \left(e^{\int_t^T r_1(s)ds} \ e^{\int_t^T \mu_1(s)ds} \right)^T$, and then the switching vector is

$$h(t)^T Q(t) = \begin{pmatrix} e^{\int_t^T r_1(s)ds} & e^{\int_t^T \mu_1(s)ds} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & e^{\int_t^T r_1(s)ds} - e^{\int_t^T \mu_1(s)ds} \end{pmatrix}.$$

Then, we can obtain the following switching function:

$$g(t, x) = e^{\int_t^T r_1(s)ds} - e^{\int_t^T \mu_1(s)ds}.$$

Hence, using (4) we get

$$m_2^*(t) = \frac{\text{sgn}\{e^{\int_t^T r_1(s)ds} - e^{\int_t^T \mu_1(s)ds}\}(\overline{m_2} - \underline{m_2}) + (\overline{m_2} + \underline{m_2})}{2} = \begin{cases} \overline{m_2} & \text{if } e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \\ \underline{m_2} & \text{if } e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \end{cases}$$

Furthermore, the optimal cash holding can be obtained as follows:

$$X_1(t) + m_2^*(t) = \begin{cases} H & \text{if } e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \\ L & \text{if } e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \end{cases} \quad (16)$$

It is not difficult to see that conclusion (16) is consistent with (14). When the future value of risky assets per unit is greater than the future value of cash assets per unit, companies will buy as many risky assets as possible. When the future value of risky assets per unit is less than the future value

of cash assets per unit, companies will hold as many cash assets as possible. This conclusion is also consistent with the actual situation.

3.2. Optimal Models with Transaction Costs

In investment markets, investors have to pay various transaction costs. In China, for example, the transaction costs usually include stamp duty, commission, transfer fees, and other expenses. In this paper we assume that the transaction cost of unit securities is θ . Next, we discuss this aspect by referring to two cases: $X_1(t) > H$ and $X_1(t) < L$.

For $X_1(t) > H$, we assume that the instantaneous turnover is $u(t) = (u_1(t) \ u_2(t))^T$. Because $X_1(t) > H$, we do not need to convert risky assets into cash assets at this time, that is, $u_2(t) = 0$.

$u_1(t)$ represents the portion of cash assets converted into risky assets, and the converted cash holdings are $X_1(t) - u_1(t) - \theta u_1(t)$. The level of cash holdings must always be within the security level and, therefore $L \leq X_1(t) - u_1(t) - \theta u_1(t) \leq H$, that is, $\frac{X_1(t)-H}{1+\theta} \leq u_1(t) \leq \frac{X_1(t)-L}{1+\theta}$. We note that $U = [\underline{u}_1(t), \overline{u}_1(t)] = [\frac{X_1(t)-H}{1+\theta}, \frac{X_1(t)-L}{1+\theta}]$. The optimal conversion model is provided as follows:

$$\begin{cases} \max_{u_1(t)} [X_1(T) + X_2(T)] \\ s.t. \begin{cases} dX_1(t) = (r_1(t)X_1(t) - u_1(t) - \theta u_1(t))dt \\ dX_2(t) = (\mu_1(t)X_2(t) + u_1(t))dt + \sigma_1(t)X_2(t)dC_t \\ u_1(t) \in U \end{cases} \end{cases} \quad (17)$$

By comparing (17) with (3), we can see that $P(t) = \begin{pmatrix} r_1(t) & 0 \\ 0 & \mu_1(t) \end{pmatrix}$, $Q(t) = \begin{pmatrix} -(1+\theta) & 0 \\ 1 & 0 \end{pmatrix}$. Then, the switching vector is

$$h(t)^T Q(t) = \begin{pmatrix} e^{\int_t^T r_1(s)ds} & e^{\int_t^T \mu_1(s)ds} \end{pmatrix} \begin{pmatrix} -(1+\theta) & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -(1+\theta)e^{\int_t^T r_1(s)ds} + e^{\int_t^T \mu_1(s)ds} & 0 \end{pmatrix}$$

Therefore, we can obtain the optimal control:

$$u_1^*(t) = \frac{\text{sgn}\{-(1+\theta)e^{\int_t^T r_1(s)ds} + e^{\int_t^T \mu_1(s)ds}\}(\overline{u}_1 - \underline{u}_1) + (\overline{u}_1 + \underline{u}_1)}{2} = \begin{cases} \overline{u}_1 & \text{if } (1+\theta)e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \\ \underline{u}_1 & \text{if } (1+\theta)e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \end{cases}$$

Furthermore, the optimal cash holding can also be described as follows:

$$X_1(t) - (1+\theta)u_1^*(t) = \begin{cases} L & \text{if } (1+\theta)e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \\ H & \text{if } (1+\theta)e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \end{cases} \quad (18)$$

Conclusion (18) shows that when the future value of $1+\theta$ units cash assets is less than the future value of risky assets per unit, the enterprise's optimal cash holding level is L. Otherwise, the optimal cash holding level is H.

For $X_1(t) < L$, we do not need to convert risky assets into cash assets at this time in this case; we assume that the instantaneous turnover is $v(t) = (v_1(t) \ v_2(t))^T$, and $v_1(t) = 0$. $v_2(t)$ represents the portion of risk assets converted into cash assets, and the converted cash holdings are $X_1(t) + v_2(t) - \theta v_2(t)$. The level of cash holdings must always be within the security level and, therefore

$$L \leq X_1(t) + v_2(t) - \theta v_2(t) \leq H,$$

that is,

$$\frac{L - X_1(t)}{1 - \theta} \leq v_2(t) \leq \frac{H - X_1(t)}{1 - \theta}.$$

We note that $V = [\underline{v}_1(t), \overline{v}_1(t)] = [\frac{L-X_1(t)}{1-\theta}, \frac{H-X_1(t)}{1-\theta}]$. The optimal conversion model can be described as follows:

$$\begin{cases} \max_{v_2(t)} [X_1(T) + X_2(T)] \\ s.t. \begin{cases} dX_1(t) = (r_1(t)X_1(t) + v_2(t) - \theta v_2(t))dt \\ dX_2(t) = (\mu_1(t)X_2(t) - v_2(t))dt + \sigma_1(t)X_2(t)dC_t \\ v \in V \end{cases} \end{cases} \quad (19)$$

By comparing model (19) with (3), we can see that $P(t) = \begin{pmatrix} r_1(t) & 0 \\ 0 & \mu_1(t) \end{pmatrix}$, $Q(t) = \begin{pmatrix} 0 & 1-\theta \\ 0 & -1 \end{pmatrix}$.

Then, the switching vector is

$$h(t)^T Q(t) = \begin{pmatrix} e^{\int_t^T r_1(s)ds} & e^{\int_t^T \mu_1(s)ds} \end{pmatrix} \begin{pmatrix} 0 & 1-\theta \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & (1-\theta)e^{\int_t^T r_1(s)ds} - e^{\int_t^T \mu_1(s)ds} \end{pmatrix}.$$

Therefore, we can obtain the optimal control:

$$v_2^*(t) = \frac{\text{sgn}\{(1-\theta)e^{\int_t^T r_1(s)ds} - e^{\int_t^T \mu_1(s)ds}\}(\overline{v}_2 - \underline{v}_2) + (\overline{v}_2 + \underline{v}_2)}{2} = \begin{cases} \overline{v}_2 & \text{if } (1-\theta)e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \\ \underline{v}_2 & \text{if } (1-\theta)e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \end{cases}$$

Furthermore, the optimal cash holding can also be obtained as follows:

$$X_1(t) + (1-\theta)v_2^*(t) = \begin{cases} H & \text{if } (1-\theta)e^{\int_t^T r_1(s)ds} > e^{\int_t^T \mu_1(s)ds} \\ L & \text{if } (1-\theta)e^{\int_t^T r_1(s)ds} < e^{\int_t^T \mu_1(s)ds} \end{cases}.$$

Conclusion (20) shows that when the future value of $1-\theta$ units of cash assets is more than the future value of risky assets per unit, the enterprise's optimal cash holding level is H. Otherwise, the optimal cash holding level is L.

4. Examples

The purpose of this section is to verify the adaptability of our models. As close as possible to business practice, at time t , we assume that the parameters involved in our models are as follows: $r_1 = 0.02$, $\mu_1 = 0.15$, $L = 5$, $H = 13$, $t = 4$, and $T = 6$, and the cost of unit turnover is $\theta = 0.008$ at time t .

If $X_1(t) = 20 > H$, the company needs to transfer some cash assets into risky assets. If transaction costs are not taken into account, due to $e^{\int_t^T r_1(s)ds} = e^{0.04} < e^{\int_t^T \mu_1(s)ds} = e^{0.3}$, according to (14), the optimal cash holding level is L. If the transaction costs are taken into account, due to $(1+\theta)e^{\int_t^T r_1(s)ds} = 1.008e^{0.04} < e^{\int_t^T \mu_1(s)ds} = e^{0.3}$, according to (18), the optimal cash holding level is L.

If $X_1(t) = 4 < L$, the company needs to transfer some risky assets into cash assets. If transaction costs are not taken into account, due to $e^{\int_t^T r_1(s)ds} = e^{0.04} < e^{\int_t^T \mu_1(s)ds} = e^{0.3}$, according to (15), the optimal cash holding level is L. If the transaction costs are taken into account, due to $(1-\theta)e^{\int_t^T r_1(s)ds} = 0.992e^{0.04} < e^{\int_t^T \mu_1(s)ds} = e^{0.3}$, according to (19), the optimal cash holding level is also L.

Through the above numerical results, we can also see that without knowing the variance in the risky assets, we can make a holding decision by simply comparing the size of the return. Because of the uncertainty of the return on risky assets, when the return on risky assets is lower than the return on cash holdings, it is obviously cost-effective to hold cash; this is in good agreement with reality.

Next, we use graphics to explain the whole holding decision-making process. First, we fix factors other than μ_1 as above, when $X_1(t) = 20 > H$. Here, we only discuss the case where transaction

costs are taken into account. In order to illustrate the problem more clearly, we expand $e^{\int_t^T \mu_1(s) ds}$ and $(1 + \theta)e^{\int_t^T r_1(s) ds}$ by 10 times at the same time, which obviously does not affect their size relationship.

In Figure 1, the dotted line represents the future value of $(1 + \theta)$ units of cash assets, and the curve represents the future value of risky assets per unit. Figure 1 shows that when $\mu_1 < 0.023984$, the dotted line is below the curve and the cash holding level is H but when $\mu_1 > 0.023984$, the dotted line is above the curve and the cash holding level is L. That is to say, when the risk rate of return reaches a certain value, cash holdings will remain at the lowest level of the safe area. Similar discussions can be made for the case where $X_1(t) < L$.

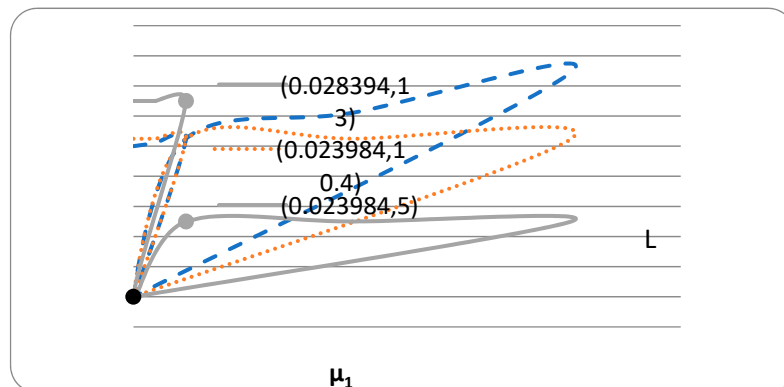


Figure 1. The impacts of μ_1 on cash holdings when $X_1(t) > H$.

Now, for case $X_1(t) > H$, let us discuss the impact of θ on the optimal cash holdings. Here, we keep the same factors as above except for θ . To simplify the problem, we can let $y = (1 + \theta)e^{\int_t^T r_1(s) ds} - e^{\int_t^T \mu_1(s) ds}$. Then, the problem becomes that if $y > 0$, the best cash holding level is H and if $y < 0$, the best cash holdings is L. The values of the parameters are the same as the previous assumptions. Thus, we have

$$y = (1 + \theta)e^{\int_t^T r_1(s) ds} - e^{\int_t^T \mu_1(s) ds} = (1 + \theta)e^{0.04} - e^{0.3}.$$

Figure 2 shows the whole decision-making process. When $y < 0$, managers will buy as many risky assets as possible, and when $y > 0$, they will hold as many cash assets as possible. A similar argument can be made for the case where $X_1(t) < L$.

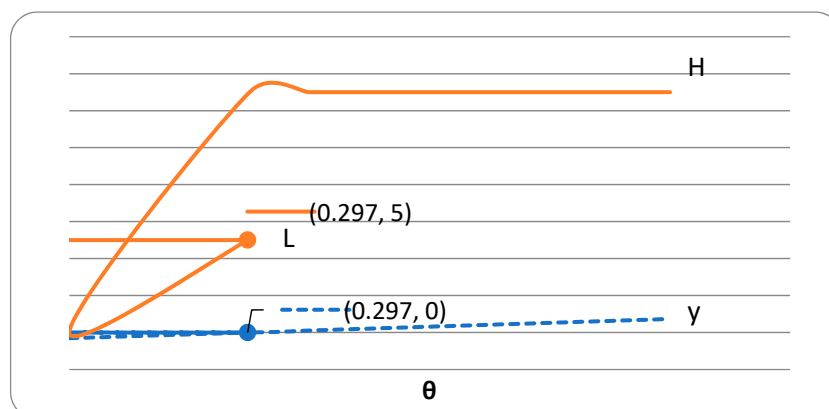


Figure 2. The impacts of θ on cash holdings when $X_1(t) > H$.

5. Conclusions

The main contributions of this paper are summarized as follows.

Firstly, the problem of cash holdings is how to determine an optimal exchange strategy between cash assets and securities. Based on the concept of uncertainty, we established an uncertain optimal control model in a time interval $[0, T]$. For the first time, a cash holding decision-making model was established based on uncertainty theory. This not only makes use of uncertainty theory in finance but also develops and supplements existing cash decision-making models.

Secondly, we proved that the model is an bang–bang control model and obtained the solutions to our cash holdings model. On the assumption that enterprises can obtain upper and lower limits for cash holdings based on their historical operating experience, our results show that without considering transaction costs, if the future value of cash assets per unit is less than the future value of risk assets per unit, the company will maximize the purchase of risky assets. On the contrary, the company will maximize the holding of cash assets. This conclusion is also a verification of reality. If transaction costs are taken into account, it can be seen from the conclusion that the result without considering transaction costs can be obtained by making the transaction cost of unit securities equal to 0 in the result obtained considering transaction costs, which is consistent with the results discussed separately in our paper.

Last but not least, using specific examples, we demonstrated that our models are feasible and can be used to make decisions. We discussed the influence of factors, including risk assets' return and the transaction cost of unit securities, on decision-making results. The results show that without considering transaction costs, managers only need to compare the return rate of investment targets, and do not even need to know the specific value. Further, the managers making decisions do not need to consider the risk of the investment objective; this is consistent with the Miller–Orr model.

Future research directions are as follows. First, this paper considered fixed transaction costs. In future studies, the model will be extended to the case where transaction costs are variable, multi-stage, and functional, etc. Second, the objective function discussed in this paper was simply to maximize the sum of cash assets and risky assets at the end of the term. In future studies, the decision model under a general utility function will be considered.

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