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Explicit Solutions of Initial Value Problems for Linear Scalar Riemann-Liouville Fractional Differential Equations With a Constant Delay

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Abstract: In this paper, we study Linear Riemann-Liouville fractional differential equations with a constant delay. The initial condition is set up similarly to the case of ordinary derivative. Explicit formulas for the solutions are obtained for various initial functions.

Keywords: Riemann-Liouville fractional derivative; constant delay; initial value problem; linear fractional equation; explicit solution

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1. Introduction

Recently, fractional differential equations have been used as more adequate models of real world problems in engineering, physics, finance, etc. ([1,2]). Usually, fractional differential equations are considered as a generalization of ordinary differential equations and delay equations. Research findings have revealed that many models based only on integer order derivatives do not provide enough information to describe the complexity of real world phenomena. In comparison to integer-order derivatives, there are several kinds of definitions for fractional derivatives. These definitions are generally not equivalent to each other; the main ones in the literature are the Caputo fractional derivative and the Riemann-Liouville (RL) fractional derivative. Comparing both definitions we mention that the set of functions which are differentiable in the RL sense is wider than the set of functions which are differentiable in the Caputo sense. A good overview of properties of fractional derivatives and potential applications of the RL derivative for modeling in science and engineering is given in [3] and physical and geometric interpretations for the RL derivative can be found in [1]. In addition, many real world processes and phenomena are characterized by the influence of past values of the state variable on the recent one and this leads to the inclusion of delays (finite, variable, state dependent, etc.) in the models. The analysis of delay fractional differential equations can be rather complex (analytical solution computation, controllability analysis, etc.). Recently, there were developments on seeking the explicit formula of solutions to linear delay fractional differential equations. Li and Wang [4] studied the linear homogenous Caputo fractional delay differential equations and gave a representation of the solution. Also, in [5,6] representations of the solution of linear non-homogeneous Caputo fractional delay differential equations are provided.

However, this is not the situation with Riemann-Louisville (RL) fractional differential equations with delays. The RL fractional differential equations with delays are not well studied. We mention the papers [7,8] where the explicit formula for the linear RL fractional equations is given but the initial condition does not correspond to the idea of the case of ordinary differential equations. In these papers the lower bound of the RL fractional derivative coincides with the left side end of the initial interval.

In this paper we study initial value problems of scalar linear RL fractional differential equations with constant delay. Similar to the case of the ordinary derivative, the differential equation is given to the right of the initial time interval. It requires the lower bound of the RL fractional derivative to coincide with the right side end of the initial interval (usually this point is zero). Note that in this case any solution of an initial problem (IVP) with RL fractional derivatives is not continuous at this point. That is why RL fractional delay differential equations are convenient for modeling process with impulsive types of initial conditions. This type of processes can be found in physics, chemistry, engineering, biology, and economics. To determine the law of the initial impulsive reaction we need to add to the usual initial condition (for example, $x(t) = \phi(t)$ on the initial interval $[-\tau, 0]$, $\tau > 0$ is the delay) a fractional condition. This conclusion is based on the results obtained in [1,9] concerning the physical interpretation of the RL fractional derivatives and initial conditions which include derivatives of the same kind. Based on the above we set up appropriate IVPs for RL linear fractional differential equations with lower limit of the RL derivative equal to the right side point of the initial interval. Explicit formulas for the solutions of the initial value problems with both zero and nonzero initial functions are obtained. Also, the cases of homogeneous as well as non-homogeneous equations are studied.

2. Preliminary Notes on Fractional Derivatives and Equations

Let $m \in L_1^{loc}([t_0, T], \mathbb{R})$ and $t_0, T \geq 0: \; t_0 < T \leq \infty$. In this paper we will use the following definitions for fractional derivatives and integrals:

- *Riemann-Liouville fractional integral* of order $q \in (0,1)$ ([10,11])

$$ {}_{t_0}I_t^q m(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} \frac{m(s)}{(t-s)^{1-q}} ds, \qquad t \in [t_0, T], $$

  where $\Gamma(.)$ is the Gamma function.

  Note sometimes the notation ${}_{t_0}D_t^{-q} m(t) = {}_{t_0}I_t^q m(t)$ is used.
- *Riemann-Liouville fractional derivative* of order $q \in (0,1)$ ([10,11])

$$ {}_{t_0}^{RL}D_t^q m(t) = \frac{d}{dt}\big({}_{t_0}I_t^{1-q} m(t)\big) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^{t} (t-s)^{-q} \, m(s) ds, \qquad t \in [t_0, T]. $$

We will give fractional integrals and RL fractional derivatives of some elementary functions which will be used later:

Proposition 1. *The following equalities are true:*

$$ {}_{t_0}^{RL}D_t^q (t-t_0)^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-q)} (t-t_0)^{\beta-q}, $$

$$
{}_{t_0}I_t^{1-q}(t-t_0)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(1+\beta-q)}(t-t_0)^{\beta-q},
$$
$$
{}_{t_0}I_t^{1-q}(t-t_0)^{q-1} = \Gamma(q),
$$
$$
{}_{t_0}^{RL}D_t^q(t-t_0)^{q-1} = 0.
$$

The definitions of the initial condition for fractional differential equations with RL-derivatives are based on the following result:

Proposition 2. *(Lemma 3.2 [12]). Let* $q \in (0,1)$, $t_0, T \geq 0 :\ t_0 < T \leq \infty$ *and* $m \in L_1^{loc}([t_0, T], \mathbb{R})$.

(a) *If there exists a.e. a limit* $\lim_{t\to t_0+}[(t-t_0)^{q-1}m(t)] = c$, *then there also exists a limit*

$$
{}_{t_0}I_t^{1-q}m(t)|_{t=t_0} := \lim_{t\to t_0+} {}_{t_0}I_t^{1-q}m(t) = c\Gamma(q).
$$

(b) *If there exists a.e. a limit* ${}_{t_0}I_t^{1-q}m(t)|_{t=t_0} = b$ *and if there exists the limit* $\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)]$ *then*

$$
\lim_{t\to t_0+}[(t-t_0)^{1-q}m(t)] = \frac{b}{\Gamma(q)}.
$$

In the case of a scalar linear RL fractional differential equation we have the following result:

Proposition 3. *(Example 4.1 [12]) The solution of the Cauchy type problem*

$$
{}_{a}^{RL}D_t^q x(t) = \lambda x(t) + f(t), \quad {}_aI_t^{1-q}x(t)|_{t=a} = b
$$

has the following form (formula 4.1.14 [12])

$$
x(t) = \frac{b}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds \tag{1}
$$

where $E_{p,q}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(jp+q)}$ *is the Mittag-Leffler function with two parameters (see, for example, [10]).*

From Proposition 3 and Proposition 2 (a) we obtain the following result for the weighted form of the initial condition:

Proposition 4. *The solution of the Cauchy type problem*

$$
{}_{a}^{RL}D_t^q x(t) = \lambda x(t) + f(t), \quad \lim_{t\to a+}\Big((t-a)^{1-q}x(t)\Big) = C
$$

has the following form

$$
x(t) = \frac{C\,\Gamma(q)}{(t-a)^{1-q}}E_{q,q}(\lambda(t-a)^q) + \int_a^t (t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)f(s)ds. \tag{2}
$$

3. Explicit Formula for the Solutions of Scalar Linear Rl Fractional Equations with Delays and Zero Initial Values

Throughout the paper we will assume $\sum_{i=n}^{l}(*) = 0$ for the integers $n, l: \ n > l$.

3.1. Homogeneous Linear RL Fractional Differential Equation

Consider scalar linear Riemann-Liouville fractional differential equations with a constant delay (HFrDE):

$$ {}_0^{RL}D_t^q x(t) = Bx(t-\tau) \text{ for } t > 0 \tag{3}$$

where $q \in (0,1)$, B, $\tau > 0$ are real constants.

We will consider the zero initial value

$$x(t) = 0 \text{ for } t \in [-\tau, 0], \tag{4}$$

and

$$\lim_{t\to 0+}\left(t^{1-q}x(t)\right) = \frac{1}{\Gamma(q)} \tag{5}$$

or

$${}_0I_t^{1-q}x(t)|_{t=0} := \lim_{t\to 0+} {}_0I_t^{1-q}x(t) = \lim_{t\to 0+}\frac{1}{\Gamma(1-q)}\int_0^t \frac{x(s)}{(t-s)^q}ds = 1. \tag{6}$$

Remark 1. *According to Proposition 2 the conditions (5) and (6) are equivalent.*

According to Remark 1 we will consider only the fractional initial condition (5).

Remark 2. *Note the IVP for HFrDE Equations (3) and (4) with zero fractional initial condition, i.e., $lim_{t\to 0+}\left(t^{1-q}x(t)\right) = 0$, has only a zero solution.*

Theorem 1. *The solution of the IVP (3)–(5) is given by*

$$x(t) = \frac{t^{q-1}}{\Gamma(q)} + \sum_{i=1}^{n}\frac{B^i}{\Gamma((i+1)q)}(t-i\tau)^{(i+1)q-1}, \quad t \in (n\tau, (n+1)\tau],\ n = 0, 1, 2, \ldots. \tag{7}$$

Remark 3. *The proof of Theorem 1 can be done by the application of Proposition 4 with $\lambda = 0$ and $E_{q,q}(0) = \frac{1}{\Gamma(q)}$ but instead we will give a direct proof.*

Proof. Let $t \in (0, \tau]$. Then from (3) we have the RL fractional differential equation ${}_0^{RL}D_t^q x(t) = 0$ for $t \in (0, \tau]$ whose solution is given by

$$x(t) = \frac{t^{q-1}}{\Gamma(q)}, \quad t \in (0, \tau], \tag{8}$$

because from Proposition 1 we have ${}_0I^{1-q}t^{q-1} = \Gamma(q)$, i.e., ${}_0I_t^{1-q}x(t)|_{t=0} = 1$ and

$${}_0^{RL}D_t^q \frac{t^{q-1}}{\Gamma(q)} = 0. \tag{9}$$

Let $t \in (\tau, 2\tau]$. Then from (3) and (8) we have the following RL fractional equation

$$
{}_{0}^{RL}D_t^q x(t) = B\frac{(t-\tau)^{q-1}}{\Gamma(q)} \text{ for } t \in (\tau, 2\tau]. \tag{10}
$$

Therefore, the solution is given by

$$
x(t) = \frac{t^{q-1}}{\Gamma(q)} + \frac{B}{\Gamma(2q)}(t-\tau)^{2q-1}, \quad t \in (\tau, 2\tau]. \tag{11}
$$

Indeed, from Proposition 1 with $\beta = 2q - 1$ we have

$$
\begin{aligned}
{}_{0}^{RL}D_t^q x(t) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^\tau (t-s)^{-q}\,\frac{s^{q-1}}{\Gamma(q)}ds \\
&+ \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_\tau^t (t-s)^{-q}\left(\frac{s^{q-1}}{\Gamma(q)} + \frac{B}{\Gamma(2q)}(s-\tau)^{2q-1}\right)ds \\
&= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t (t-s)^{-q}\,\frac{s^{q-1}}{\Gamma(q)}ds + \frac{B}{\Gamma(2q)\Gamma(1-q)}\frac{d}{dt}\int_\tau^t (t-s)^{-q}\,(s-\tau)^{2q-1}ds \\
&= \frac{B}{\Gamma(q)}(t-\tau)^{q-1}.
\end{aligned} \tag{12}
$$

Let $t \in (2\tau, 3\tau]$. Then from (3), (8) and (11) we have

$$
{}_{0}^{RL}D_t^q x(t) = B\frac{(t-\tau)^{q-1}}{\Gamma(q)} + \frac{B^2}{\Gamma(2q)}(t-2\tau)^{2q-1} \text{ for } t \in (2\tau, 3\tau]. \tag{13}
$$

Therefore,

$$
x(t) = \frac{t^{q-1}}{\Gamma(q)} + \frac{B}{\Gamma(2q)}(t-\tau)^{2q-1} + \frac{B^2}{\Gamma(3q)}(t-2\tau)^{3q-1}, \quad t \in (2\tau, 3\tau], \tag{14}
$$

because from Proposition 1 with $\beta = 2q - 1$ and the equality

$$
\frac{d}{dt}\int_a^t (t-s)^{-q}(s-a)^{kq-1}ds = \frac{(t-a)^{(k-1)q-1}\Gamma(1-q)\Gamma(kq)}{\Gamma((k-1)q)} \tag{15}
$$

we have

$$
\begin{aligned}
{}^{RL}_{0}D^q_t x(t) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int\limits_0^\tau (t-s)^{-q}\frac{s^{q-1}}{\Gamma(q)}ds \\
&+ \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int\limits_\tau^{2\tau}(t-s)^{-q}\Big(\frac{s^{q-1}}{\Gamma(q)}+\frac{B}{\Gamma(2q)}(s-\tau)^{2q-1}\Big)ds \\
&+ \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int\limits_{2\tau}^{t}(t-s)^{-q}\Big(\frac{s^{q-1}}{\Gamma(q)}+\frac{B}{\Gamma(2q)}(s-\tau)^{2q-1}+\frac{B^2}{\Gamma(3q)}(s-2\tau)^{3q-1}\Big)ds \\
&= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int\limits_0^t(t-s)^{-q}\frac{s^{q-1}}{\Gamma(q)}ds + \frac{B}{\Gamma(2q)\Gamma(1-q)}\frac{d}{dt}\int\limits_\tau^t(t-s)^{-q}(s-\tau)^{2q-1}ds \\
&+ \frac{B^2}{\Gamma(3q)\Gamma(1-q)}\frac{d}{dt}\int\limits_{2\tau}^t(t-s)^{-q}(s-2\tau)^{3q-1}ds \\
&= \frac{B}{\Gamma(q)}(t-\tau)^{q-1}+\frac{B^2}{\Gamma(2q)}(t-2\tau)^{2q-1}.
\end{aligned}
\tag{16}
$$

Continue this process and the claim is established. □

3.2. Non-Homogeneous Linear RL Fractional Differential Equation

Consider non-homogeneous scalar linear Riemann-Liouville fractional differential equations with a constant delay (NFrDE):

$$
{}^{RL}_{0}D^q_t x(t) = Bx(t-\tau) + f(t) \text{ for } t > 0, \tag{17}
$$

with the zero initial condition (4) and fractional condition

$$
\lim_{t\to 0+}\Big(t^{1-q}x(t)\Big) = 0 \tag{18}
$$

or

$$
{}_0I_t^{1-q}x(t)|_{t=0} := \lim_{t\to 0+}\, {}_0I_t^{1-q}x(t) = \lim_{t\to 0+}\frac{1}{\Gamma(1-q)}\int\limits_0^t \frac{x(s)}{(t-s)^q}ds = 0. \tag{19}
$$

where $f \in C(\mathbb{R}_+, \mathbb{R})$, and $B, \tau > 0$ are real constants.

According to Remark 1 we will consider only the initial condition (19).

Using a direct proof we will obtain an explicit formula for the solution of the IVP (4), (17) and (19).

Theorem 2. *The solution of the IVP (4), (17) and (19) is given by*

$$
\begin{aligned}
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + \sum_{i=1}^n \frac{B^i}{\Gamma((i+1)q)}\int_{i\tau}^t (t-s)^{(i+1)q-1}f(s-i\tau)ds \\
\textit{for } t \in (n\tau, (n+1)\tau],\ n = 0, 1, 2, \ldots.
\end{aligned}
\tag{20}
$$

Proof. Let $t \in [0, \tau]$. Use the variation of constants method we will search for solutions in the form

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} c(s) ds \tag{21}$$

where $c \in C([0,\tau], \mathbb{R})$ is the unknown function to be obtained.

Then according to (17) we have

$${}_0^{RL}D_t^q x(t) = Bx(t-\tau) + f(t) = f(t) \text{ for } t \in (0,\tau]. \tag{22}$$

Also, applying $\int_\xi^t (t-s)^{-q}(s-\xi)^{q-1} ds = \Gamma(1-q)\Gamma(q)$ we obtain

$$\begin{aligned} {}_0^{RL}D_t^q x(t) &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} \int_0^s \frac{(s-\xi)^{q-1}}{\Gamma(q)} c(\xi) d\xi ds \\ &= \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t \int_\xi^t (t-s)^{-q} \frac{(s-\xi)^{q-1}}{\Gamma(q)} c(\xi) ds d\xi \\ &= \frac{d}{dt} \int_0^t c(\xi) \Big(\frac{1}{\Gamma(1-q)} \int_\xi^t (t-s)^{-q} \frac{(s-\xi)^{q-1}}{\Gamma(q)} ds \Big) d\xi \\ &= \frac{d}{dt} \int_0^t c(\xi) d\xi = c(t). \end{aligned} \tag{23}$$

From (22) and (23) we obtain $c(t) \equiv f(t)$, i.e., the solution is given by

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds, \quad t \in (0, \tau]. \tag{24}$$

Note that it is easy to check the validity of condition (19) for the solution defined by (24).

Let $t \in (\tau, 2\tau]$. Use the variation of constants method we will search for solutions in the form

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s) ds + \frac{B}{\Gamma(2q)} \int_\tau^t (t-s)^{2q-1} c(s) ds \text{ for } t \in (\tau, 2\tau] \tag{25}$$

where $c \in C([\tau, 2\tau], \mathbb{R})$ is the unknown function to be obtained.

Then according to (17) and (24) we have

$$\begin{aligned} {}_0^{RL}D_t^q x(t) = Bx(t-\tau) + f(t) &= B \int_0^{t-\tau} \frac{(t-\tau-s)^{q-1}}{\Gamma(q)} f(s) ds + f(t) \\ &= B \int_\tau^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s-\tau) ds + f(t) \text{ for } t \in (\tau, 2\tau]. \end{aligned} \tag{26}$$

Also, applying $-q\Gamma(-q) = \Gamma(1-q)$, equality ${}_0I_{t-\xi}^{1-(q+1)}(t-\xi)^{2q-1} = \frac{\Gamma(2q)}{\Gamma(q)}(t-\xi)^{q-1}$ (see Proposition 1), (15) with $a = \xi$, $k = 2$, and

$$\begin{aligned} &\frac{d}{dt} \int_\tau^t c(\xi) \Big(\int_\xi^t (t-s)^{-q} (s-\xi)^{2q-1} ds \Big) \xi = \int_\tau^t c(\xi) \frac{d}{dt} \Big(\int_\xi^t (t-s)^{-q} (s-\xi)^{2q-1} ds \Big) d\xi \\ &= \frac{\Gamma(1-q)\Gamma(2q)}{\Gamma(q)} \int_\tau^t c(\xi)(t-\xi)^{q-1} d\xi \end{aligned} \tag{27}$$

we obtain

$$\begin{aligned}
{}_0^{RL}D_t^q x(t) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\Big(\int_0^\tau (t-s)^{-q}\int_0^s \frac{(s-\xi)^{q-1}}{\Gamma(q)}f(\xi)d\xi ds \\
&+ \int_\tau^t (t-s)^{-q}\int_0^s \frac{(s-\xi)^{q-1}}{\Gamma(q)}f(\xi)d\xi ds + \int_\tau^t (t-s)^{-q}\frac{B}{\Gamma(2q)}\int_\tau^s (s-\xi)^{2q-1}c(\xi)d\xi ds\Big) \\
&= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\Big(\int_0^t (t-s)^{-q}\int_0^s \frac{(s-\xi)^{q-1}}{\Gamma(q)}f(\xi)d\xi ds \\
&+ \frac{B}{\Gamma(2q)}\int_\tau^t c(\xi)\int_\xi^t (t-s)^{-q}(s-\xi)^{2q-1}ds d\xi\Big) \\
&= f(t) + \frac{B}{\Gamma(2q)\Gamma(1-q)}\frac{d}{dt}\int_\tau^t c(\xi)\Big(\int_\xi^t (t-s)^{-q}(s-\xi)^{2q-1}ds\Big)d\xi \\
&= f(t) + B\int_\tau^t c(\xi)\frac{(t-\xi)^{q-1}}{\Gamma(q)}d\xi,
\end{aligned} \tag{28}$$

i.e., from (26) and (28) we get $c(s) = f(s-\tau),\ s \in [\tau, t]$ and

$$x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + \frac{B}{\Gamma(2q)}\int_\tau^t (t-s)^{2q-1}f(s-\tau)ds \text{ for } t \in (\tau, 2\tau]. \tag{29}$$

Let $t \in (2\tau, 3\tau]$. Use the variation of constants method we will search for solutions in the form

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds + \frac{B}{\Gamma(2q)}\int_\tau^t (t-s)^{2q-1}f(s-\tau)ds \\
&+ \frac{B^2}{\Gamma(3q)}\int_{2\tau}^t (t-s)^{3q-1}c(s)ds \quad \text{for } t \in (2\tau, 3\tau].
\end{aligned} \tag{30}$$

Then according to (17), (24) and (29) we have

$$\begin{aligned}
{}_0^{RL}D_t^q x(t) &= Bx(t-\tau) + f(t) \\
&= B\int_0^{t-\tau} \frac{(t-\tau-s)^{q-1}}{\Gamma(q)}f(s)ds + \frac{B^2}{\Gamma(2q)}\int_\tau^{t-\tau} (t-\tau-s)^{2q-1}f(s-\tau)ds + f(t) \\
&= B\int_\tau^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s-\tau)ds + \frac{B^2}{\Gamma(2q)}\int_{2\tau}^t (t-s)^{2q-1}f(s-2\tau)ds + f(t) \\
&\quad \text{for } t \in (2\tau, 3\tau].
\end{aligned} \tag{31}$$

Similar to (28) we obtain

$$
\begin{aligned}
{}_0^{RL}D_t^q x(t) &= \frac{1}{\Gamma(1-q)}\frac{d}{dt}\Big(\int_0^t (t-s)^{-q}\int_0^s \frac{(s-\xi)^{q-1}}{\Gamma(q)}f(\xi)d\xi ds \\
&+ \frac{B}{\Gamma(2q)}\int_\tau^t (t-s)^{-q}\int_\tau^s (s-\xi)^{2q-1}f(\xi-\tau)d\xi ds \\
&+ \frac{B^2}{\Gamma(3q)}\int_{2\tau}^t (t-s)^{-q}\int_{2\tau}^s (s-\xi)^{3q-1}c(\xi)d\xi ds\Big) \\
&= f(t) + B\int_\tau^t \frac{(t-\xi)^{q-1}}{\Gamma(q)}f(\xi-\tau)d\xi \\
&+ \frac{B^2}{\Gamma(3q)\Gamma(1-q)}\int_{2\tau}^t c(\xi)\frac{d}{dt}\Big(\int_\xi^t (t-s)^{-q}(s-\xi)^{3q-1}ds\Big)d\xi \\
&= f(t) + B\int_\tau^t \frac{(t-\xi)^{q-1}}{\Gamma(q)}f(\xi-\tau)d\xi + B\int_{2\tau}^t c(\xi)\frac{(t-\xi)^{2q-1}}{\Gamma(2q)}d\xi,
\end{aligned} \tag{32}
$$

i.e., from (31) and (32) we get $c(s) = f(s-2\tau),\ s \in [2\tau, t]$ and

$$
\begin{aligned}
x(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds &+ \frac{B}{\Gamma(2q)}\int_\tau^t (t-s)^{2q-1}f(s-\tau)ds \\
&+ \frac{B^2}{\Gamma(3q)}\int_{2\tau}^t (t-s)^{3q-1}f(s-2\tau)ds \text{ for } t \in (2\tau, 3\tau].
\end{aligned} \tag{33}
$$

Continue this process and the claim is established. □

Remark 4. *Note the formula for the solution in the homogeneous case does not follow from the one in the non-homogeneous case because of the fractional conditions (5) and (18) (respectively, (6) and (19)).*

4. Explicit Formula for the Solutions of Scalar Linear Rl Fractional Equations with Delays and Non-Zero Initial Values

Consider the linear non-homogeneous RL fractional differential Equation (17) with nonzero initial value:

$$
\begin{aligned}
&x(t) = g(t) \text{ for } t \in [-\tau, 0) \\
&{}_0I_t^{1-q}x(t)|_{t=0} := \lim_{t\to 0+} {}_0I_t^{1-q}x(t) = \lim_{t\to 0+}\frac{1}{\Gamma(1-q)}\int_0^t \frac{x(s)}{(t-s)^q}ds = g(0),
\end{aligned} \tag{34}
$$

where $g \in C([-\tau, 0], \mathbb{R}),\ |g(0)| < \infty$.

Remark 5. *Note that the function $g(t) = t^{q-1}$ is not applicable in this case as an initial function.*

Remark 6. *According to Remark 1 the fractional condition in (34) could be replaced by* $\lim_{t\to 0+}\Big(t^{1-q}x(t)\Big) = \frac{g(0)}{\Gamma(q)}$.

Define the function

$$\Psi_q(t) = \begin{cases} g(t) & t \in [-\tau, 0] \\ \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds \\ \quad + \frac{B}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{i\tau}^{(i+1)\tau} (t-s)^{q-1} \Psi_q(s-\tau) ds \\ \quad + \frac{B}{\Gamma(q)} \int_{n\tau}^t (t-s)^{q-1} \Psi_q(s-\tau) ds & t \in (n\tau, (n+1)\tau], n = 0, 1, 2, \ldots. \end{cases}$$

Theorem 3. *The solution of the IVP (17), (34) is given by*

$$x(t) = \Psi_q(t), \quad t \in (n\tau, (n+1)\tau], \ n = 0, 1, 2, \ldots. \tag{35}$$

Proof. Let $t \in (0, \tau]$. Then from the Equation (3) and the initial condition (34) we have

$$\begin{aligned} & {}_0^{RL}D_t^q x(t) = Bg(t-\tau) + f(t) \ \text{ for } t \in (0, \tau] \\ & {}_0I_t^{1-q} x(t)|_{t=0} = g(0). \end{aligned} \tag{36}$$

According to Proposition 3 with $\lambda = 0$ and the equality $E_{q,q}(0) = \frac{1}{\Gamma(q)}$, the solution is

$$x(t) = \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{B}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s-\tau) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds = \Psi_q(t), \quad t \in (0, \tau]. \tag{37}$$

Let $t \in (\tau, 2\tau]$. Then from (17), (34) and (37) we have

$${}_0^{RL}D_t^q x(t) = Bx(t-\tau) + f(t) = \begin{cases} Bg(t-\tau) + f(t) & t \in (0, \tau] \\ B\Psi_q(t-\tau) + f(t) & t \in (\tau, 2\tau]. \end{cases}$$

According to Proposition 3 and Equation $\int_\xi^t (t-s)^{-q} (s-\xi)^{q-1} ds = \Gamma(1-q)\Gamma(q)$ the solution is

$$\begin{aligned} x(t) = & \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{B}{\Gamma(q)} \int_0^\tau (t-s)^{q-1} g(s-\tau) ds \\ & + \frac{B}{\Gamma(q)} \int_\tau^t (t-s)^{q-1} \Psi_q(s-\tau) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t \in (\tau, 2\tau]. \end{aligned} \tag{38}$$

Let $t \in (2\tau, 3\tau]$. Then from (17), (34), (37) and (38) we have

$${}_0^{RL}D_t^q x(t) = \begin{cases} Bg(t-\tau) = B\Psi_q(t-\tau) & t \in (0, \tau] \\ \frac{Bg(0)}{\Gamma(q)} (t-\tau)^{q-1} + \frac{B^2}{\Gamma(q)} \int_0^{t-\tau} (t-\tau-s)^{q-1} g(s-\tau) ds = B\Psi_q(t-\tau) & t \in (\tau, 2\tau] \\ B\Psi_q(t-\tau) & t \in (2\tau, 3\tau]. \end{cases}$$

According to Proposition 3 the solution is

$$\begin{aligned} x(t) = & \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{B}{\Gamma(q)} \int_0^\tau (t-s)^{q-1} \Psi_q(s-\tau) ds + \frac{B}{\Gamma(q)} \int_\tau^{2\tau} (t-s)^{q-1} \Psi_q(s-\tau) ds \\ & + \frac{B}{\Gamma(q)} \int_{2\tau}^t (t-s)^{-q} \Psi_q(s-\tau) ds, \quad t \in (2\tau, 3\tau]. \end{aligned} \tag{39}$$

Continue the process, and the proof is complete. □

Special Case: In the homogeneous case of $f(t) \equiv 0$ the solution of the IVP (3), (34) is given by the function

$$\Xi_q(t) = \begin{cases} g(t) & t \in [-\tau, 0] \\ \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{B}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Xi_q(s-\tau) ds & t \in (0, \tau] \\ \frac{g(0)}{\Gamma(q)} t^{q-1} + \frac{B}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{i\tau}^{(i+1)\tau} (t-s)^{q-1} \Xi_q(s-\tau) ds & \\ \quad + \frac{B}{\Gamma(q)} \int_{n\tau}^t (t-s)^{q-1} \Xi_q(s-\tau) ds & t \in (n\tau, (n+1)\tau], n = 1, 2, \dots. \end{cases}$$

5. Explicit Formula for the Solutions of the General Scalar Linear Rl Fractional Equations with Delays and Non-Zero Initial Values

5.1. Zero Initial Function

Consider the non-homogeneous scalar linear Riemann-Liouville fractional differential equations with a constant delay :

$$ {}_0^{RL}D_t^q x(t) = Ax(t) + Bx(t-\tau) + f(t) \text{ for } t > 0, \tag{40}$$

with the initial conditions

$$x(t) = 0, \quad t \in [-\tau, 0], \tag{41}$$

$$\lim_{t \to 0+} \left(t^{1-q} x(t) \right) = \frac{C}{\Gamma(q)}, \tag{42}$$

where $f \in C(\mathbb{R}_+, \mathbb{R})$, $A, B, C, \tau > 0$ are real constants. Define the function

$$\Phi_q(t) = \begin{cases} 0 & t \in [-\tau, 0] \\ C\, E_{q,q}(At^q) t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds & t \in (0, \tau] \\ C\, E_{q,q}(At^q) t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds & \\ \quad + B \sum_{i=0}^{n-1} \int_{i\tau}^{(i+1)\tau} (t-s)^{q-1} E_{q,q}(A(t-s)^q) \Phi_q(s-\tau) ds & \\ \quad + B \int_{n\tau}^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \Phi_q(s-\tau) ds & \\ \qquad \text{for } \; t \in (n\tau, (n+1)\tau], n = 1, 2, \dots. & \end{cases}$$

Theorem 4. *The solution of the IVP (40) and (41), (42) is given by*

$$x(t) = \Phi_q(t), \quad t \in (n\tau, (n+1)\tau], \; n = 0, 1, 2, \dots. \tag{43}$$

Proof. Let $t \in (0, \tau]$. Then from the Equation (40) we have

$$ {}_0^{RL}D_t^q x(t) = Ax(t) + f(t) \; \text{ for } t \in (0, \tau], \tag{44}$$

with initial condition (42) .

According to Proposition 3 with $\lambda = A$ the solution is

$$x(t) = C\, E_{q,q}(At^q) t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds = \Phi_q(t), \quad t \in (0, \tau]. \tag{45}$$

Let $t \in (\tau, 2\tau]$. Then from (40)–(42) and (45) we have

$$ {}_0^{RL}D_t^q x(t) = Ax(t) + \begin{cases} f(t) & t \in (0, \tau] \\ B\Phi(t-\tau) + f(t) & t \in (\tau, 2\tau]. \end{cases} $$

According to Proposition 3 the solution is

$$ \begin{aligned} x(t) &= C\, E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds \\ &+ B \int_\tau^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Phi(s-\tau) ds = \Phi_q(t). \end{aligned} \tag{46} $$

Let $t \in (2\tau, 3\tau]$. Then from (40)–(42) and (46) we have

$$ {}_0^{RL}D_t^q x(t) = Ax(t) + \begin{cases} f(t) & t \in (0, \tau] \\ B\Phi_q(t-\tau) + f(t) & t \in (\tau, 2\tau] \\ B\Phi_q(t-\tau) + f(t) & t \in (2\tau, 3\tau]. \end{cases} $$

According to Proposition 3 the solution is

$$ \begin{aligned} x(t) &= C\, E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds \\ &+ B \int_\tau^{2\tau} (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Phi(s-\tau) ds \\ &+ B \int_{2\tau}^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Phi(s-\tau) ds = \Phi_q(t). \end{aligned} \tag{47} $$

Continue this process, and the claim is established. □

5.2. Non-Zero Initial Function

Consider non-homogeneous scalar linear Riemann-Liouville fractional differential equations with a constant delay (40) with the initial conditions

$$ x(t) = g(t), \quad t \in [-\tau, 0], \tag{48} $$

$$ \lim_{t \to 0+} \left(t^{1-q} x(t) \right) = g(0), \tag{49} $$

where $f \in C(\mathbb{R}_+, \mathbb{R})$, $g \in C([-\tau, 0], \mathbb{R})$.

Define the function

$$ \Lambda_q(t) = \begin{cases} g(t) & t \in (-\tau, 0] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Big(Bg(s-\tau) + f(s)\Big) ds & t \in (0, \tau] \\ g(0)\Gamma(q)E_{q,q}(At^q)t^{q-1} + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds & \\ \quad + B\sum_{i=0}^{n-1} \int_{i\tau}^{(i+1)\tau} (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Lambda_q(s-\tau) ds & \\ \quad + B\int_{n\tau}^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Lambda_q(s-\tau) ds & \\ \qquad \text{for} \quad t \in (n\tau, (n+1)\tau], n = 1, 2, \dots. & \end{cases} $$

Theorem 5. *The solution of the IVP (40) and (48), (49) is given by*

$$x(t) = \Lambda_q(t), \quad t \in (n\tau, (n+1)\tau], \ n = 0, 1, 2, \ldots.$$

Proof. The proof is similar to the one of Theorem 3 so we omit it. □

6. Conclusions

In fractional models finding exact solutions is an important question and it can be quite complicated even in the linear scalar case when considering RL fractional differential equations. In this paper we study initial value problems of scalar linear RL fractional differential equations with constant delay and an initial value problem is set up in an appropriate way. On the one hand, it is similar to the case of the integer order derivative, and on the other hand, it is related to the definition and properties of the RL derivative. This allows the corresponding initial value problem to be used for modeling processes with impulsive type initial conditions which can be found in physics, chemistry, engineering, biology, and economics. Explicit formulas for the solutions of initial value problems with both zero and nonzero initial functions are obtained and homogeneous and non-homogeneous equations are studied. The formulas given will be very helpful in the theoretical study of linear scalar fractional models, for linearization of nonlinear models, and for the monotone-iterative technique for RL fractional differential equations.

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