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Fixed Point Theory for Digital k-Surfaces and Some Remarks on the Euler Characteristics of Digital Closed Surfaces

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Abstract: The present paper studies the fixed point property (*FPP*) for closed k-surfaces. We also intensively study Euler characteristics of a closed k-surface and a connected sum of closed k-surfaces. Furthermore, we explore some relationships between the *FPP* and Euler characteristics of closed k-surfaces. After explaining how to define the Euler characteristic of a closed k-surface more precisely, we confirm a certain consistency of the Euler characteristic of a closed k-surface and a continuous analog of it. In proceeding with this work, for a simple closed k-surface in $\mathbb{Z}^3$, say S_k, we can see that both the minimal 26-adjacency neighborhood of a point $x \in S_k$, denoted by $M_k(x)$, and the geometric realization of it in $\mathbb{R}^3$, denoted by $D_k(x)$, play important roles in both digital surface theory and fixed point theory. Moreover, we prove that the simple closed 18-surfaces MSS_{18} and MSS'_{18} do not have the almost fixed point property (*AFPP*). Consequently, we conclude that the triviality or the non-triviality of the Euler characteristics of simple closed k-surfaces have no relationships with the *FPP* in digital topology. Using this fact, we correct many errors in many papers written by L. Boxer et al.

Keywords: fixed point property; almost (approximate) fixed point property; digital surface; digital connected sum; geometric realization; Euler characteristic; minimal $(3^n - 1)$-neighborhood; digital topology

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1. Introduction

In $\mathbb{Z}^3$, the concept of closed k-surface was established in References [1,2] and its digital topological characterizations were also studied [3–7]. Many explorations of various properties of closed k-surfaces have been proceeded from the viewpoints of digital topology and digital geometry [1–9]. Based on the studies of the earlier works [3–5,10,11], given (digital) closed k-surfaces and connected sums of closed k-surfaces, we will investigate the fixed point property (*FPP*) or the almost fixed point property (*AFPP*) of them. Moreover, after explaining the Euler characteristic of a closed k-surface in Reference [5] more precisely, we confirm strong relationships between the Euler characteristic of a closed k-surface and that of the continuous analog (or geometric realization) of a closed k-surface.

Indeed, there are several kinds of approaches to establish a digital k-surface [3–6,12,13]. In digital surface theory, we need to consider a binary digital image structure. To be precise, in the case of $X \subset \mathbb{Z}^3$, we often assume a digital image X in the digital picture P,

$$P \in \{(\mathbb{Z}^3, 26, 6, X), (\mathbb{Z}^3, 18, 6, X), (\mathbb{Z}^3, 6, 26, X)\}. \tag{1}$$

Thus, we can study a (digital) closed k-surface with one of the above picture P. Moreover, for a digital image (X, k), the notion of the Euler characteristic of (X, k) was proposed in several ways [3–5,14–19]. The concept of digital connected sum of closed k-surfaces in $\mathbb{Z}^n$ was firstly introduced in Reference [3] by using several types of simple closed k-curves in $\mathbb{Z}^2, k \in \{4, 8\}$.

Hereafter, we denote a simple closed k-surface in $\mathbb{Z}^3$ (for more details, see Definition 6) with S_k. Indeed, given an S_k, the studies of its Euler characteristic, an efficient formulation of a continuous analog of it, and the *FPP* for S_k play important roles in digital geometry. Thus, we have the following queries:

(Q1) How to establish a geometric realization of an S_k?
(Q2) Does the geometric realization transform an S_k into a certain spherical (or a sphere-like) polyhedron in $\mathbb{R}^3$?
(Q3) How to define the Euler characteristic of an S_k?
(Q4) Are there certain relationships between the Euler characteristic of an S_k and that of a geometric realization of an S_k?
(Q5) What about the *FPP* or the *AFPP* for an S_k?

To address these issues, Reference [5] introduced the Euler characteristic of an S_k, which can facilitate the studies of both digital and typical surface theories. This paper continues a series of studies of Euler characteristics of digital surfaces [5]. In order to prevent a certain misunderstanding or wrong interpretation of the Euler characteristic of an S_k, after referring to several essential notions associated with the Euler characteristic of an S_k, the present paper corrects some assertions in Reference [14] involving the Euler characteristics of an S_k and a connected sum of closed k-surfaces. To be precise, we will more precisely explain how to define the Euler characteristics of an S_k already introduced in Reference [5] and a digital connected sum introduced in Reference [3]. Indeed, the Euler characteristic of an S_k suggested in Reference [5] is proved to be consistent with the typical Euler characteristic of a closed surface from the viewpoints of algebraic topology and polyhedral geometry.

The rest of the paper is organized as follows: Section 2 refers to some notions involving a digital k-surface and a connected sum of two digital k-surfaces. Moreover, it confirms the pointed 18-contractibility of MSS'_{18} which will be used in the paper. Section 3 establishes the sets $M_k(x)$ and $D_k(x)$ (see Definitions 8 and 9) to develop a 2-dimensional simplicial complex as a geometric realization of a simple closed k-surface S_k. Section 4 studies the Euler characteristics of a closed k-surface and a connected sum of two closed k-surfaces proposed in Reference [5]. In particular, given an S_k, using the set $\{D_k(x) \mid x \in S_k\}$, we can characterize the Euler characteristic of an S_k. Section 5 studies the *FPP* or the *AFPP* for several kinds of simple closed k-surfaces in $\mathbb{Z}^3$, MSS_{18}, MSS'_{18}, MSS_6, and so on. Finally, we prove that the simple closed 18-surfaces MSS_{18} and MSS'_{18} do not have the *AFPP*. Hence, we conclude that, in digital topology, the triviality or the non-triviality of the Euler characteristics of simple closed k-surfaces are irrelevant to the *FPP* or the *AFPP*. Furthermore, we corrects many errors in the paper written by Boxer et al. in Reference [14] (see Remarks 11 and 12) and some mistakes in Reference [3,4] (see Remark 10). Section 6 concludes the paper with some remarks.

2. Basic Notions Related to Digital k-Surfaces and a Connected Sum for k-Surfaces

In order to make the paper self-contained, let us now recall some terminology from digital curve and digital surface theories. Let $\mathbb{N}$ and $\mathbb{Z}$ represent the sets of natural numbers and integers, respectively.

Rosenfeld [20] called a set $X(\subset \mathbb{Z}^n)$ with a k-adjacency a digital image, denoted by (X, k). In particular, in digital surface theory, let us consider a binary digital image (X, k) with a k-adjacency in a digital picture $(\mathbb{Z}^n, k, \bar{k}, X)$ [17,20], where $n \in \mathbb{N}$. Then, we call the pair (X, k) a digital image with a k-adjacency (for short, *digital image*). In order to study (X, k) in $\mathbb{Z}^n, n \geq 1$, we need k-adjacency relations of $\mathbb{Z}^n$ which are generalizations of the commonly used 4- and 8-adjacency of $\mathbb{Z}^2$, and 6-, 18-, and 26-adjacency of $\mathbb{Z}^3$. To be precise, we will say that distinct points $p, q \in \mathbb{Z}^n$ are k-(or $k(t, n)$-)adjacent if they satisfy the following property [10] (for more details, see also Reference [21] as an advanced representation of the k-adjacency relations of $\mathbb{Z}^n$ in Reference [10]):

For a natural number t, $1 \leq t \leq n$, we say that distinct points

$$p = (p_1, p_2, \cdots, p_n) \text{ and } q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n \tag{2}$$

are $k(t,n)$-(k-, for short)adjacent if at most t of their coordinates differs by ± 1 and all others coincide.

Concretely, these $k(t,n)$-adjacency relations of $\mathbb{Z}^n$ are determined according to the number $t \in \mathbb{N}$ [10] (see also Reference [21]). In the present paper, we will use the symbol " $:=$" to introduce new notions without proving the fact.

Using the operator of Equation (2), the k-adjacency relations of $\mathbb{Z}^n$ are obtained [10] (see also References [21,22]), as follows

$$\left\{ \begin{array}{l} (a)\, k := k(t,n) = \sum\limits_{i=n-t}^{n-1} 2^{n-i} C_i^n, \text{where } C_i^n = \dfrac{n!}{(n-i)!\; i!} \\ \text{or equivalently} \\ (b)\, k := k(t,n) = \sum\limits_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n = \dfrac{n!}{(n-i)!\; i!}. \end{array} \right\} \tag{3}$$

A digital image (X,k) in $\mathbb{Z}^n$ can indeed be considered to be a set $X(\subset \mathbb{Z}^n)$ with the k-adjacency relation of Equation (3). Using the k-adjacency relations of $\mathbb{Z}^n$ of Equation (3), we say that a digital k-neighborhood of p in $\mathbb{Z}^n$ is the set [20]

$$N_k(p) := \{q \mid p \text{ is } k\text{-adjacent to } q\} \cup \{p\}.$$

Furthermore, we often use the notation [17]

$$N_k^*(p) := N_k(p) \setminus \{p\}.$$

For $a, b \in \mathbb{Z}$ with $a \lneq b$, the set $[a,b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \mid a \leq n \leq b\}$ with 2-adjacency is called a digital interval [17].

Let us now recall some terminology and notions which are used in this paper.

- We say that two subsets (A,k) and (B,k) of (X,k) are k-adjacent if $A \cap B = \emptyset$ and that there are points $a \in A$ and $b \in B$ such that a and b are k-adjacent [17]. In particular, in case B is a singleton, say $B = \{x\}$, we say that A is k-adjacent to x.
- For a k-adjacency relation of $\mathbb{Z}^n$, a k-path with $l+1$ elements in $\mathbb{Z}^n$ is assumed to be a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if $|\, i - j\,| = 1$ [17].
- A digital image (X,k) is said to be k-connected if, for any distinct points such as x, y in (X,k), there is a k-path $(x_i)_{i \in [0,l]_{\mathbb{Z}}} \subset X$ such that $x = x_0$ and $y = x_l$.
- For a digital image (X,k), the k-component of $x \in X$ is defined to be the largest k-connected subset of (X,k) containing the point x.
- We say that a simple k-path is a finite set $(x_i)_{i \in [0,m]_{\mathbb{Z}}} \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j\,| = 1$ [17]. In the cases $x_0 = x$ and $x_m = y$, we denote the length of the simple k-path with $l_k(x,y) := m$.
- A simple closed k-curve (or simple k-cycle) with l elements in $\mathbb{Z}^n$ [10], denoted by $SC_k^{n,l}, l \geq 4, l \in \mathbb{N}_0 \setminus \{2\}$, $\mathbb{N}_0$ is the set of even natural numbers [10,17] and is the finite set $(x_i)_{i \in [0,l-1]_{\mathbb{Z}}}$ such that x_i and x_j are k-adjacent if and only if $|\, i - j\,| = \pm 1 (mod\, l)$ [10].
- For a digital image (X,k), a digital k-neighborhood of $x_0 \in X$ with radius ε is defined in X as the following subset [10] of X:

$$N_k(x_0, \varepsilon) := \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \tag{4}$$

where $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x and $\varepsilon \in \mathbb{N}$. For instance, for $X \subset \mathbb{Z}^n$, we obtain [10]

$$N_k(x, 1) = N_k(x) \cap X. \tag{5}$$

- Rosenfeld [20] defined the notion of digital continuity of a map $f : (X, k_0) \to (Y, k_1)$ by saying that f maps every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1).

Motivated by the digital continuity proposed by Rosenfeld, in terms of the digital k-neighborhood of a point with radius 1 (see Equation (5)), the digital continuity of a map between digital images was represented, as follows:

Proposition 1 ([10,11])**.** *Let (X, k_0) and (Y, k_1) be digital images in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : (X, k_0) \to (Y, k_1)$ is (digitally) (k_0, k_1)-continuous if and only if, for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 1 in case $n_0 = n_1$ and $k_0 = k_1 := k$, the map f is called a 'k-continuous map. Since an n-dimensional digital image (X, k) is considered to be a set X in $\mathbb{Z}^n$ with one of the k-adjacency relations of Equation (3) (or a digital k-graph [23]), regarding a classification of n-dimensional digital images, we use the term a (k_0, k_1)-isomorphism (or k-isomorphism) as in Reference [23] (see also Reference [11]) rather than a (k_0, k_1)-homeomorphism (or k-homeomorphism) as in Reference [24].

Definition 1 ([23] (see also a (k_0, k_1)-homeomorphism in Reference [24]))**.** *Consider two digital images (X, k_0) and (Y, k_1) in $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Then, a map $h : X \to Y$ is called a (k_0, k_1)-isomorphism if h is a (k_0, k_1)-continuous bijection, and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. Then, we use the notation $X \approx_{(k_0,k_1)} Y$. Moreover, in the case $k_0 = k_1 := k$, we use the notation $X \approx_k Y$.*

In References [23,25,26], we developed many notions from the viewpoint of digital graph theory, such as graph (k_0, k_1)-homomorphism, graph (k_0, k_1)-isomorphism, and graph (k_0, k_1)-homotopy which are, respectively, digital graphical versions of the (k_0, k_1)-continuity, (k_0, k_1)-homeomorphism, and (k_0, k_1)-homotopy in digital topology. Since a digital image (X, k) can be recognized as a digital k-graph [5,23], we mainly use the digital k-graphical method to study Euler characteristics of a closed k-surface in this paper.

The following notion of *interior* is often used in establishing the notion of digital connected sum.

Definition 2 ([3])**.** *Let $c^* := (x_0, x_1, \cdots, x_n)$ be a closed k-curve in $(\mathbb{Z}^2, k, \bar{k}, c^*)$. A point x of $\bar{c^*}$, the complement of c^* in $\mathbb{Z}^2$, is said to be interior to c^* if it belongs to the bounded $\bar{k}$-connected component of $\bar{c^*}$.*

The following digital images MSC_8^*, MSC_4^*, and $MSC_8'^*$ [3,4,10] play important roles in establishing a connected sum and in studying the digital fundamental group of a digital connected sum of closed k-surfaces. Thus, we will recall them.

($\star$) $MSC_8^* := MSC_8 \cup Int(MSC_8)$ [4], where MSC_8 is a digital image 8-isomorphic to the digital image, i.e., $MSC_8 := SC_8^{2,6} := \{c_0 = (0,0), c_1 = (1,1), c_2 = (1,2), c_3 = (0,3), c_4 = (-1,2), c_5 = (-1,1)\}$;

($\star$) $MSC_4^* := MSC_4 \cup Int(MSC_4)$ [4], where MSC_4 is a digital image 4-isomorphic to the digital image, i.e., $MSC_4 := SC_4^{2,8} := \{v_0 = (0,0), v_1 = (1,0), v_2 = (2,0), v_3 = (2,1), v_4 = (2,2), v_5 = (1,2), v_6 = (0,2), v_7 = (0,1)\}$; and

($\star$) $MSC_8'^* := MSC_8' \cup Int(MSC_8')$ [4], where MSC_8' is a digital image 8-isomorphic to the digital image, i.e., $MSC_8' := SC_8^{2,4} := \{w_0 = (0,0), w_1 = (1,1), w_2 = (0,2), w_3 = (-1,1)\}$.

Based on the pointed digital homotopy in Reference [27] (see also Reference [24]), the following notion of k-homotopy relative to a subset $A \subset X$ is often used in studying k-homotopic properties of digital images (X, k) in $\mathbb{Z}^n$. For a digital image (X, k) and $A \subset X$, we often call $((X, A), k)$ a digital image pair.

Definition 3 ([10] (see also [3])). *Let $((X,A),k_0)$ and (Y,k_1) be a digital image pair and a digital image, respectively. Let $f,g : X \to Y$ be (k_0,k_1)-continuous functions. Suppose there exist $m \in \mathbb{N}$ and a function $H : X \times [0,m]_{\mathbb{Z}} \to Y$ such that*

- *for all $x \in X, H(x,0) = f(x)$ and $H(x,m) = g(x)$;*
- *for all $x \in X$, the induced function $H_x : [0,m]_{\mathbb{Z}} \to Y$ given by $H_x(t) = H(x,t)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,k_1)$-continuous;*
- *for all $t \in [0,m]_{\mathbb{Z}}$, the induced function $H_t : X \to Y$ given by $H_t(x) = H(x,t)$ for all $x \in X$ is (k_0,k_1)-continuous. Then, we say that H is a (k_0,k_1)-homotopy between f and g [24].*
- *Furthermore, for all $t \in [0,m]_{\mathbb{Z}}$, assume that the induced map H_t on A is a constant which follows the prescribed function from A to Y. To be precise, $H_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0,m]_{\mathbb{Z}}$. Then, we call H a (k_0,k_1)-homotopy relative to A between f and g and we say that f and g are (k_0,k_1)-homotopic relative to A in Y, $f \simeq_{(k_0,k_1)rel A} g$ in symbols.*

In Definition 3, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0,k_1)-homotopy at $\{x_0\}$ [24]. When f and g are pointed (k_0,k_1)-homotopic in Y, we use the notation $f \simeq_{(k_0,k_1)} g$. In the cases $k_0 = k_1 := k$ and $n_0 = n_1$, f and g are said to be pointed k-homotopic in Y and we use the notations $f \simeq_k g$ and $f \in [g]$, which denote the k-homotopy class of g. If, for some $x_0 \in X$, 1_X is k-homotopic to the constant map in the space x_0 relative to $\{x_0\}$, then we say that (X,x_0) is pointed k-contractible [24]. The notion of digital homotopy equivalence was firstly introduced in Reference [25] (see also Reference [26]), as follows:

Definition 4 ([25] (see also Reference [26])). *For two digital images (X,k) and (Y,k) in $\mathbb{Z}^n$, if there are k-continuous maps $h : X \to Y$ and $l : Y \to X$ such that the composite $l \circ h$ is k-homotopic to 1_X and the composite $h \circ l$ is k-homotopic to 1_Y, then the map $h : X \to Y$ is called a k-homotopy equivalence and is denoted by $X \simeq_{k \cdot h \cdot e} Y$. Moreover, we say that (X,k) is k-homotopy equivalent to (Y,k).*

Using the concept of digital k-homotopy equivalence, we can classify digital images [25]. Now, we recall the notion of *k-contractibility* to be used later in this paper.

Definition 5 ([10,24,27]). *For a digital image (X,k), if the identity map 1_X is k-homotopic relative to $\{x_0\}$ in X to a constant map with image consisting of some point $x_0 \in X$, then (X,x_0) is said to be pointed k-contractible.*

The following are proven in References [3,5,10,11,24].

- In case X is pointed k-contractible, the k-fundamental group $\pi^k(X,x_0)$ is trivial [24].
- MSC_8 is not 8-contractible and MSC_4 is not 4-contractible either [3,10].
- MSC_8' are 8-contractible [3,5,24].
- Owing to the digital (k_0,k_1)-covering theory in References [10,11], the k-fundamental groups of $SC_k^{n,l}$ were proven such that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group [10,11].
- Motivated by the calculation of the digital k-fundamental group of $SC_k^{2,l}$, i.e., $\pi^k(SC_k^{2,l}), k \in \{4,8\}$ in References [10,11], it turns out that $SC_k^{n,l}$ is not k-contractible if $l \geq 6$.

In particular, both the non-8-contractibility of MSC_8 and the non-4-contractibility of MSC_4 play important roles in formulating a connected sum of two closed k-surfaces.

In order to study a closed k-surface in $\mathbb{Z}^n$, let us recall some terminology, such as a k-corner, a generalized simple closed k-curve, and so on. A point $x \in (X,k)$ is called a *k-corner* if x is k-adjacent to two and only two points y and $z \in X$ such that y and z are k-adjacent to each other [2]. The k-corner x is called *simple* if y and z are not k-corners and if x is the only point k-adjacent to both y and z. (X,k) is called a *generalized simple closed* k-curve if what is obtained by removing all simple k-corners of X is a simple closed k-curve [2,6]. For a k-connected digital image (X,k) in $X \subset \mathbb{Z}^3$, we recall

$$|X|^x := N_{26}^*(x) \cap X = N_{26}(x,1) \setminus \{x\}, \tag{6}$$

where $N_{26}^*(x) := \{x' \mid x \text{ and } x' \text{ are 26-adjacent}\}$ [1,2]. In general, for a k-connected digital image (X, k) in $\mathbb{Z}^n, n \geq 3$, we can state [5]

$$|X|^x = N_{3^n-1}^*(x) \cap X = N_{3^n-1}(x, 1) \setminus \{x\}, \tag{7}$$

where

$$N_{3^n-1}^*(x) = \{x' \mid x \text{ and } x' \text{ are } (3^n-1)\text{-adjacent}\}.$$

Hereafter, for a k-surface in $\mathbb{Z}^n, n \in \mathbb{N} \setminus \{1, 2, 3\}$ [3,4], we call the set $|X|^x$ of Equation (7) the *minimal* $(3^n - 1)$*-adjacency neighborhood* of x in X.

Reference [5] introduced the notion of a closed k-surface in $\mathbb{Z}^n, n \geq 3$. However, in the present paper, we will stress the study of closed k-surfaces in $\mathbb{Z}^3$ with the following approach in References [6,7].

Definition 6 ([3,7])**.** *Let (X, k) be a digital image in $\mathbb{Z}^3$, and $\bar{X} := \mathbb{Z}^3 \setminus X$. Then, X is called a closed k-surface if it satisfies the following:*

(1) In case $(k, \bar{k}) \in \{(26, 6), (6, 26)\}$,

- *(a) for each point $x \in X$, $|\underline{X}|^x$ has exactly one k-component k-adjacent to x;*
- *(b) $|\bar{X}|^x$ has exactly two $\bar{k}$-components $\bar{k}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and*
- *(c) for any point $y \in N_k(x) \cap X$ (or $N_k(x, 1)$ in (X, k)), $N_{\bar{k}}(y) \cap C^{xx} \neq \phi$ and $N_{\bar{k}}(y) \cap D^{xx} \neq \phi$. Furthermore, if a closed k-surface X does not have a simple k-point, then X is called simple.*

(2) In case $(k, \bar{k}) = (18, 6)$,

- *(a) X is k-connected,*
- *(b) for each point $x \in X$, $|X|^x$ is a generalized simple closed k-curve. Furthermore, if the image $|X|^x$ is a simple closed k-curve, then the closed k-surface X is called simple.*

From now on, we denote a closed k-surface in $\mathbb{Z}^3$ with S_k, $k \in \{6, 18, 26\}$, which will be used in this paper. Namely, we will consider only simple closed k-surface in $\mathbb{Z}^3$ in the picture as referred to in Equation (1), i.e.,

$$\{(\mathbb{Z}^3, 26, 6, S_{26}), (\mathbb{Z}^3, 18, 6, S_{18}), (\mathbb{Z}^3, 6, 26, S_6)\}.$$

Definition 7 ([3])**.** *In $\mathbb{Z}^3$, let S_{k_0} (resp. S_{k_1}) be a closed k_0-(resp. a closed k_1-)surface, where $k_0 = k_1 \in \{6, 18, 26\}$.*

- *Consider $A'_{k_0} \subset A_{k_0} \subset S_{k_0}$ and take $A_{k_0} \setminus A'_{k_0} \subset S_{k_0}$, where $A_{k_0} \approx_{(k_0,4)} MSC_4^*$, $A_{k_0} \approx_{(k_0,8)} MSC_8^*$, or $A_{k_0} \approx_{(k_0,8)} MSC_8'^*$ and, further, $A'_{k_0} \approx_{(k_0,4)} Int(MSC_4)$, $A'_{k_0} \approx_{(k_0,8)} Int(MSC_8)$, or $A'_{k_0} \approx_{(k_0,8)} Int(MSC'_8)$, respectively.*
- *Let $f : A_{k_0} \to f(A_{k_0}) \subset S'_{k_1}$ be a (k_0, k_1)-isomorphism. Remove A'_{k_0} and $f(A'_{k_0})$ from S_{k_0} and S_{k_1}, respectively.*
- *Identify $A_{k_0} \setminus A'_{k_0}$ and $f(A_{k_0} \setminus A'_{k_0})$ by using the (k_0, k_1)-isomorphism f. Then, the quotient space $S'_{k_0} \cup S'_{k_1} / \sim$ is obtained by $i(x) \sim f(x) \in S'_{k_1}$ for $x \in A_{k_0} \setminus A'_{k_0}$ and is denoted by $S_{k_0} \sharp S_{k_1}$, where $S'_{k_0} = S_{k_0} \setminus A'_{k_0}$, $S'_{k_1} = S_{k_1} \setminus f(A'_{k_0})$, and the map $i : A_{k_0} \setminus A'_{k_0} \to S'_{k_0}$ is the inclusion map.*

Owing to Definition 7, $S_{k_0} \sharp S_{k_1}$ is obtained in $\mathbb{Z}^3$. Moreover, the digital topological type of $S_{k_0} \sharp S_{k_1}$ absolutely depends on the choice of the subset $A_{k_0} \subset S_{k_0}$ [5]. Furthermore, the k-adjacency of $S_{k_0} \sharp S_{k_1}$ is required as follows:

Remark 1 ([3])**.** *In the quotient space $S_{k_0} \sharp S_{k_1} := S'_{k_0} \cup S'_{k_1} / \sim$, the subsets $S'_{k_0} \setminus (A_{k_0} \setminus A'_{k_0})$ and $S'_{k_1} \setminus f(A_{k_0} \setminus A'_{k_0})$ in $S_{k_0} \sharp S_{k_1}$ are assumed to be disjoint and are not k-adjacent, where $k_0 = k_1 := k$. Then, the digital image $(S_{k_0} \sharp S_{k_1}, k)$ is called a (digital) connected sum of S_{k_0} and S_{k_1}.*

Hereafter, we denote by MSS_k a *minimal simple closed k-surface* in $\mathbb{Z}^3$ (see Figure 1). Furthermore, we recall the following closed k-surfaces, $k \in \{6, 18, 26\}$ [3]:

- $MSS_6 \approx_6 (MSC_4 \times [0,2]_{\mathbb{Z}}) \cup (Int(MSC_4) \times \{0,2\})$ [3,4].

Then, MSS_6 is the minimal simple closed 6-surface which is not 6-contractible (see Figure 1c). Namely, we obtain $(MSS_6, 6, 26, \mathbb{Z}^3)$ according to Equation (1).

Let us now recall two types of simple closed 18-surfaces which are pointed 18-contractible, e.g., MSS_{18} and MSS'_{18}, as follows:

- $MSS'_{18} \approx_{18} (MSC'_8 \times \{1\}) \cup (Int(MSC'_8) \times \{0,2\})$ [3,4].

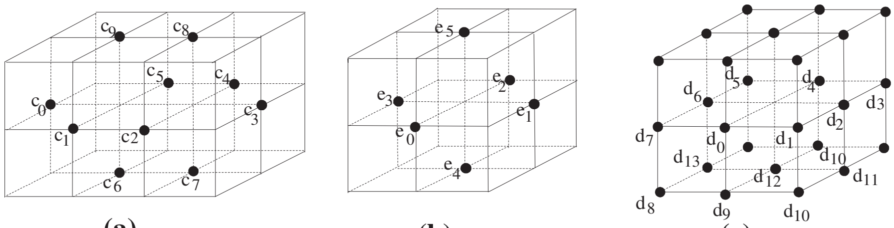

Figure 1. (**a**) MSS_{18} [3,4]; (**b**) $MSS'_{18} = MSS'_{26}$ [3,4]; (**c**) MSS_6 [3].

Then, Reference [3,4] stated that MSS'_{18} is 18-contractible and that it is the minimal simple closed 18-surface (see Figure 1b), i.e., we obtain $(MSS'_{18}, 18, 6, \mathbb{Z}^3)$. Here, the term "minimal" comes from the minimal cardinality of the given digital image as a closed 18-surface.

In order to use the pointed 18-contractibility of MSS'_{18} in this paper, we prove it more precisely, as follows:

Lemma 1. *MSS'_{18} is pointed 18-contractible.*

Proof. Consider the map $H : MSS'_{18} \times [0,2]_{\mathbb{Z}} \to MSS'_{18}$ (see Figure 2a) such that

$$\left\{\begin{array}{l} H(x,0) = x \text{ for any } x \in MSS'_{18}, \\ H(x,1) = e_5 \text{ if } x \in \{e_0, e_3, e_5\}, \text{ and } H(x,1) = e_2 \text{ if } x \in \{e_1, e_2, e_4\}, \\ H(x,2) = \{e_5\}, \text{ for any } x \in MSS'_{18}, \end{array}\right\} \quad (8)$$

Then, the map H is an 18-homotopy relative to the set $\{e_5\}$ since it satisfies the following:

(1) for all $x \in MSS'_{18}, H(x,0) = 1_{MSS'_{18}}$ as an identity map on the set MSS'_{18}, say $1_{MSS'_{18}}$, and $H(x,2) = C_{\{e_5\}}$ as the constant map at the set $\{e_5\}$
(2) for all $x \in MSS'_{18}$, the induced function $H_x : [0,2]_{\mathbb{Z}} \to MSS'_{18}$ given by $H_x(t) = H(x,t)$ for all $t \in [0,2]_{\mathbb{Z}}$ is $(2,18)$-continuous;
(3) for all $t \in [0,2]_{\mathbb{Z}}$, the induced function $H_t : MSS'_{18} \to MSS'_{18}$ given by $H_t(x) = H(x,t)$ for all $x \in MSS'_{18}$ is 18-continuous.
Thus, we obtain H which is an 18-homotopy between $1_{MSS'_{18}}$ and $C_{\{e_5\}}$.
(4) Furthermore, for all $t \in [0,2]_{\mathbb{Z}}$, assume that the induced map H_t on $\{e_5\}$ is a constant.

Owing to properties (1)–(4), we prove that MSS'_{18} is 18-homotopy equivalent to $\{e_5\}$ and we complete the proof. □

In view of the proof of Lemma 1, although we proved the 18-contractibility of MSS'_{18} relative to the set $\{e_5\}$, we find that MSS'_{18} is indeed 18-homotopy equivalent to any singleton $\{x\} \subset MSS'_{18}$.

Let us further introduce two simple closed k-surface, $k \in \{18, 26\}$, as follows:

- $MSS_{18} \approx_{18} (MSC_8 \times \{1\}) \cup (Int(MSC_8) \times \{0,2\})$ [3,4].
Then, MSS_{18} is indeed pointed 18-contractible (correction of the "*non-18-contractibility*" of MSS_{18} in Theorem 4.3(3) of Reference [3] and Theorem 4.2(3) of Reference [4]). Moreover, it is proved to be a simple closed 18-surface (see Figure 1a) [3,4]. Using a method similar to the 18-homotopy

of Equation (8), we observe that there is indeed an 18-homotopy relative to the set $\{c_9\}$ between $1_{MSS_{18}}$ and $C_{\{c_9\}}$, which is the constant map at $\{c_9\}$ (see Figure 2b),

$$H : MSS_{18} \times [0,3]_{\mathbb{Z}} \to MSS_{18}, \tag{9}$$

which implies the pointed 18-contractibility of MSS_{18}. More precisely, starting with $H(x,0)$ as the identity map $1_{MSS_{18}}$, Figure 2(b1) shows the process of $H(x,1)$ and Figure 2(b2) explains the process of $H(x,2)$. Moreover, $H(x,3)$ means the constant map $C_{\{c_9\}}$. In addition, we observe that MSS_{18} is indeed 18-homotopy equivalent to any singleton $\{x\} \subset MSS_{18}$. Furthermore, Reference [3] also already proved that MSS_{18} is simply 18-connected [4].

- $MSS'_{26} := MSS'_{18}$, which is 26-contractible [3,4] and is the minimal simple closed 26-surface (see Figure 1b). Finally, we obtain $(MSS'_{26}, 26, 6, \mathbb{Z}^3)$ according to Equation (1). Moreover, the proof of the 26-contractibility of MSS'_{26} is trivially proceeded with the homotopy in Equation (8).

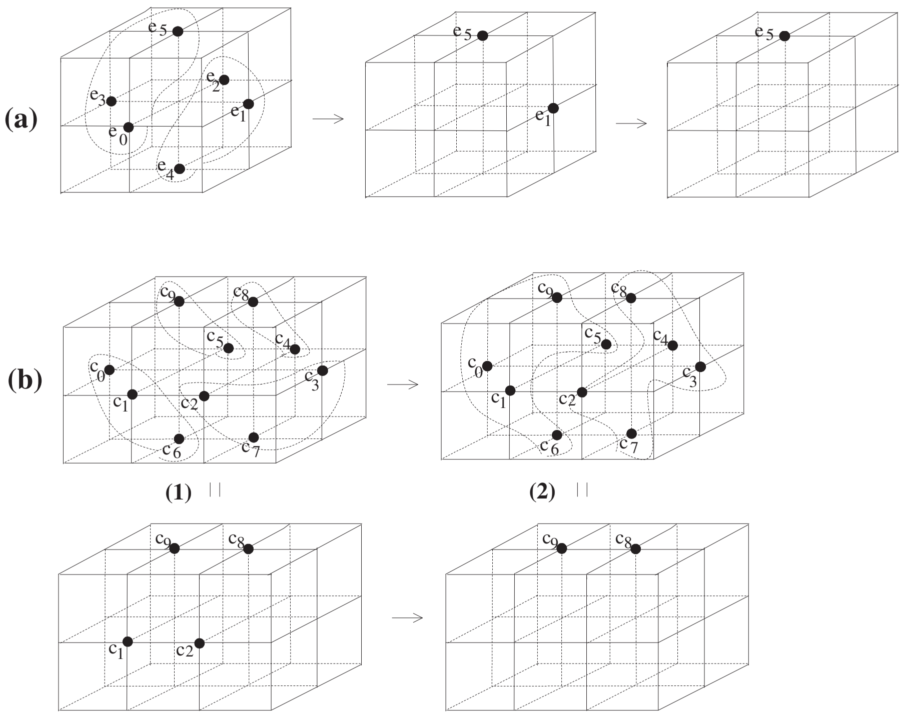

Figure 2. Configuration of the pointed 18-contractibility of MSS'_{18} (**a**) and MSS_{18} (**b**).

Indeed, we point out that the digital 6-, 18-, and 26-sphere-like models MSS_6, MSS_{18}, and $MSS'_{18} := MSS'_{26}$ in Figure 1 were firstly introduced in References [3,4].

Remark 2. *Let T be the set $X \times [0,1]_{\mathbb{Z}}$, where $X = \{c_0 = (0,0), c_1 = (1,0), c_2 = (1,1), c_3 = (0,1)\}$. Then, we obtain the following:*
(1) the digital image $(T,6)$ is not a closed 6-surface.
(2) $(T,6)$ is pointed 6-contractible.

Proof. (1) For any point $t \in T$, the set $\mid T \mid^t$ does not satisfy the properties Definition 6(1) (b) and (c).

(2) Using a method similar to the homotopy of Equation (8), we observe that there is a 6-homotopy relative to any singleton $\{t\} \subset T$ between the identity map 1_T and the constant map $C_{\{t\}}$. Thus, we can conclude that $(T,6)$ is pointed 6-contractible relative to any singleton $\{t\} \subset T$. $\square$

3. A Geometric Realization of a Simple Closed k-Surface

In order to address questions (Q1) and (Q2) in Section 1, given an S_k and for each point $x \in S_k$, we need to establish a special kind neighborhood of x matching an open neighborhood of a certain point of a typical surface (or a 2-dimensional topological manifold). Indeed, given a digital image (X, k) in $\mathbb{Z}^3$, for each point $x \in X$, the set $| X|^x$ (see Equation (6)) plays an important role in examining if (X, k) is a simple closed k-surface in $\mathbb{Z}^3$ (see Definition 6). This approach is quite different from one examining if a topological space becomes a typical surface from the viewpoint of manifold theory. However, motivations of the two approaches are similar to each other. Roughly saying, each point x of a 2-dimensional topological manifold (or a surface) (X, T_X) has an open neighborhood in (X, T_X) which is homeomorphic to an open disc in the 2-dimensional Euclidian topological space $(\mathbb{R}^2, U)$. In digital surface theory, we also follow this kind of approach under a certain digital situation.

In order to study the Euler characteristics of a simple closed k-surface and a connected sum of two simple closed k-surfaces (see Reference [5]), let us now recall a geometric realization of a digital image (X, k). For a digital image (X, k) and each point $x(\in X)$, owing to the set $| X|^x \cup \{x\}$, a special kind of geometric realization can be considered. However, in digital surface theory, we have some difficulties in establishing the so-called '*digital k-neighborhood of a point*' in (X, k) matching an open neighborhood of a point in a typical surface. Thus, motivated by the fact that, for an S_k and $x \in S_k$, we observe that $| S_k |^x$ is an essentially important set guaranteeing the closed k-surface structure of the S_k (see also Remarks 5 and 6). Motivated by this observation, let us now treat this issue with a special kind of idea overcoming the difficulties. Roughly saying in advance, given a simple closed k-surface S_k in $\mathbb{Z}^3$, consider it as a digital k-graph, denoted by G_k. First of all, let us take all minimal k-cycles in

$$(N_{26}(x)^* \cup \{x\}) \cap S_k, \tag{10}$$

denoted by $M_k(x)$ (see Definition 8). Hereafter, we need to remind that the set of Equation (10) is the set $N_{26}(x, 1)$ in S_k. Next, we formulate a certain 2-dimensional simplicial complex in the 3-dimensional real space, $\mathbb{R}^3$, say $D_k(x)$, inherited from $M_k(x)$ (see Definition 9). More precisely, each 2-dimensional simplex in $D_k(x)$ is a polygon in $\mathbb{R}^3$ formulated by the corresponding minimal k-cycle in $M_k(x)$ (see Definitions 8 and 9). Consequently, we have a geometric realization of S_k, denoted by $| S_k |$, which is the union of all $D_k(x), x \in S_k$ (see Definition 10). Then, we can observe that $| S_k |$ is indeed a closed 2-dimensional simplicial complex, i.e., a sphere-like polygon in $\mathbb{R}^3$ (see Proposition 2).

Since the present paper focuses on the study of several types of connected sums of the simple closed k-surfaces MSS_6, MSS_{18}, and MSS'_{18}, hereafter, we only deal with simple closed k-surfaces in $\mathbb{Z}^3$, denoted by S_k. As mentioned above, given an S_k, let us now propose the sets $M_k(x)$ (see Definition 8) and $D_k(x)$ (see Definition 9) derived by the set $| S_k |^x, x \in S_k$.

Definition 8. *Given a simple closed k-surface S_k in $\mathbb{Z}^3$, for $x \in S_k$, let*

$$M_k(x) := \{C_k \mid C_k \textit{ be a minimal k-cycle in } (N_{26}(x)^* \cup \{x\}) \cap S_k.\}. \tag{11}$$

The set $M_k(x)$ has its own features, as follows:

Remark 3. *(1) Each of the minimal k-cycles in $M_k(x)$, say C_k, associated with Equation (11) need not be a simple closed k-cycle in S_k (see the 18-curve consisting of c_0, c_1, c_9 of MSS_{18} in Figure 3a).*
(2) The term "minimal" comes from the 'minimal k-cycles' in S_k taken from the only one of the eight digital cubes

$$\prod_{i=1}^{3} [x_i, x_i \pm 1]_{\mathbb{Z}},$$

where $x = (x_1, x_2, x_3) \in S_k$.
(3) The element C_k in $M_k(x)$ need not contain the point x (see Example 1(3)).

(4) Not every k-cycle C_k in $M_k(x)$ is $SC_k^{3,l}$, $l \geq 4$ (see Example 1).
(5) $M_k(x)$ may contain several k-cycles with different types depending on the situation (see Example 1).

Example 1. *(1) In MSS_{18}, for the point c_1 in MSS_{18} in Figure 3a, we have $M_{18}(c_1)$ as the set consisting of four 18-cycles,*

$$(c_0, c_1, c_9), (c_0, c_1, c_6), (c_1, c_2, c_8, c_9), \text{ and } (c_1, c_2, c_7, c_6).$$

(2) In MSS'_{18}, for the point e_0 in MSS'_{18} in Figure 3b, we obtain $M_{18}(e_0)$ as the set being composed of four 18-cycles,

$$(e_0, e_3, e_5), (e_0, e_3, e_4), (e_0, e_1, e_5), \text{ and } (e_0, e_1, e_4).$$

(3) In MSS_6, for the point d_0 in Figure 3c, we have $M_6(d_0)$ as the set consisting of twelve 6-cycles with four elements,

$$(d_0, d_7, d_8, d_9), (d_0, d_1, d_{10}, d_9), (d_1, d_2, d_{11}, d_{10}), \text{ and so on.}$$

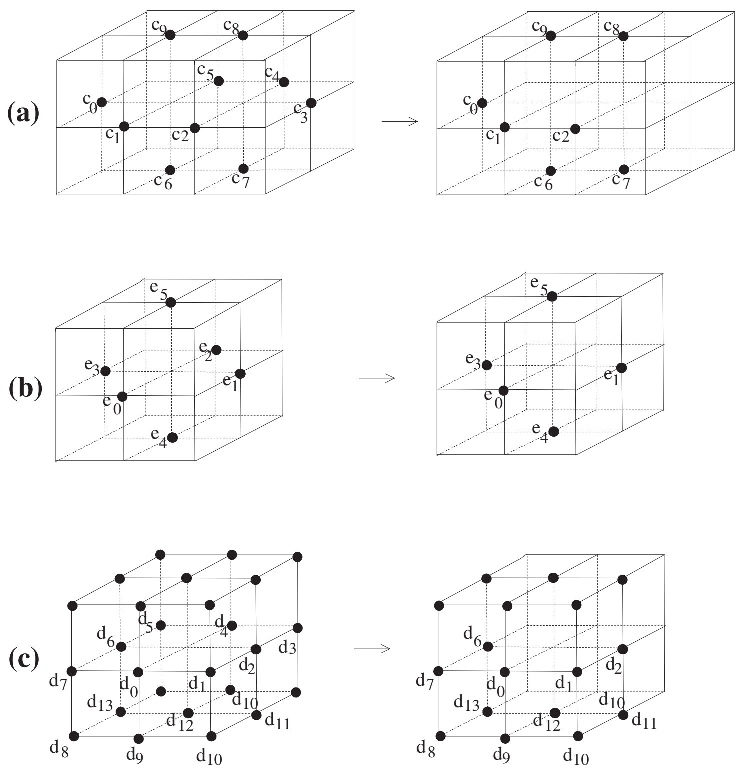

Figure 3. (**a**) Configuration of the elements of $M_{18}(c_1)$ in MSS_{18} for the point $c_1 \in MSS_{18}$; (**b**) explanation of the elements of $M_{18}(e_0)$ in MSS'_{18} for the point $e_0 \in MSS'_{18}$ (or MSS'_{26}); and (**c**) configuration of the elements of $M_6(d_0)$ for the point $d_0 \in MSS_6$.

Indeed, using each minimal k-cycle in $M_k(x)$, we can produce a certain polygon (a solid triangle or a solid rectangle) in $\mathbb{R}^3$. For instance, in Example 1(a)(1), the given 18-cycle (c_0, c_1, c_9) produces a solid triangle, say $< c_0, c_1, c_9 > := \overline{(c_0, c_1, c_9)}$, and further, the 18-cycle (c_1, c_2, c_8, c_9) leads to a solid rectangle, say $< c_1, c_2, c_8, c_9 > := \overline{(c_1, c_2, c_8, c_9)}$ in $\mathbb{R}^3$. Motivated by this approach, we can define the following:

Definition 9. *Given a simple closed k-surface S_k in $\mathbb{Z}^3$, for a point $x \in S_k$, let*

$$D_k(x) := \bigcup_{C_k \in M_k(x)} \overline{C_k},$$

where $\overline{C_k}$ means the polygon formulated by the minimal k-cycle $C_k \in M_k(x)$. Indeed, $D_k(x)$ is the set as the union of polygons (solid triangles or solid rectangles) formulated by the minimal k-cycles in $M_k(x)$. Then, we say that $D_k(x)$ is a geometric realization of $M_k(x)$.

In Definition 9, we observe that each minimal k-cycle C_k in $M_k(x)$ produces only a polygon as a subset of $D_k(x)$ in $\mathbb{R}^3$. Thus, it turns out that $D_k(x)$ is a simplicial complex inherited from $M_k(x)$ (see Example 2).

Owing to the definition of $M_k(x)$, for an $S_k, x \in S_k$ in $\mathbb{Z}^3$, it is obvious that the set $D_k(x)$ consists of triangles or rectangles with boundary in the subspace $(D_k(x), U_{D_k(x)})$, where $U_{D_k(x)}$ is the subspace topology induced by the 3-dimensional Euclidean topological space $(\mathbb{R}^3, U)$. For instance, based on the set $M_k(x)$ in Figure 3, we obtain the following:

Example 2. *Depending on the points x in MSS_{18}, MSS'_{18} (or MSS'_{26}), or MSS_6, according to $M_k(x)$ in Figure 3, we have $D_k(x), k \in \{6, 18\}$, as follows:*

(1) Based on MSS_{18}, we observe that $D_{18}(c_1)$ is the set as the union of polygons formulated by the 18-cycles in $M_{18}(c_1)$, i.e., the union of the two triangles with boundary generated by the two 18-cycles (c_0, c_1, c_9) and (c_0, c_1, c_6) and the two rectangles with boundary formulated by the 18-cycles (c_1, c_2, c_8, c_9) and (c_1, c_2, c_7, c_6).

(2) Based on MSS'_{18}, we observe that $D_{18}(e_0)$ is the set which is the union of four triangles with boundary formulated by four 18-cycles in $M_{18}(e_0)$.

(3) In terms of the methods used in Equations (1) and (2), based on MSS_6, we observe that $D_6(d_0)$ is the set as the union of twelve polygons (or regular rectangles) formulated by the twelve 6-cycles in $M_6(d_0)$.

Given an S_k, using $D_k(x), x \in S_k$, let us now establish a geometric realization of S_k, as follows:

Definition 10. *Given a simple closed k-surface S_k in $\mathbb{Z}^3$, let*

$$\mid S_k \mid = \bigcup_{x \in S_k} D_k(x). \tag{12}$$

Then, we call the set $\mid S_k \mid$ the geometric realization of S_k.

Proposition 2. *Given a simple closed k-surface S_k in $\mathbb{Z}^3$, the geometric realization $\mid S_k \mid$ is uniquely determined as a connected 2-dimensional simplicial complex (or a sphere-like polyhedron) in $\mathbb{R}^3$.*

Proof. Given a simple closed k-surface S_k in $\mathbb{Z}^3$, for each point $x \in S_k$, it is obvious that the set $D_k(x) (\subset \mathbb{R}^3)$ is a simplicial complex consisting of triangles or rectangles with boundaries.

For two k-adjacent points x_1 and x_2 in S_k, $D_k(x_1)$ and $D_k(x_2)$ have a non-empty intersection, i.e.,

$$D_k(x_1) \cap D_k(x_2) \neq \emptyset, \tag{13}$$

which implies that $\mid S_k \mid$ is connected. To be precise, the intersection $D_k(x_1) \cap D_k(x_2)$ of Equation (13) is the union of the 2-dimensional simplexes (or polygons) derived from the minimal k-cycles in

$$M_k(x_1) \cap M_k(x_2) = [N_{26}(x_1)^* \cup \{x_1\}] \cap [N_{26}(x_2)^* \cup \{x_2\}] \cap S_k.$$

Namely, we observe the identity

$$M_k(x_1) \cap M_k(x_2) = N_{26}(x_1, 1) \cap N_{26}(x_2, 1) \text{ in } S_k.$$

Thus, $D_k(x_1) \cap D_k(x_2)$ has some 2-dimensional simplexes (or polygons) in common from each of them. Since S_k is k-connected, for any two k-adjacent points in S_k, using the property of Equation (13), we can formulate a connected 2-dimensional simplicial complex because $D_k(x)$ is a 2-dimensional simplicial complex (see Definition 9), as follows

$$|\,S_k\,| := \bigcup_{x \in S_k} D_k(x) \text{ as in (12)}$$

from the given S_k according to Definition 9. To be precise, $|\,S_k\,|$ has 0-dimensional simplexes derived from each of all elements in S_k. The 1-dimensional simplexes of $|\,S_k\,|$ are all line segments formulated by all two k-adjacent points in S_k. Finally, the 2-dimensional simplexes of $|\,S_k\,|$ come from the polygons in $D_k(x), x \in S_k$. Obviously, owing to the definition of S_k and the notion of $|\,X\,|^x$ (see Definition 6), there is no n-dimensional simplex in $|\,S_k\,|, n \geq 3$. $\square$

According to Proposition 2, we obtain the following:

Remark 4. *Given a simple closed k-surface S_k in $\mathbb{Z}^3$, $k \in \{6, 18, 26\}$, the geometric realization $|\,S_k\,|$ is a sphere-like polyhedron in $\mathbb{R}^3$.*

Eventually, given an S_k, $|\,S_k\,|$ is obtained in terms of the following process.

$$S_k \to G_k \to \{M_k \,|\, x \in S_k\} \to \{D_k \,|\, x \in S_k\} \to |\,S_k\,|. \tag{14}$$

The term "generated" (see Definition 12 of Reference [5]) in Reference [5] means the process in Equation (14).

Remark 5. *In view of Definitions 9 and 10, we observe that, given an S_k and a point $x \in S_k$, the set $Int(D_k(x))(\subset \mathbb{R}^3)$ can be considered a open neighborhood of x in $|\,S_k\,|$, where the term as "Int" means the interior operator in the subspace $(|\,S_k\,|, U_{|\,S_k\,|})$, where $U_{|\,S_k\,|}$ is the subspace topology on $|\,S_k\,|$ induced by the 3-dimensional Euclidean topological space $(\mathbb{R}^3, U)$. This approach can facilitate the study of some objects in digital surface theory.*

Remark 6 (Importance of the sets $M_k(x)$ and $D_k(x)$ with respect to a geometric realization of an S_k)**.** *Unlike a typical surface (or a 2-dimensional topological manifold) in the Euclidean topological space $(\mathbb{R}^3, U)$, we observe that, given an S_k and $x \in S_k$ motivated by the set, $|\,S_k\,|^x$, the sets $M_k(x)$ and $D_k(x), x \in S_k$ play important roles in establishing a geometric realization of the given S_k.*

Owing to Proposition 2 and the process of Equation (14), given an S_k in $\mathbb{Z}^3$, we obtain $|\,S_k\,|$ (see Equation (13)) as a typical polyhedron without boundary in the subspace $(|\,S_k\,|, U_{|\,S_k\,|})$ (see Remark 4).

Example 3. *For MSS_{18} in Figure 1a, we see that $|\,MSS_{18}|$ seems like to be a small rugby ball.*

4. Euler Characteristics for Digital k-Surfaces and Connected Sums of Closed k-Surfaces

In order to address questions (Q3) and (Q4) in Section 1 and, further, to exactly understand the notion of Euler characteristic of a simple closed k-surface S_k in $\mathbb{Z}^3$ (see Reference [5]), we now stress that the geometric realization of S_k, $|\,S_k\,|$, is a sphere-like 2-dimensional simplicial complex generated by the set $\{D_k(x) \,|\, x \in S_k\}$ (see Proposition 2).

Given an S_k in $\mathbb{Z}^3$, a '2-*dimensional digital k-simplex*' in $\mathbb{Z}^3$ is obviously defined as the set $\{x_0, x_1, x_2\}$ contained in $N_k(x_i, 1) \subset S_k, i \in \{0, 1, 2\}$ (see Equation (5)) such that each of two elements of $\{x_0, x_1, x_2\}$ are k-adjacent (see Reference [23]). Moreover, a '2-*dimensional k-simplex*' in $\mathbb{R}^3$ is said to be a solid triangle formulated by a "2-dimensional digital k-simplex". Then, we can recognize some differences between a 2-dimensional digital k-simplex (*resp.* a 2-dimensional k-simplex) and an element of $M_k(x)$ (*resp.* $D_k(x)$), as follows:

Remark 7. *Given an $S_k, k \in \{6, 18, 26\}$, not every element of $M_k(x)$ becomes a 2-dimensional digital k-simplex in S_k.*

Proof. Consider MSS_{18} in Figure 3a. Whereas the set $M_{18}(c_1)$ contains the minimal 18-cycle (c_1, c_2, c_8, c_9) in MSS_{18} (see Figure 3a), the 18-cycle (c_1, c_2, c_8, c_9) is not a 2-dimensional digital 18-simplex in MSS_{18}. □

Based on the $M_{18}(x), x \in MSS_{18}$ in Figure 3a, we observe that, although the set $D_{18}(c_1)$ contains a rectangle (or a polygon), say $< c_1, c_2, c_8, c_9 > := \overline{(c_1, c_2, c_8, c_9)}$, generated by the minimal 18-cycle (c_1, c_2, c_8, c_9) in MSS_{18}, the rectangle $< c_1, c_2, c_8, c_9 >$ in $D_{18}(c_1)$ is not formulated by any 2-dimensional digital 18-simplex in MSS_{18}.

Example 4. *For the simple closed 18-surface MSS_{18} in Figure 1a, we have twelve polygons in $\mid MSS_{18} \mid = \bigcup_{c_i \in MSS_{18}} D_{18}(c_i)$ generated by the set consisting of the following 18-cycles in $\bigcup_{c_i \in MSS_{18}} M_{18}(c_i)$*

$$\left\{ \begin{array}{l} \{c_0, c_1, c_9\}, \{c_1, c_2, c_8, c_9\}, \{c_2, c_3, c_8\}, \{c_3, c_4, c_8\}, \\ \{c_4, c_5, c_9, c_8\}, \{c_0, c_5, c_9\}, \{c_0, c_1, c_6\}, \{c_1, c_2, c_7, c_6\}, \\ \{c_2, c_3, c_7\}, \{c_3, c_4, c_7\}, \{c_4, c_5, c_6, c_7\}, \{c_0, c_5, c_6\}. \end{array} \right\} \tag{15}$$

However, in MSS_{18}, we have only eight 2-dimensional digital 18-simplices, such as

$$\left\{ \begin{array}{l} \{c_0, c_1, c_9\}, \{c_2, c_3, c_8\}, \{c_3, c_4, c_8\}, \{c_0, c_5, c_9\}, \\ \{c_0, c_1, c_6\}, \{c_2, c_3, c_7\}, \{c_3, c_4, c_7\}, \{c_0, c_5, c_6\}. \end{array} \right\}$$

Thus, the simplicial complex generated by the eight 2-dimensional 18-simplex is quite different from the geometric realization $\mid MSS_{18} \mid$ of [5] (or the current geometric realization $\mid MSS_{18} \mid$).

Remark 8 (Limitations of the approach of an Euler characteristic in Reference [14])**.** *Given an S_k referred to in Example 4, Reference [14] considered only the simplicial complexes formulated by only 2-dimensional digital k-simplexes on S_k. Then, given an S_k in $\mathbb{Z}^3$, it is obvious that it need not produce a polyhedron in $\mathbb{R}^3$. To be precise, according to the approach of Reference [14], since each of the sets*

$$\{c_1, c_2, c_8, c_9\}, \{c_1, c_2, c_7, c_6\}, \{c_4, c_5, c_9, c_8\}, \{c_4, c_5, c_6, c_7\}$$

is not a 2-dimensional digital 18-simplex, the simplicial simplex induced by the 2-dimensional 18-simplices in MSS_{18} is not even a polyhedron in $\mathbb{R}^3$.

For instance, suppose the set generated by only the 2-dimensional 18-simplices in Reference [14]

$$\left\{ \begin{array}{l} \{c_0, c_1, c_9\}, \{c_2, c_3, c_8\}, \{c_3, c_4, c_8\}, \{c_0, c_5, c_9\}, \\ \{c_0, c_1, c_6\}, \{c_2, c_3, c_7\}, \{c_3, c_4, c_7\}, \{c_0, c_5, c_6\}. \end{array} \right\} \tag{16}$$

Then, the union of all polygons inherited from these eight 18-cycles is not a polyhedron in $\mathbb{R}^3$. This implies that, given an S_k, the geometric approximation referred to in Reference [14] does not support a transformation from S_k to a certain sphere-like polyhedron in $\mathbb{R}^3$. Hence, comparing Definition 10, the approach of Reference [14] is very restrictive.

Hence, the current notions of $M_k(x)$ and $D_k(x)$ in Definitions 8 and 9 are substantially required in digital surface theory.

Using both the digital k-graph theoretical method in References [4,10] and the notions of $D_k(x)$ and $M_k(x), x \in S_k$, inherited from G_k (see Definitions 8 and 10 of the present paper), we can define the Euler characteristic of an S_k. Owing to Definition 10 and Proposition 2, we can make the definition for Euler characteristic of S_k in Reference [5] clear in the following way.

Definition 11 ([5])**.** *For an S_k, the Euler characteristic of S_k is defined by*

$$\mathcal{E}(S_k) = \mathcal{E}(|S_k|),$$

where $\mathcal{E}(|S_k|) = V - E + F$ and V means the number of the vertexes of $|S_k|$, E is the number of the k-edges of $|S_k|$, and F is the number of the polygons in $|S_k| = \bigcup_{x \in S_k} D_k(x)$.

Remark 9 (Advantages of the approach of the current Euler characteristic of an S_k (see also Reference [5])**.** *The approach using Definition 11 is consistent with the research of the Euler characteristic of a typical closed surface from algebraic topology and polyhedron geometry.*

Namely, for simple closed k-surfaces in $\mathbb{Z}^3$, the following assertion in Reference [5] is right with the same proof as in Reference [5] according to the Definition 11 and Proposition 2.

Proposition 3. *Given an S_k, we obtain $\mathcal{E}(S_k) = 2$.*

Proof. Using Proposition 2, given an S_k, since $|\, S_k \,|$ is a sphere-like polyhedron in $\mathbb{R}^3$, we obtain $\mathcal{E}(S_k) = 2$. □

Example 5. *(1) $\mathcal{E}(MSS_{18}) = 10 - 20 + 12 = 2$.*
(2) $\mathcal{E}(MSS_6) = 26 - 48 + 24 = 2$.

Owing to Proposition 2, the digital analogue of the Euler characteristic of a connected sum in typical topology in References [28,29] was developed. Indeed, in Reference [3], we stated the simple closed k-surface structure of a connected sum of two simple closed k-surfaces (see Theorem 5.4 of Reference [3]). However, in order to use this fact in the present paper, we need to prove it more precisely, as follows:

Theorem 1. *Given two simple closed k-surfaces S_k and S'_k, $S_k \sharp S'_k$ is a simple closed k-surface in $\mathbb{Z}^3$.*

Proof. (Case 1) In the case $(k, \bar{k}) \in \{(26,6), (6,26)\}$, we observe that, for each point $x \in S_k \sharp S'_k$, $|\, S_k \sharp S'_k \,|^x$ has exactly one k-component k-adjacent to x. Moreover, $|\overline{S_k \sharp S'_k}|^x$ has exactly two $\bar{k}$-components $\bar{k}$-adjacent to x. We denote these two components with C^{xx} and D^{xx}. Finally, for any point $y \in N_k(x) \cap (S_k \sharp S'_k)$ (or $N_k(x,1)$ in $S_k \sharp S'_k$), we obtain

$$N_{\bar{k}}(y) \cap C^{xx} \neq \phi \text{ and } N_{\bar{k}}(y) \cap D^{xx} \neq \phi.$$

Since $S_k \sharp S'_k$ does not have any simple k-point, $S_k \sharp S'_k$ is a simple closed k-surface.

(Case 2) In the case $(k, \bar{k}) = (18, 6)$, $S_k \sharp S'_k$ is obviously k-connected. Moreover, for each point $x \in S_k \sharp S'_k$, in view of the process of $S_k \sharp S'_k$ (see Definition 7 and Remark 1), $|S_k \sharp S'_k|^x$ is exactly a simple closed k-curve, which $S_k \sharp S'_k$ is a simple closed k-surface. □

Using Definition 11 and Theorem 1, as roughly proved in Reference [3], we obtain the following:

Corollary 1. *(1) $\mathcal{E}(S_k \sharp S'_k) = \mathcal{E}(S_k) + \mathcal{E}(S'_k) - 2$ [4].*
(2) $\mathcal{E}(S_k \sharp S'_k) = \mathcal{E}(S_k) = \mathcal{E}(S'_k)$.

Proof. Owing to Theorem 1, the proofs of Equations (1) and (2) are completed. □

In view of Corollary 1, it turns out that the calculations of the Euler characteristics of connected sums of simple closed k-surfaces suggested in Reference [5] obviously hold, as follows:

Example 6. *(1)* $\mathcal{E}(MSS_6 \sharp MSS_6) = 14 - 28 + 16 = 2$.
(2) $\mathcal{E}(MSS_{18} \sharp MSS_{18}) = \mathcal{E}(MSS_{18}) = 2$.
(3) $\mathcal{E}(MSS'_{18} \sharp MSS_{18}) = \mathcal{E}(MSS'_{18}) = \mathcal{E}(MSS_{18}) = 2$.

In digital surface theory, Reference [5] already proved that $MSS_{18} \sharp MSS_{18}$ is simply 18-connected. However, we now need to correct some errors in Reference [5] relating to the calculations of the digital 6-fundamental groups of MSS_6 and $MSS_6 \sharp MSS_6$ in Reference [4]. Indeed, using trivial extensions in Reference [24], the calculations should be proceeded, as follows:

Remark 10. *(1) The* 18*-fundamental group of* MSS_6 *should be calculated as a trivial group as in Reference [14] instead of the free group generated by two cyclic groups (correction of Lemma 3.3(3) of Reference [5]).*

(2) The 6*-fundamental group of* $MSS_6 \sharp MSS_6$ *should be calculated as a trivial group as in Reference [14] instead of the free group generated by two cyclic groups (correction of Theorem 3.4(1) of Reference [5]).*

5. The (Almost) Fixed Point Property for Digital *k*-Surfaces and Connected Sums of Closed *k*-Surfaces

In order to address the query (Q5) in Section 1, let us now recall the fixed point property and the almost fixed point property from the viewpoint of digital topology.

- We say that a digital image (X, k) in $\mathbb{Z}^n$ has the *fixed point property* (*FPP*) [30] if, for every k-continuous map $f : (X, k) \to (X, k)$, there is a point $x \in X$ such that $f(x) = x$.
- We say that a digital image (X, k) in $\mathbb{Z}^n$ has the *almost fixed point property* (*AFPP*) [30,31] if, for every k-continuous self-map f of (X, k), there is a point $x \in X$ such that $f(x) = x$ or $f(x)$ is k-adjacent to x. In general, we observe that the *AFPP* is a more generalized concept than the *FPP*.

By Proposition 3 and Example 5, it turns out that each of $\mathcal{E}(MSS_6)$, $\mathcal{E}(MSS_{18})$, and $\mathcal{E}(MSS'_{18})$ is not trivial. Despite this situation, in this section, we prove that each of MSS_6, MSS_{18}, and MSS'_{18} does not have the *FPP* (see Theorem 2).

Theorem 2. *(1)* MSS_{18} *does not have the AFPP.*
(2) MSS'_{18} *does not have the AFPP.*

Proof. (1) Let us consider the self-bijection f of MSS_{18} using the composite of the three different types of reflections F_1, F_2, and F_3 of MSS_{18}, as follows (see Figure 4a):

$$F_1, F_2, F_3 : MSS_{18} \to MSS_{18}$$

is defined as

$$\left\{ \begin{array}{l} F_1 : c_6 \leftrightarrow c_9, c_7 \leftrightarrow c_8 \text{ and } F_1(x) = x, \text{ for } x \in MSS_{18} \setminus \{c_6, c_7, c_8, c_9\}, \\ F_2 : c_1 \leftrightarrow c_5, c_2 \leftrightarrow c_4, \text{ and } F_2(x) = x, \text{ for } x \in MSS_{18} \setminus \{c_1, c_2, c_4, c_5\}, \\ F_3 : c_0 \leftrightarrow c_3, F_1(c_6) \leftrightarrow F_1(c_7), F_1(c_9) \leftrightarrow F_1(c_8), F_2(c_1) \leftrightarrow F_2(c_2), F_2(c_5) \leftrightarrow F_2(c_4), \end{array} \right\} \quad (17)$$

where the notation "$p \leftrightarrow q$" in Equation (17) means a mapping from p to q and vice versa by using the given maps, F_1, F_2, and F_3, where $p, q \in MSS_{18}$.

Then, we observe that the maps F_1, F_2, and F_3 are special kinds of reflections which are 18-continuous self-bijections of MSS_{18}. Then, we obtain the composite

$$f := F_3 \circ F_2 \circ F_1 : MSS_{18} \to MSS_{18}$$

which is also an 18-continuous self-bijection of MSS_{18} (see Figure 4a). However, the map f does not support the *AFPP* for MSS_{18}. To be precise, we observe that there is no point $x \in MSS_{18}$ such that $f(x) = x$ or $f(x)$ is 18-adjacent to x.

(2) Let us consider the self-bijection g of MSS'_{18} in the following way (see Figure 4b):

$$g : MSS'_{18} \to MSS'_{18}$$

is defined as

$$\begin{Bmatrix} g(e_0) = e_2, g(e_2) = e_0, g(e_1) = e_3, g(e_3) = e_1, g(e_4) = e_5, g(e_5) = e_4, \\ i.e., e_0 \leftrightarrow e_2, e_1 \leftrightarrow e_3, e_4 \leftrightarrow e_5. \end{Bmatrix} \tag{18}$$

Then, we observe that the map g is an 18-continuous bijection on MSS'_{18}. However, we find that the map g does not support the *AFPP* for MSS'_{18}. To be specific, we observe that there is no point $x \in MSS'_{18}$ such that $g(x) = x$ or $g(x)$ is 18-adjacent to x. □

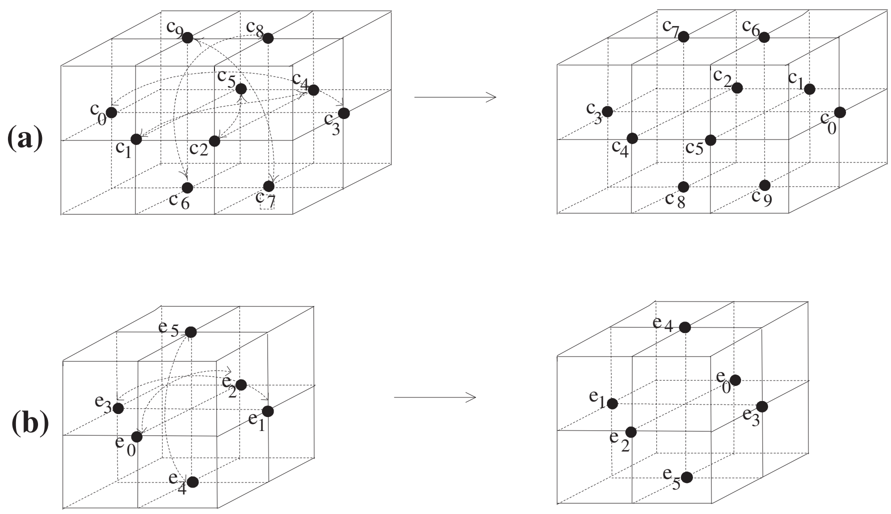

Figure 4. Explanations of the non-almost fixed point property (*AFPP*) of MSS_{18} (**a**) MSS'_{18} (or MSS'_{26}) (**b**).

It is obvious that MSS_6 does not have the *FPP*. However, owing to Proposition 2 and Definition 11, we obtain $\mathcal{E}(MSS_6) = 2$. Thus, owing to Proposition 3, Theorem 2 and Examples 5 and 6, we have the following:

Corollary 2. *For an S_k, the non-triviality of the Euler characteristic of an S_k implies neither the FPP nor the AFPP of the given S_k.*

As stated above, in view of the feature of the Euler characteristics of digital k-surfaces, we can stress that the study of fixed point theory using the current Euler characteristic is quite different from the approach of the typical fixed point theory. Moreover, based on Remarks 5 and 6, we can also point out the following:

Remark 11. *(1) The authors of Reference [14] studied a certain Euler characteristic of a digital image (X, k) using digital homology groups of (X, k) (see Section 6 of Reference [14]). Moreover, they urged to establish some connection between the Euler characteristic of a k-surface in Reference [5] (or the current version of Definition 11 in the present paper) and their Euler characteristic using the digital homology referred to in Reference [14]. However, it turns out that they are totally different. Thus, in view of Theorem 2 and Corollary 2, their assertions in Reference [14] involving the Euler characteristic of S_k of Reference [5] are too far from the approach of Reference [5] (or the current one). Indeed, we find that their approach in Reference [14] is irrelevant to the current Euler characteristic in Reference [5] (or the current one).*

(2) In view of Remarks 8 and 9, Definition 11, and Theorem 2, the current Euler characteristic of S_k facilitates the study of digital surfaces of from the viewpoints of digital surface and typical surface theories.

Remark 12. *(1) The digital homology in Reference [14] is indeed quite different from the typical homology group in algebraic topology (for more details, see Section 1 of Reference [32]). Furthermore, the digital homology referred to in Reference [14] is also different from the simplicia homology in algebraic topology. Moreover, in view of this situation, the comment in Reference [14] involving his approach to the Euler characteristic of an S_k with the Euler characteristic in Reference [5] (or the current approach) can be incorrect.*

(2) The digital 6-, 18-, and 26-sphere-like models, MSS_6, MSS_{18}, MSS'_{18} (or MSS'_{26}, in Figure 1 were firstly introduced in Reference [3,4]. However, the authors used them in Reference [14] with their attribution.

6. Conclusions and a Further Work

The present paper intensively explained the process of a geometric realization of an S_k in $\mathbb{Z}^3$. Using this frame, we showed that the current Euler characteristic of a simple closed k-surface is consistent with that of the typical surface in algebraic topology. Indeed, we also confirmed that the set $Int(D_k(x))$ (see Definition 9) plays important role in establishing $\mid S_k \mid$ (see Remark 5). Moreover, we also have proved that the simple closed 18-surfaces MSS_{18} and MSS'_{18} do not have the *AFPP*. Finally, it turns out that the non-triviality of the Euler characteristics of simple closed k-surfaces, MSS_6, MSS_{18}, and MSS'_{18}, implies neither the *FPP* nor the *AFPP* (see Theorem 2).

The recent paper [33] established many kinds of digital topological structures on $\mathbb{Z}^n$ which are not homeomorphic to the n-dimensional Khalimsky topological space. Moreover, References [34,35] developed the notion of digital rough approximations using Khalimsky and Marcus–Wyse topological structure. As a further work, using the methods in References [33,36], we can further study the following:

- a development of a new type digital surface associated with a Khalimsky manifold.
- fixed point theory for many kinds of digital topological structures on $\mathbb{Z}^n$ in [33].
- given a typical surface X in pure topology and geometry, after developing a new type of LF-topological structure on X, $T(X)$, we can explore some connections related to Euler characteristics between X and $T(X)$.
- after improving the earlier digital homology groups [14] for digital images, we can propose some relationships between the current Euler characteristic and a certain invariant involving new homology groups for digital closed k-surfaces.

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