mathematics

Article

On the Construction of Some Fractional Stochastic Gompertz Models

Giacomo Ascione † and **Enrica Pirozzi** *,†

Dipartimento di Matematica e Applicazioni "Renato Caccioppoli", Universitá degli Studi di Napoli Federico II, I-80126 Naples, Italy; giacomo.ascione@unina.it

* Correspondence: enrica.pirozzi@unina.it

† These authors contributed equally to this work.

Received: 22 November 2019; Accepted: 22 December 2019; Published: 2 January 2020

Abstract: The aim of this paper is the construction of stochastic versions for some fractional Gompertz curves. To do this, we first study a class of linear fractional-integral stochastic equations, proving existence and uniqueness of a Gaussian solution. Such kinds of equations are then used to construct fractional stochastic Gompertz models. Finally, a new fractional Gompertz model, based on the previous two, is introduced and a stochastic version of it is provided.

Keywords: Caputo fractional derivative; Gaussian processes; fractional-integral equations

---

1. Introduction

Fractional calculus is presently applied to a lot of scientific fields. Despite the problem of defining fractional derivatives being quite old (see, for instance, [1,2]), it has mainly been developed in recent times (see [3]). Due to its versatility in describing slower or also different time scales, fractional derivatives and fractional-order differential equations are very often used in applications, so that also different books have been written on the argument (see, for instance, [4–6]). The main generalization of the classical Cauchy problems to the fractional order is achieved via the so-called Caputo-fractional derivative, introduced by Michele Caputo in [7]. In such paper, the fractional derivative is used to study the Q-factor of some non-ferromagnetic solids, thus being introduced in an applicative context. From such moment, fractional calculus has been used to address a lot of different models: from epidemics [8] to osmosis [9], from neurophysiology [10] to viscoelasticity [11] and many others [12].

Here we focus on fractional-order population growth models. A first model of population growth can be achieved by modifying the classical Malthus model by introducing a fractional-order derivative in place of the classical one (see [12,13]). As a second step, one could ask for a fractional-order generalization of a Gompertz model. Gompertz model are quite popular growth model. Such models take into account a time-varying birth rate, which describes the fact that a person's resistance to death decreases with age. Such models have been used in particular to model cancer growth, starting from [14] and then used to describe a single species growth (see for instance [15]). For this and other reasons, Gompertz curves have been widely studied. For instance, knowing that some species of cancer evolved following a Gompertz law, optimal control of it has become necessary (see for instance [16]). At the same time, stochastic models became necessary to describe eventual environmental (and thus unpredictable) effects (see [17–20] and many others).

Concerning fractional-order Gompertz models, the first one has been introduced in [21], but it is not achieved by simply substituting the fractional derivative in place of the classical one. To understand how the fractional-order model is introduced, let us recall that the classical Gompertz curve $x(t)$ can be defined as the solution of the non-linear Cauchy problem

$$\begin{cases} \frac{dx}{dt}(t) = \alpha x(t) - \beta x(t) \log\left(\frac{x(t)}{x_0}\right), \\ x(0) = x_0 \end{cases} \tag{1}$$

where $\alpha, \beta > 0$ are dynamical parameters and $x_0 > 0$ is the initial population density. It is also well known that the solution is given by

$$x(t) = x_0 \exp\left(\frac{\alpha}{\beta}(1 - e^{-\beta t})\right)$$

and then the model admits a carrying capacity

$$x(\infty) = \lim_{t \to +\infty} x(t) = x_0 e^{\frac{\alpha}{\beta}}.$$

If we define $y(t) = \log\left(\frac{x(t)}{x_0}\right)$, the Cauchy problem (1) can be rewritten as

$$\begin{cases} \frac{dx}{dt}(t) = (\alpha - \beta y(t))x(t), \\ \frac{dy}{dt}(t) = \alpha - \beta y(t), \\ x(0) = x_0, \\ y(0) = 0. \end{cases} \tag{2}$$

In [21], the fractional-order Gompertz model is achieved by substituting the Caputo-fractional derivative in place of the classical one only in the linear equation. In particular, the function $y_\nu(t)$ is defined as the solution of the fractional Cauchy problem

$$\begin{cases} \frac{d^\nu y_\nu}{dt^\nu}(t) = \alpha - \beta y_\nu(t), \\ y_\nu(0) = 0, \end{cases} \tag{3}$$

where $\frac{d^\nu}{dt^\nu}$ is the fractional Caputo derivative of order $\nu \in (0,1)$, and then defining the fractional Gompertz curve as $x_\nu(t) = x_0 e^{y_\nu(t)}$. In such case, we have

$$y_\nu(t) = \frac{\alpha}{\beta}(1 - E_\nu(-\beta t^\nu)), \qquad x_\nu(t) = x_0 \exp\left(\frac{\alpha}{\beta}(1 - E_\nu(-\beta t^\nu))\right) \tag{4}$$

where $E_\nu(t)$ is the Mittag-Leffler function (defined in Equation (8)).

In [22] another type of fractional-order Gompertz model has been introduced. To understand how such model is defined, let us recall that for the classical model we have $y(t) = \frac{\alpha}{\beta}(1 - e^{-\beta t})$ and then we can rewrite the first equation of Equations (2) as

$$\begin{cases} e^{\beta t} \frac{d}{dt} x(t) = \alpha x(t), \\ x(0) = x_0. \end{cases}$$

In [22], they use the Caputo-fractional derivative with respect to another function, as defined in [23], to define the improved fractional Gompertz curve $x_\nu(t)$ the solution of

$$\begin{cases} \left(e^{\beta t} \frac{d}{dt}\right)^\nu x_\nu(t) = \alpha x_\nu(t), \\ x_\nu(0) = x_0, \end{cases} \tag{5}$$

given by

$$x_\nu(t) = x_0 E_\nu\left(\frac{\alpha}{\beta^\nu}(1-e^{-\beta t})^\nu\right).$$

In this paper, we aim to define a class of stochastic Gompertz models that generalize the two proposed fractional Gompertz curves. To do this, we first need to investigate some results related to a class of stochastic linear fractional-integral equations, concerning in particular the existence of Gaussian solutions. Such equations generalize the Caputo-fractional stochastic differential equations studied for instance in [24,25].

In particular this approach leads to a method of construction for general fractional growth models with noise that preserves normal or log-normal one-dimensional distributions. The preservation of such laws permits recognition in some macroscopical observable functions (the mean in the normal case and the median in the log-normal case) of the original growth models. Thus, these stochastic models work as a noisy perturbation of the original deterministic ones. This procedure could not be achieved by using the classical tools of fractionalization via time-change (see for instance [26–29]) for different reasons. For instance, if we apply a time-change to the stochastic Gompertz model, since the stochastic differential equation that drives the model is non-linear, its mean does not solve Equation (1) with the Caputo derivative in place of the classical one. However, such time-changed process can be still seen as the exponential of a time-changed Ornstein-Uhlenbeck process (in the sense of [28]), but, being the latter not a Gaussian one, the time-changed Gompertz model is not log-normal and its median does not coincide with the function x_ν given in Equation (4), despite the mean of the time-changed Ornstein-Uhlenbeck process is still solution of Equation (3). Our new approach overtakes such problems, giving then some log-normal or normal processes whose dynamics are given by perturbation of the deterministic ones.

The paper is structured as follows:

- In Section 2 we give some basic definitions and preliminaries on fractional calculus;
- In Section 3 we study a class of linear fractional-integral stochastic equations: we will need them to define the stochastic models for fractional Gompertz curves. In particular, we focus on existence and almost surely uniqueness of Gaussian solutions. Moreover, since they are obtained via a Picard approximation method, we also give an estimate of the speed of convergence of the method in terms of the distribution of the maximum of the chosen noise.
- In Section 4 we give some examples on possible choices of noise. In particular in Section 4.4 we show that such fractional-integral equations are indeed a generalization of the fractional stochastic differential equations discussed in [24,25].
- In Section 5 we use the results from the previous sections to introduce stochastic models for fractional Gompertz curves. In particular in Section 5.1 we give some generalities on the classical stochastic Gompertz model, while in Sections 5.2 and 5.3 we give a stochastic version of the fractional Gompertz curve introduced in [21] and of the improved fractional Gompertz curve introduced in [22]. Finally, in Section 5.4 we construct a new fractional Gompertz model obtained by merging the approach of the previous two models and we describe a stochastic counterpart for it.

2. Some Preliminaries on Fractional Calculus

Concerning the main properties of fractional integrals and derivatives, we refer to [30]. Let us give the following definition of the fractional-integral.

Definition 1. *Given $\nu > 0$ the fractional-integral $\mathcal{I}^\nu_t$ of order ν is defined as*

$$\mathcal{I}^\nu_t f = \frac{1}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} f(\tau)d\tau$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$.

It is easy to see that for instance, for any $f \in L^\infty$ the fractional-integral $\mathcal{I}_t^\nu f$ is defined. Moreover, for any ν_1, ν_2 it holds $\mathcal{I}_t^{\nu_1} \mathcal{I}_t^{\nu_2} = \mathcal{I}_t^{\nu_1+\nu_2}$. It is also interesting to notice that the fractional-integral $\mathcal{I}_t^\nu$ is a convolution operator. Indeed if we define the kernel $I_\nu(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}\chi_{[0,+\infty)}$, then, for any function $f : [0, +\infty) \to \mathbb{R}$

$$\mathcal{I}_t^\nu f = (I_\nu * f)(t)$$

where $*$ is the convolution product and f is extended to the whole real line by setting $f(t) = 0$ for any $t < 0$. If $\nu \in (0, 1)$, then the convolution kernel I_ν is singular, but still in $L^1(0, T)$ for any $T > 0$. Therefore, while for any $\nu \in [1, +\infty)$, one only needs f to be in L^1_{loc}, it is not enough if $\nu \in (0, 1)$.

Now we can define the Riemann-Liouville derivative.

Definition 2. *Given* $\nu \in (0, 1)$*, the Riemann-Liouville fractional derivative* D_t^ν *of order* ν *is defined as*

$$D_t^\nu f = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t (t-\tau)^{-\nu} f(\tau) d\tau$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$.

From the definition of D_t^ν, one easily obtains that

$$D_t^\nu = \frac{d}{dt} \mathcal{I}_t^{1-\nu} \tag{6}$$

for any $\nu \in (0, 1)$. Thus, by the semigroup property of the fractional-integral and the fact that $\mathcal{I}_t^1$ is the classical integral, we have

$$D_t^\nu(\mathcal{I}_t^\nu f) = \frac{d}{dt} \mathcal{I}_t^{1-\nu} \mathcal{I}_t^\nu f = f(t)$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$. In particular we have that D_t^ν is the left inverse of $\mathcal{I}_t^\nu$ and thus, vice versa, $\mathcal{I}_t^\nu$ is the right inverse of D_t^ν for any $\nu \in (0, 1)$.

However, we also have another fractional derivative.

Definition 3. *Given* $\nu \in (0, 1)$*, the Caputo-fractional derivative* $\frac{d^\nu}{dt^\nu}$ *of order* ν *is defined as*

$$\frac{d^\nu f}{dt^\nu}(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-\tau)^{-\nu} f'(\tau) d\tau$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$.

The class of functions for which $\frac{d^\nu f}{dt^\nu}$ is defined is smaller than the one for which $D_t^\nu f$ is: indeed, one has at least to ask that f is absolutely continuous. Moreover, we have that

$$\frac{d^\nu}{dt^\nu} = \mathcal{I}_t^{1-\nu} \left(\frac{d}{dt}\right)$$

hence, working as before, we have

$$\mathcal{I}_t^\nu \frac{d^\nu f}{dt^\nu} = \mathcal{I}_t^\nu \mathcal{I}_t^{1-\nu} \frac{df}{dt} = f(t)$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$. We can conclude that I_t^ν is the left inverse of $\frac{d^\nu}{dt^\nu}$, and then $\frac{d^\nu}{dt^\nu}$ is the right inverse of I_t^ν. There is also a relation between Riemann-Liouville and Caputo derivative:

$$\frac{d^\nu f}{dt^\nu} = D_t^\nu(f - f(0+)). \tag{7}$$

From now on we will denote $f(0+) = f(0)$. This relation lets us also define the Caputo-fractional derivative for any Riemann-Liouville derivable function, hence for a much wider class of functions. Concerning Caputo derivatives, we can define fractional Cauchy problems by using them. Indeed, under suitable assumptions, the fractional Cauchy problem

$$\begin{cases} \frac{d^\nu y}{dt^\nu}(t) = F(t, y(t)), \\ y(0) = y_0 \end{cases}$$

is well-posed. In particular, the relaxation problem

$$\begin{cases} \frac{d^\nu y}{dt^\nu}(t) = ay(t), \\ y(0) = y_0 \end{cases}$$

admits as unique solution the function

$$y(t) = y_0 E_\nu(at^\nu)$$

where $E_\nu(t)$ is the Mittag-Leffler function, defined as

$$E_\nu(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\nu k + 1)}, \ t \in \mathbb{R}, \tag{8}$$

which is a generalization of the exponential function (observe that if $\nu = 1$, $E_1(t) = e^t$).

We need also to introduce fractional calculus with respect to other functions. Riemann-Liouville type fractional derivative of a function with respect of another function were introduced to deal with Leibniz rule and chain rule for fractional derivatives (see, for instance, [31,32]). For this part, we mainly refer to [23]. Let us first give the definition of fractional-integral with respect to another function.

Definition 4. *Given $\nu > 0$ and an increasing function $\Psi \in C^1(0, +\infty)$ such that $\Psi'(t) \neq 0$ for any $t > 0$, the fractional-integral $\mathcal{I}_t^{\nu,\Psi}$ with respect to Ψ is given by*

$$\mathcal{I}_t^{\nu,\Psi} f := \frac{1}{\Gamma(\nu)} \int_0^t \Psi'(\tau)(\Psi(t) - \Psi(\tau))^{\nu-1} f(\tau) d\tau$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$.

Observe that if $\Psi(t) = t$, we achieve the classical fractional-integral. Let us now define the Riemann-Liouville type fractional derivative.

Definition 5. *Given $\nu \in (0, 1)$ and an increasing function $\Psi \in C^1(0, +\infty)$ such that $\Psi'(t) \neq 0$ for any $t > 0$, the Riemann-Liouville fractional derivative $D_t^{\nu,\Psi}$ with respect to Ψ is given by*

$$D_t^{\nu,\Psi} f := \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t \Psi'(\tau)(\Psi(t) - \Psi(\tau))^{-\nu} f(\tau) d\tau$$

for any suitable function $f : [0, +\infty) \to \mathbb{R}$.

Observe that for $\Psi(t) = t$, we achieve the classical Riemann-Liouville fractional derivative. Moreover, we have in this case

$$D_t^{\nu,\Psi} = \frac{1}{\Psi'(t)} \frac{d}{dt} \mathcal{I}_t^{1-\nu,\Psi}. \tag{9}$$

Let us also give the definition of the Caputo type fractional derivative.

Definition 6. *Given $\nu \in (0,1)$ and an increasing function $\Psi \in C^1(0,+\infty)$ such that $\Psi'(t) \neq 0$ for any $t > 0$, the Caputo-fractional derivative $\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu$ with respect to Ψ is given by*

$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu f(t) = \frac{1}{\Gamma(1-\nu)}\int_0^t (\Psi(t)-\Psi(\tau))^{-\nu} f'(\tau)d\tau$$

for any suitable function $f : [0,+\infty) \to \mathbb{R}$.

In [23] (Theorem 3) the following relation is shown

$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu f(t) = D_t^{\nu,\Psi}(f - f(0)). \tag{10}$$

Using this relation, one can extend the definition of Caputo-fractional derivative of a function with respect to another function to the whole class of the Riemann-Liouville derivable (with respect to Ψ) functions. Moreover, under suitable assumptions, the following fractional Cauchy problem is well-posed

$$\begin{cases} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu y(t) = F(t, y(t)), \\ y(0) = y_0. \end{cases}$$

In the spirit of [31,32], let us show a chain rule for Caputo-fractional derivatives of a function with respect to another function.

Proposition 1. *Let g be a Caputo-derivable function and Ψ be an increasing function in $C^1(J)$ such that $\Psi'(t) \neq 0$ for any $t \in J$ and $\Psi(0) = 0$. Define $f(t) = g(\Psi(t))$. Then*

$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu f(t) = \frac{d^\nu g}{dt^\nu}(\Psi(t))$$

Proof. First, let us observe that

$$\begin{aligned} \mathcal{I}_t^{1-\nu,\Psi} f &= \frac{1}{\Gamma(1-\nu)}\int_0^t \Psi'(\tau)(\Psi(t)-\Psi(\tau))^{-\nu} f(\tau)d\tau \\ &= \frac{1}{\Gamma(1-\nu)}\int_0^t \Psi'(\tau)(\Psi(t)-\Psi(\tau))^{-\nu} g(\Psi(\tau))d\tau \\ &= \frac{1}{\Gamma(1-\nu)}\int_0^{\Psi(t)} (\Psi(t)-z)^{-\nu} g(z)dz = \mathcal{I}_{\Psi(t)}^{1-\nu} g. \end{aligned}$$

Deriving both sides of this relation and dividing by $\Psi'(t)$, by using Equation (9), we have

$$D_t^{\nu,\Psi} f = D_{\Psi(t)}^\nu g.$$

Finally, by substituting $f(t) - f(0)$ and $g(t) - g(0)$ in place of f and g and using Equation (10) we conclude the proof. □

This proposition leads us to easily give the solution for the relaxation equation

$$\begin{cases} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu y(t) = ay(t), \\ y(0) = y_0 \end{cases} \tag{11}$$

whenever $\Psi(0) = 0$. Indeed, if we define $z(t)$ as the solution of the relaxation equation for the Caputo derivative, hence $z(t) = y_0 E_\nu(at^\nu)$, and $y(t) = z(\Psi(t))$, we have, by the previous proposition

$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^\nu y(t) = \frac{d^\nu z}{dt^\nu}(\Psi(t)) = az(\Psi(t)) = ay(t)$$

thus $y(t)$ is the solution of Equation (11).

3. Stochastic Linear fractional-integral Equations with Constant Coefficients and Gaussian Solutions

From now on let us fix a complete filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

In this section, we want to study existence and uniqueness of solutions of stochastic linear fractional-integral equations in the form

$$Y_\nu(t) = y_0 + \mathcal{I}_t^\nu(aY_\nu(t) + b) + G(t) \tag{12}$$

where $y_0, a, b \in \mathbb{R}$, $a \neq 0$, and $G(t)$ is a given $\mathcal{F}_t$-adapted Gaussian process. From now on, as shorthand notation, let us denote

$$\mathcal{G}(J) := \{G : \Omega \times J \to \mathbb{R} \mid G \text{ is a } \mathcal{F}_t\text{-adapted Gaussian process with a.s. continuous paths}\}$$

where $J = [0, T]$ for some $T > 0$ or $J = [0, +\infty)$, and

$$\mathcal{G}_r(J) := \{G : \Omega \times J \to \mathbb{R} \mid G \text{ is a } \mathcal{F}_t\text{-adapted Gaussian process with a.s. } r\text{-Hölder continuous paths}\}.$$

Remark 1. *Obviously, for any $f, g \in C^0(J)$ and $Z \in \mathcal{G}(J)$, we have $fZ + g \in \mathcal{G}(J)$.*

Moreover, let us denote $\|f\|_{L^\infty(J)} = \sup_{t \in J} |f(t)|$ for any $f \in L^\infty(J)$, where $J = [0, T]$ with $T > 0$.

3.1. The fractional-integral of a Gaussian Process

First, one could ask if the fractional-integral of a Gaussian process is still a Gaussian process. Concerning this problem, we have the following Lemma.

Lemma 1. *Let $Z \in \mathcal{G}(J)$ for some time interval J and define $Z_\nu(t) := \mathcal{I}_t^\nu Z$ for $t \in J$. Then $Z_\nu \in \mathcal{G}(J)$. Moreover, if $J = [0, T]$ for some $T > 0$, then $Z_\nu \in \mathcal{G}_\nu(J)$.*

Proof. Let us consider

$$A = \{\omega \in \Omega : \ t \mapsto Z(t, \omega) \text{ is continuous}\}$$

and recall that $\mathbb{P}(\Omega \setminus A) = 0$. Fix $\omega \in A$ and observe that $\mathcal{I}_t^\nu Z(\cdot, \omega)$ is well-defined and continuous (in t). We need to show that it is a $\mathcal{F}_t$-adapted Gaussian process. Let us define

$$\mathcal{I}_t^{\nu,\varepsilon} Z(\omega) = \frac{1}{\Gamma(\nu)} \int_0^{t-\varepsilon} (t - \tau)^{\nu-1} Z(\tau, \omega) d\tau$$

which is well-defined as Riemann integral since, for fixed $t > 0$, $\tau \mapsto (t - \tau)^{\nu-1} Z(\tau, \omega)$ is continuous in $[0, t - \varepsilon]$. To show that $Z_\nu(t)$ is $\mathcal{F}_t$-adapted, let us observe that, by definition of Riemann integral,

$$\mathcal{I}_t^{\nu,\varepsilon} Z(\omega) = \lim_{n \to +\infty} \sum_{j=1}^{n} (t - s_{j,n})^{\nu-1} Z(s_{j,n}, \omega) \delta_n^\varepsilon$$

for any $\omega \in A$ where $\delta_n^\varepsilon = \frac{t-\varepsilon}{n}$ and $s_{j,n} = \left(j - \frac{1}{2}\right)\delta_n^\varepsilon$, with $j = 1, \dots, n$. Hence we have that almost surely

$$\mathcal{I}_t^{\nu,\varepsilon} Z = \lim_{n \to +\infty} \sum_{j=1}^{n} (t - s_{j,n})^{\nu-1} Z(s_{j,n}) \delta_n^\varepsilon.$$

Since Z is $\mathcal{F}_t$-adapted and with a.s. continuous paths, it is progressively measurable and then, for any $n \in \mathbb{N}$ and $j = 1, \dots, n$, $Z(s_{j,n})$ is $\mathcal{F}_{s_{j,n}}$-measurable and thus, being $s_{j,n} < t$, $\mathcal{F}_t$-measurable. Hence the variable $\sum_{j=1}^{n}(t - s_{j,n})^{\nu-1} Z(s_{j,n})\delta_n$ is $\mathcal{F}_t$-measurable for any $n \in \mathbb{N}$ and so it is its limit as $n \to +\infty$, concluding that for any $\varepsilon > 0$, $\mathcal{I}_t^{\nu,\varepsilon} Z$ is $\mathcal{F}_t$-measurable. Now let us consider, for $k \in \mathbb{N}$, $\varepsilon_k = \frac{1}{k}$. Let us observe that for fixed $\omega \in A$ we have, for $\tau \in (0, t)$,

$$|(t-\tau)^{\nu-1} Z(\tau, \omega) \chi_{(0,t-\varepsilon_k)}(\tau)| \le (t-\tau)^{\nu-1} \max_{\tau \in [0,t]} Z(\tau, \omega)$$

which is a $L^1(0,t)$ function. Hence, we have, by Lebesgue dominated convergence theorem, that

$$Z_\nu(t, \omega) = \lim_{k \to +\infty} \mathcal{I}_t^{\nu,\varepsilon_k} Z(\omega)$$

thus also $Z_\nu(t)$ (being a.s. limit of $\mathcal{F}_t$-measurable r.v.) is $\mathcal{F}_t$-measurable.

Now let us show that Z_ν is a Gaussian process. Let us fix $m \in \mathbb{N}$, $(a_1, \dots, a_m) \in \mathbb{R}^m$, $(t_1, \dots, t_m) \in J^m$ and let us consider the random variable $\mathcal{Z}^\varepsilon$ given by

$$\mathcal{Z}^\varepsilon := \sum_{i=1}^{m} a_i \mathcal{I}_{t_i}^{\nu,\varepsilon} Z.$$

As before, if we define, for fixed $i \le m$ and $n \in \mathbb{N}$, $\delta_{i,n}^\varepsilon := \frac{t_i - \varepsilon}{n}$ and $s_{i,j,n} := \left(j - \frac{1}{2}\right)\delta_{i,n}^\varepsilon$ for $j = 1, \dots, n$, we have that, by definition of Riemann integral,

$$\mathcal{I}_{t_i}^{\nu,\varepsilon} Z(\omega) = \lim_{n \to +\infty} \sum_{j=1}^{n} (t_i - s_{i,j,n})^{\nu-1} Z(s_{i,j,n}, \omega) \delta_{i,n}^\varepsilon$$

for any $\omega \in A$. Hence we have, for any $\omega \in A$,

$$\mathcal{Z}^\varepsilon(\omega) = \lim_{n \to +\infty} \sum_{i=1}^{m} a_i \sum_{j=1}^{n} (t_i - s_{i,j,n})^{\nu-1} Z(s_{i,j,n}, \omega) \delta_{i,n}^\varepsilon.$$

Since $Z(t)$ is a Gaussian process the random variable $\sum_{i=1}^{m} a_i \sum_{j=1}^{n} (t_i - s_{i,j,n})^{\nu-1} Z(s_{i,j,n}) \delta_{i,n}^\varepsilon$ is Gaussian for any $n \in \mathbb{N}$. Hence $\mathcal{Z}^\varepsilon$ is almost surely limit of Gaussian random variables, hence it must be Gaussian.

As before, if we consider $\varepsilon_k = 1/k$, we have that for $\omega \in A$

$$\sum_{i=1}^{m} a_i Z_\nu(t_i, \omega) = \lim_{k \to +\infty} \mathcal{Z}^{\varepsilon_k}(\omega)$$

hence $\sum_{i=1}^{m} a_i Z_\nu(t_i, \omega)$ is almost surely limit of Gaussian random variables and must be Gaussian itself. The arbitrariness of $(a_1, \dots, a_m) \in \mathbb{R}^m$ and $m \in \mathbb{N}$ gives us the fact that Z_ν is a Gaussian process.

Finally, suppose that $J = [0, T]$ and let us consider $t_1, t_2 \in J$. Suppose, without loss of generality, that $t_1 < t_2$ and set $t_2 = t_1 + h$. Hence we have for $\omega \in A$

$$\begin{aligned}
|Z_\nu(t_1+h,\omega) - Z_\nu(t_1,\omega)| &\le \frac{1}{\Gamma(\nu)}\int_0^{t_1}((t_1-\tau)^{\nu-1} - (t_1+h-\tau)^{\nu-1})|Z(\tau,\omega)|d\tau \\
&+ \frac{1}{\Gamma(\nu)}\int_{t_1}^{t_1+h}(t_1+h-\tau)^{\nu-1}|Z(\tau,\omega)|d\tau \\
&\le \frac{\|Z(\cdot,\omega)\|_{L^\infty(J)}}{\Gamma(\nu+1)}(t_1^\nu - (t_1+h)^\nu + h^\nu) \\
&+ \frac{\|Z(\cdot,\omega)\|_{L^\infty(J)}}{\Gamma(\nu+1)}h^\nu \\
&\le \frac{3\,\|Z(\cdot,\omega)\|_{L^\infty(J)}}{\Gamma(\nu+1)}h^\nu,
\end{aligned}$$

concluding the proof. □

Let us remark that fractionally integrated Gauss-Markov processes have been also studied in [33].

3.2. Compatibility between Fractionally Integrated Gaussian Processes

Now we want to study the behavior of a fractionally integrated Gaussian process with respect to other Gaussian processes. To do this let us first give the following shorthand notation.

Definition 7. *Let $Z(t)$ and $G(t)$ be two $\mathcal{F}_t$-adapted Gaussian processes with a.s. continuous paths. We say that $Z(t)$ and $G(t)$ are compatible if for any $n, m \in \mathbb{N}$, any $(a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}) \in \mathbb{R}^{n+m}$ and any $(t_1, \dots, t_n, t_{n+1}, \dots, t_{n+m}) \in \mathbb{R}^{n+m}$ the random variable*

$$\sum_{i=1}^{n} a_i Z(t_i) + \sum_{i=n+1}^{m} a_i G(t_i)$$

is still a Gaussian random variable. This obviously implies that $Z(t) + G(t) \in \mathcal{G}(J)$. Let us denote, for any $G \in \mathcal{G}(J)$,

$$\mathcal{G}(J, G) := \{Z \in \mathcal{G}(J) : Z \text{ and } G \text{ are compatible}\}.$$

It is obvious that if $Z(t)$ and $G(t)$ are independent $\mathcal{F}_t$-adapted Gaussian processes with a.s. continuous paths, then $Z(t)$ and $G(t)$ are compatible.

Now let us show the following Lemma.

Lemma 2. *Let $Z, G \in \mathcal{G}(J)$ such that $Z \in \mathcal{G}(J, G)$. Then, setting $Z_\nu(t) := \mathcal{I}_t^\nu Z$ for $t \in J$, $Z_\nu(t) \in \mathcal{G}(J, G)$.*

Proof. Let us consider

$$A = \{\omega \in \Omega : \ t \mapsto (Z(t,\omega), G(t,\omega)) \text{ is continuous}\}$$

and recall that $\mathbb{P}(\Omega \setminus A) = 0$. Fix $\omega \in A$ and observe that $Z_\nu(t) \in \mathcal{G}(J)$ by the previous Lemma and that for $\omega \in A$, $Z_\nu(\cdot, \omega)$ is continuous. Thus, we have that $t \mapsto Z_\nu(t, \omega) + G(t, \omega)$ is continuous for any $\omega \in A$. Moreover, since Z_ν and G are $\mathcal{F}_t$-adapted, $Z_\nu + G$ is $\mathcal{F}_t$-adapted. Now we need to show the compatibility property.

Let us fix $m_1, m_2 \in \mathbb{N}$, $(t_1, \dots, t_{m_1}, t_{m_1+1}, \dots, t_{m_2}) \in J^{m_1+m_2}$ and $(a_1, \dots, a_{m_1}, a_{m_1+1}, \dots, a_{m_2}) \in \mathbb{R}^{m_1+m_2}$ and let us define the random variables

$$\mathcal{Z} := \sum_{i=1}^{m_1} a_i Z_\nu(t_i) + \sum_{i=m_1+1}^{m_2} a_i G(t_i),$$
$$\mathcal{Z}^\varepsilon := \sum_{i=1}^{m_1} a_i \mathcal{I}_{t_i}^{\nu,\varepsilon} Z + \sum_{i=m_1+1}^{m_2} a_i G(t_i).$$

Let us work on the second one. Fix $\omega \in A$. Thus, by recalling the definition of $\delta^\varepsilon_{i,n}$ and $s_{i,j,n}$ for $i \le m$ and $j = 1, \dots, n$ given in the previous lemma, we have that

$$\mathcal{Z}^\varepsilon(\omega) = \lim_{n\to+\infty} \sum_{j=1}^{n} \sum_{i=1}^{m} a_i (t_i - s_{i,j,n})^{\nu-1} Z(s_{i,j,n}, \omega) \delta^\varepsilon_{i,n} + \sum_{i=1}^{m} a_i G(t_i, \omega)$$

hence we have that almost surely

$$\mathcal{Z}^\varepsilon = \lim_{n\to+\infty} \sum_{j=1}^{n} \sum_{i=1}^{m_1} a_i (t_i - s_{i,j,n})^{\nu-1} Z(s_{i,j,n}) \delta^\varepsilon_{i,n} + \sum_{i=m_1+1}^{m_2} a_i G(t_i)$$

where on the RHS we have Gaussian random variables since $Z(t)$ and $G(t)$ are compatible. Hence $\mathcal{Z}^\varepsilon$ is a Gaussian random variable. Moreover, if we define $\varepsilon_k = 1/k$ for $k \in \mathbb{N}$, one has that $\mathcal{Z} = \lim_{k\to+\infty} \mathcal{Z}^{\varepsilon_k}$ almost surely, thus $\mathcal{Z}$ is a Gaussian random variable and then Z_ν and G are compatible. □

Remark 2. *Obviously, for any $f, g \in C^0(J)$ and $Z \in \mathcal{G}(J, G)$, we have $fZ + g \in \mathcal{G}(J, G)$. Moreover, for any $f, g \in C^0(J)$ and $Z \in \mathcal{G}(J, G)$, we also have $fZ + gG \in \mathcal{G}(J, G)$.*

3.3. Main Result

Now we are ready to show an existence and uniqueness result in $\mathcal{G}(J)$ for the solution of Equation (12) in the fashion of [6] (Theorem 3.3).

Theorem 1. *For any $T > 0$, $J = [0, T]$, and $G \in \mathcal{G}(J)$, Equation (12) admits a unique solution $Y_\nu \in \mathcal{G}(J)$. Moreover, if $G \in \mathcal{G}_\nu(J)$, then $Y_\nu \in \mathcal{G}_\nu(J)$.*

Proof. Let us consider

$$A = \{\omega \in \Omega : \ t \mapsto G(t, \omega) \text{ is continuous}\} \tag{13}$$

and recall that $\mathbb{P}(\Omega \setminus A) = 0$. Fix $\omega \in A$ and define the operator $\mathcal{A}_\omega : C^0_\gamma(J) \to C^0_\gamma(J)$ as

$$(\mathcal{A}_\omega f)(t) = y_0 + \mathcal{I}^\nu_t(af + b) + G(t, \omega), \ f \in C^0(J)$$

where $C^0_\gamma(J)$ is the Banach space $(C^0(J), \|\cdot\|_\gamma)$ of the continuous functions equipped with the Bielecki norm $\|f\|_\gamma = \max_{t\in J} |f(t)|e^{-\gamma t}$ for some $\gamma > 0$, which is equivalent to the classical L^∞ norm.

Let us show that $\mathcal{A}_\omega$ is well-posed, i.e., $\mathcal{A}_\omega f$ is continuous. To do this consider $t_1, t_2 \in J$ and suppose, without loss of generality, that $t_1 < t_2$. Then, we can set $t_2 = t_1 + \delta$. We have

$$
\begin{aligned}
|\mathcal{A}_\omega f(t_1+\delta)-\mathcal{A}_\omega f(t_1)| &\le \frac{1}{\Gamma(\nu)}\int_0^{t_1}((t_1-\tau)^{\nu-1}-(t_1+\delta-\tau)^{\nu-1})|af(\tau)+b|d\tau \\
&+\frac{1}{\Gamma(\nu)}\int_{t_1}^{t_1+\delta}(t_1+\delta-\tau)^{\nu-1}|af(\tau)+b|d\tau \\
&+|G(t_1+\delta,\omega)-G(t_1,\omega)| \\
&\le \frac{|a|\,\|f\|_{L^\infty(J)}+|b|}{\Gamma(\nu+1)}(t_1^\nu-(t_1+\delta)^\nu+\delta^\nu) \\
&+\frac{|a|\,\|f\|_{L^\infty(J)}+|b|}{\Gamma(\nu+1)}\delta^\nu+|G(t_1+\delta,\omega)-G(t_1,\omega)| \\
&\le \frac{3(|a|\,\|f\|_{L^\infty(J)}+|b|)}{\Gamma(\nu+1)}\delta^\nu+|G(t_1+\delta,\omega)-G(t_1,\omega)|
\end{aligned} \tag{14}
$$

hence, being $G(\cdot,\omega)$ continuous, sending $\delta\to 0^+$, we have $\lim_{\delta\to 0^+}|\mathcal{A}_\omega f(t+\delta)-\mathcal{A}_\omega f(t)|=0$ and then $\mathcal{A}_\omega f\in C^0(J)$.

Now consider $f_1,f_2\in C^0(J)$, choose $q\in\left(1,\frac{1}{1-\nu}\right)$ and set p such that $\frac{1}{q}+\frac{1}{p}=1$. We have

$$
\begin{aligned}
|\mathcal{A}_\omega f_1(t)-\mathcal{A}_\omega f_2(t)| &\le \frac{|a|}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}|f_1(\tau)-f_2(\tau)|d\tau \\
&= \frac{|a|}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}|f_1(\tau)-f_2(\tau)|e^{-\gamma\tau}e^{\gamma\tau}d\tau \\
&\le \frac{|a|\,\|f_1-f_2\|_\gamma}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1}e^{\gamma\tau}d\tau \\
&\le \frac{|a|\,\|f_1-f_2\|_\gamma}{\Gamma(\nu)}\left(\int_0^t (t-\tau)^{q(\nu-1)}d\tau\right)^{\frac{1}{q}}\left(\int_0^t e^{p\gamma\tau}d\tau\right)^{\frac{1}{p}} \\
&\le \frac{|a|\,\|f_1-f_2\|_\gamma\, T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}}\Gamma(\nu)}\frac{1}{(p\gamma)^{\frac{1}{p}}}(e^{p\gamma t}-1)^{\frac{1}{p}} \\
&\le \frac{|a|\,\|f_1-f_2\|_\gamma\, T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}}\Gamma(\nu)}\frac{1}{(p\gamma)^{\frac{1}{p}}}e^{\gamma t}
\end{aligned}
$$

and then

$$
|\mathcal{A}_\omega f_1(t)-\mathcal{A}_\omega f_2(t)|e^{-\gamma t}\le \frac{|a|T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}}\Gamma(\nu)(p\gamma)^{\frac{1}{p}}}\|f_1-f_2\|_\gamma .
$$

Taking the maximum, we have

$$
\|\mathcal{A}_\omega f_1-\mathcal{A}_\omega f_2\|_\gamma\le \frac{|a|T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}}\Gamma(\nu)(p\gamma)^{\frac{1}{p}}}\|f_1-f_2\|_\gamma .
$$

Thus, we can choose $\gamma>0$ big enough to have

$$
\frac{|a|T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}}\Gamma(\nu)(p\gamma)^{\frac{1}{p}}}<1.
$$

With this choice of γ, we have that $\mathcal{A}_\omega$ is a contraction and thus admits a unique fixed point (see [34], Theorem 3.1): let us denote it as $Y_\nu(\cdot,\omega)$.

Moreover, let us consider the sequence (for fixed $\omega \in A$)

$$\begin{cases} f_0(t,\omega) \equiv \ 0 \\ f_n(t,\omega) = \mathcal{A}_\omega f_{n-1}(t,\omega) \quad n \geq 1. \end{cases}$$

This sequence is such that $f_n(\cdot,\omega) \to Y_\nu(\cdot,\omega)$ in C^0 by contraction theorem (see [34]).

Now let us define a stochastic operator $\mathcal{A}$. For $\omega \in A$, let us define it as

$$\mathcal{A} f(t,\omega) = \mathcal{A}_\omega f(t,\omega)$$

for any stochastic process such that $t \mapsto f(t,\omega)$ is continuous for any $\omega \in A$, while for $\omega \notin A$ let us complete it as we wish, since $\Omega \setminus A$ is a null set.

We can re-interpret our sequence as a sequence of stochastic processes given by

$$\begin{cases} f_0(t) \equiv 0 \\ f_n(t) = \mathcal{A} f_{n-1}(t) \quad n \geq 1. \end{cases} \tag{15}$$

Now let us observe that f_0 is a (degenerate) $\mathcal{F}_t$-adapted Gaussian process with a.s. continuous paths. For f_1, we have that

$$f_1(t) = y_0 + G(t)$$

which obviously belongs to $\mathcal{G}(J,G)$ by Remark 2. Let us suppose that $f_{n-1} \in \mathcal{G}(J,G)$. By using Remark 2 and Lemmas 1 and 2, we have that $f_n \in \mathcal{G}(J,G)$. Hence we have that for any $n \in \mathbb{N}$, $f_n \in \mathcal{G}(J,G)$.

Now, we have that

$$Y_\nu(t) = \lim_{n \to +\infty} f_n(t)$$

where the limit is in the a.s. sense, thus it is easy to see that $Y_\nu \in \mathcal{G}(J)$ (a.s. continuity of the paths follows from the continuity of $t \mapsto Y_\nu(t,\omega)$ for $\omega \in \mathcal{A}$, since $Y_\nu(\cdot,\omega)$ are fixed points of $\mathcal{A}_\omega$). Finally, a.s. uniqueness follows from the fact that $\mathcal{A}_\omega$ are contractions for $\omega \in A$, hence their fixed point is unique.

Now, if G is a.s. ν-Hölder continuous, let us define

$$A_\nu = \{\omega \in \Omega : \ t \mapsto G(t,\omega) \text{ is } \nu\text{-Hölder continuous}\}$$

and let us recall that $\mathbb{P}(\Omega \setminus A_\nu) = 0$. Consider $\omega \in A_\nu$ and $f \in C^0(J)$ and observe that, from (14), we have

$$\begin{aligned} |\mathcal{A}_\omega f(t_1+\delta) - \mathcal{A}_\omega f(t_1)| &\leq \frac{3(|a| \, \|f\|_{L^\infty(J)} + |b|)}{\Gamma(\nu+1)}\delta^\nu + |G(t_1+\delta,\omega) - G(t_1,\omega)| \\ &\leq \left(\frac{3(|a| \, \|f\|_{L^\infty(J)} + |b|)}{\Gamma(\nu+1)} + C(\omega)\right)\delta^\nu \end{aligned}$$

where $C(\omega)$ is such that $|G(t+\delta,\omega) - G(t,\omega)| \leq C(\omega)\delta^\nu$ (that exists since we have chosen $\omega \in A_\nu$). In particular, we have that for any $f \in C^0(J)$, $\mathcal{A}_\omega f \in C^\nu(J)$. Almost surely ν-Hölder continuity of the paths of Y_ν thus follows from the fact that, for any $\omega \in A_\nu$, $Y_\nu(\cdot,\omega) = A_\omega Y_\nu(\cdot,\omega)$. $\square$

Remark 3. *Let us observe that the fractional-integral operator $\mathcal{I}_t^\nu$ is a compact Hilbert-Schmidt operator in $L^2([0,T])$ (see, for instance [35]) if $\nu > \frac{1}{2}$. Indeed, the integral kernel $k(t,\tau) := (t-\tau)^{\nu-1}\chi_{(0,t)}(\tau)$ (where $\chi_{(0,t)}$ is the indicator function of the interval $(0,t)$) is such that*

$$\int_0^T \int_0^T |k(t,\tau)|^2 dt d\tau < +\infty$$

if and only if $\nu > \frac{1}{2}$. In such case, one can use the structure of the equation to show that there exists a unique Gaussian solution. Setting for instance $a = 1$ and $b = 0$, we have for fixed $\omega \in \Omega$

$$(I - \mathcal{I}_t^\nu)Y_\nu(\cdot,\omega) = y_0 + G(\cdot,\omega),$$

where I is the identity operator; hence we have

$$Y_\nu(\cdot,\omega) = (I - \mathcal{I}_t^\nu)^{-1}(y_0 + G(\cdot,\omega)).$$

In such a way, for $\nu > \frac{1}{2}$, one has the characterization of the solution Y as

$$Y_\nu = \sum_{i=0}^{+\infty} (\mathcal{I}_t^\nu)^i (y_0 + G)$$

and then Y is given by a linear operator applied to a Gaussian process, hence it is Gaussian.

3.4. Speed of Convergence

We could also investigate the speed of convergence of the sequence f_n defined in Equation (15) to Y_ν. The following proposition is an easy consequence of the contraction theorem.

Proposition 2. *Consider $J = [0,T]$ for some $T > 0$, $\nu \in (0,1)$ and $q \in \left(1, \frac{1}{1-\nu}\right)$. Moreover, consider Y_ν solution of Equation (12). Set p such that $\frac{1}{q} + \frac{1}{p} = 1$ and*

$$L = \frac{|a| T^{\frac{1+q(\nu-1)}{q}}}{(1+q(\nu-1))^{\frac{1}{q}} \Gamma(\nu) (p\gamma)^{\frac{1}{p}}}$$

and fix $\gamma > 0$ such that $L < 1$. Finally, define $\widetilde{G}(t) = y_0 + G(t)$. Then, for the sequence f_n defined in Equation (15) we have

$$\|f_n - Y_\nu\|_{L^\infty(J)} \le \frac{L^n}{1-L} e^{\gamma T} \left\|\widetilde{G}\right\|_{L^\infty(J)} \tag{16}$$

almost surely. As a consequence it holds

$$\mathbb{P}(\|f_n - Y_\nu\|_{L^\infty(J)} > \varepsilon) \le \mathbb{P}\left(\max_{t\in[0,T]} |\widetilde{G}(t)| > \frac{(1-L)\varepsilon}{L^n e^{\gamma T}}\right).$$

Proof. Fix $\omega \in A$ (where the set A is defined in Equation (13)). By contraction theorem (see [34]) we have, since L is the Lipschitz constant of $\mathcal{A}_\omega : C_\gamma^0(J) \to C_\gamma^0(J)$,

$$\|f_n(\cdot,\omega) - Y_\nu(\cdot,\omega)\|_\gamma \le \frac{L^n}{1-L} \|f_1(\cdot,\omega) - f_0(\cdot,\omega)\|_\gamma .$$

Now let us recall that for any function $f \in C_\gamma^0(J)$

$$e^{-\gamma T} \|f\|_{L^\infty(J)} \le \|f\|_\gamma \le \|f\|_{L^\infty(J)}$$

thus, we have

$$\|f_n(\cdot,\omega)-Y_\nu(\cdot,\omega)\|_{L^\infty(J)} \le \frac{L^n}{1-L}e^{\gamma T}\|f_1(\cdot,\omega)-f_0(\cdot,\omega)\|_{L^\infty(J)}.$$

Now let us recall that $f_0(t,\omega)=0$ and $f_1(t,\omega)=y_0+G(t,\omega)=\widetilde{G}(t,\omega)$, thus we have

$$\|f_n(\cdot,\omega)-Y_\nu(\cdot,\omega)\|_{L^\infty(J)} \le \frac{L^n}{1-L}e^{\gamma T}\left\|\widetilde{G}(\cdot,\omega)\right\|_{L^\infty(J)}.$$

Since $\mathbb{P}(\Omega\setminus A)=0$, Equation (16) holds. □

3.5. The Mean of Y_ν

Let us introduce another class of Gaussian processes

$$\mathcal{G}_0(J) := \{G\in\mathcal{G}(J):\ \mathbb{E}[G(t)]=0,\ \forall t\in J\}.$$

We want to investigate the mean of Y_ν, solution of (12), when $G\in\mathcal{G}_0(J)$. We have the following result.

Proposition 3. *Fix $G\in\mathcal{G}_0(J)$ and let us suppose that $y_\nu(t):=\mathbb{E}[Y_\nu(t)]$ is in $L^\infty(J)$, where $Y_\nu(t)$ is solution of (12) in J. Then, $y_\nu(t)$ is solution of the fractional Cauchy problem*

$$\begin{cases}\frac{d^\nu y_\nu}{dt^\nu}(t)=ay_\nu(t)+b,\ t\in\ J\\ y_\nu(0)=y_0.\end{cases}$$

Proof. First, let us observe that

$$Y_\nu(0)=y_0+G(0)$$

hence $y_\nu(0)=y_0$. Now, let us notice that

$$\int_0^t(t-\tau)^{\nu-1}|y_\nu(\tau)|d\tau\le\|y_\nu\|_{L^\infty(J)}\frac{t^\nu}{\nu}.$$

Hence we can use Fubini's theorem to achieve

$$y_\nu(t)=y_0+\mathcal{I}_t^\nu(ay_\nu+b).$$

Rearranging the equation and applying $\mathcal{I}_t^{1-\nu}$ on both sides we have

$$\mathcal{I}_t^{1-\nu}(y_\nu-y_0)=\int_0^t(ay_\nu(\tau)+b)d\tau.$$

Since on the RHS we have a C^1 function, we can differentiate both terms and use (6) and (7) to achieve

$$\frac{d^\nu y_\nu}{dt^\nu}(t)=ay_\nu(t)+b.$$

□

Remark 4. *It is not difficult to show that if $G\in\mathcal{G}(J)$, $g(t):=\mathbb{E}[G(t)]$ is Riemann-Liouville derivable and y_ν is in $L^\infty(J)$, then y_ν is solution of the Cauchy problem*

$$\begin{cases}\frac{d^\nu y_\nu}{dt^\nu}(t)=ay_\nu(t)+b+D_t^\nu g(t),\ t\in\ J\\ y_\nu(0)=y_0+g(0).\end{cases}$$

The proof of such result is analogous to the previous one.

4. The Choice of G: Some Examples

In this section, we will give some example concerning the choice of the process $G \in \mathcal{G}(J)$. Actually, these kinds of equations are noisy versions of the Cauchy problems

$$\begin{cases} \frac{d^\nu y_\nu}{dt^\nu} = ay_\nu + b \\ y_\nu(0) = y_0 \end{cases} \tag{17}$$

and the choice of the noise depends on the choice of $G \in \mathcal{G}(J)$. Moreover, if we take $G \in \mathcal{G}_0(J)$ and $\mathbb{E}[Y_\nu(t)]$ is in $L^\infty(J)$, we are considering a process with an assigned mean and we can modulate covariance by changing G. Let us give some examples.

4.1. Brownian Motion and White Noise

A first simple case is given by choosing $G(t)$ as a Brownian motion $W(t)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Concerning the regularity of the solution Y_ν of Equation (12), we have the following result.

Corollary 1. *Consider in Equation (12) $\nu \in (0, 1/2)$ and $G = W$ a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Then the solution Y_ν of Equation (12) belongs to $\mathcal{G}_\nu(J) \cap \mathcal{G}(J, W)$ for any $J = [0, T]$.*

Actually, we could imagine writing (only formally) our integral equation in differential form. We have, by (formally) using the relation (6),

$$dY_\nu(t) = D_t^{1-\nu}(aY_\nu + b)dt + dW(t).$$

Writing in this way, we can see what the role is of $W(t)$: it works like a white noise introduced in Equation (17).

In particular, if $\nu = 1$, $Y_1(t)$ is a classical Ornstein-Uhlenbeck process.

4.2. Fractional Brownian Motion and Fractional White Noise

We could also choose $G(t)$ to be a fractional Brownian motion $W_H(t)$ with Hurst index $H \in (0, 1)$, introduced in [36]. We will not focus on the features of such process, but for them we refer to [37,38]. Let us recall that the definition of W_H already involves fractional integrals. Indeed, for $\nu \in (0, 1)$, we can define the operators ${}_-\mathcal{I}_t^\nu$ and ${}_-D_t^\nu$ as

$${}_-\mathcal{I}_t^\nu f = \frac{1}{\Gamma(\nu)} \int_t^{+\infty} (\tau - t)^{\nu-1} f(\tau) d\tau$$

and

$${}_-D_t^\nu f = -\frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_t^{+\infty} (\tau - t)^{-\nu} f(\tau) d\tau$$

for any suitable function $f : \mathbb{R} \to \mathbb{R}$. In such case, if $H \in \left(\frac{1}{2}, 1\right)$ and we fix $\nu = H - \frac{1}{2}$, then there exists a (normalizing) constant C_H such that

$$W_H(t) = C_H \int_{\mathbb{R}} {}_-\mathcal{I}_s^\nu \chi_{(0,t)} dW(s)$$

while if $H \in \left(0, \frac{1}{2}\right)$, if we fix $\nu = \frac{1}{2} - H$, we have

$$W_H(t) = C_H \int_{\mathbb{R}} {}_-D_s^\nu \chi_{(0,t)} dW(s)$$

Concerning the regularity of the solution Y_ν of Equation (12), we have the following result:

Corollary 2. *Consider $H \in (0,1)$ and let (in Equation (12)) $G = W_H$ be a fractional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with Hurst index H; then, if $\nu \in (0, H)$, the solution Y_ν of Equation (12) belongs to $\mathcal{G}_\nu(J) \cap \mathcal{G}(J, W_H)$ for any $J = [0, T]$.*

Such corollary is linked to the fact that the paths of $W_H(t)$ are γ-Hölder continuous for any $\gamma < H$, as shown in [38].

As before, we could formally write Equation (12) in differential form, by using Equation (6), to achieve

$$dY_\nu(t) = D_t^{1-\nu}(aY_\nu + b)dt + dW_H(t)$$

where the fractional white noise dW_H must be carefully interpreted. Thus, we have that our equation is a perturbation of (17) with a fractional white noise.

For $\nu = 1$, we obtain the fractional Ornstein-Uhlenbeck process ([39,40]).

4.3. Ornstein-Uhlenbeck Process and Colored Noise

We get another example by choosing $G(t)$ to be an Ornstein-Uhlenbeck process $U(t)$, solution of

$$dU(t) = (\lambda U(t) + \mu)dt + \sigma dW(t) \tag{18}$$

for some $\lambda, \mu \in \mathbb{R}$, $\sigma > 0$ and W a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. In such case, we have the following regularity result:

Corollary 3. *Consider, in Equation (12), $\nu \in \left(0, \frac{1}{2}\right)$ and $G := U$ an Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Then, the solution Y_ν of Equation (12) belongs to $\mathcal{G}_\nu(J) \cap \mathcal{G}(J, U)$ for any $J = [0, T]$.*

As before we can write the differential form of the equation obtaining

$$dY_\nu(t) = D_t^{1-\nu}(aY_\nu + b)dt + dU(t)$$

that, by using Equation (18), becomes

$$dY_\nu(t) = (D_t^{1-\nu}(aY_\nu + b) + \lambda U(t) + \mu)dt + \sigma dW(t),$$

thus observing the effect of a colored noise (for $\nu = 1$ see, for instance, [41,42]).

Eventually, we could also use a fractional Ornstein-Uhlenbeck U_H in place of U, obtaining the following regularity result:

Corollary 4. *Consider $H \in (0,1)$ and let (in Equation (12)) $G = U_H$ be a fractional Ornstein-Uhlenbeck process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with Hurst index H; then, if $\nu \in (0, H)$, the solution Y_ν of Equation (12) belongs to $\mathcal{G}_\nu(J) \cap \mathcal{G}(J, U_H)$ for any $J = [0, T]$.*

4.4. Fractional Itô Integral

There is another particular choice of G that can be done. Let us suppose that $\nu \in \left(\frac{1}{2}, 1\right)$ and observe that, for fixed $t > 0$, the function $(t-\tau)^{\nu-1}$ is in $L^2(0,t)$. Thus, the following process is well-defined and belongs to $\mathcal{G}_0(J)$ for any $J = [0, T]$:

$$G(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} dW(\tau) \tag{19}$$

where W is a Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and the integral must be interpreted in the Itô sense. With this noise, Equation (12) is the integral version of a Caputo-fractional stochastic differential equation, as studied for instance in [24,25]. For such equations, closed form of the solutions can be obtained via a variation of constant formula, as shown in [24]. In this particular case it is known that $y_\nu(t) := \mathbb{E}[Y_\nu(t)]$ is a continuous function.

5. Stochastic Models for Fractional Gompertz Curves

In this section, we will construct two classes of stochastic models for fractional Gompertz curves: one for the fractional Gompertz curve given in [21], the other for the improved one given in [22]. Moreover, we will introduce a third model that combines the two previous approaches. However, let us first recall how the classical stochastic Gompertz model is constructed.

5.1. The Stochastic Gompertz Model

Here we will recall some basics of the stochastic Gompertz model. We will follow the lines of [43]. Let us consider a stochastic process $X(t)$ solution of the following stochastic differential equation

$$dX(t) = \left(\alpha - \beta \log\left(\frac{X(t)}{X_0}\right)\right) X(t)dt + \sqrt{2\alpha}X(t)dW(t),\ X(0) = x_0 \tag{20}$$

where $x_0 > 0$ is a constant, $W(t)$ is a Brownian motion (with respect to the filtration $\mathcal{F}_t$) and $\alpha, \beta > 0$ are the growth parameters we defined in Section 1.

Now, if we define the process $Y(t) = \log\left(\frac{X(t)}{x_0}\right)$, a simple application of the Itô formula leads to

$$dY(t) = (-\beta Y(t) + \alpha)dt + \sqrt{2\alpha}dW(t),\ Y(0) = 0 \tag{21}$$

which is the Stochastic Differential Equation of an Ornstein-Uhlenbeck process. It will be useful to write such equation in integral form

$$Y(t) = \int_0^t (-\beta Y(t) + \alpha)dt + \sqrt{2\alpha}W(t). \tag{22}$$

In particular, $Y(t)$ is a Gaussian process and thus $X(t) = x_0 e^{Y(t)}$ is a log-normal process. Moreover, it is easy to see that, setting $y(t) = \mathbb{E}[Y(t)]$, we have

$$\begin{cases} \frac{dy}{dt}(t) = -\beta y(t) + \alpha \\ y(0) = 0 \end{cases}$$

that is the second equation of (2). However, since (20) is non-linear, we cannot conclude that the mean of $X(t)$ solves the Gompertz Equation (1). Let us then denote with $x(t)$ the median of $X(t)$, i.e., for each $t > 0$

$$x(t) := \inf\left\{z \in \mathbb{R} :\ \mathbb{P}(X(t) \le z) > \frac{1}{2}\right\}.$$

Since $X(t)$ is absolutely continuous for $t > 0$, $x(t)$ is the unique solution of the equation (in z)

$$\mathbb{P}(X(t) \le z) = \frac{1}{2}$$

while, since $X(0) = x_0$, $x(0) = x_0$.

It is well known that the median of a log-normal variable coincides with the exponential of the mean of its logarithm, i.e.,

$$x(t) = x_0 e^{y(t)}.$$

In particular this shows that $x(t)$ solves Gompertz Equation (1).

For this reason, we can consider $X(t)$ a stochastic version of the Gompertz curve: the deterministic model is recovered via the median of the process, which is an observable function that describes the macroscopic behavior of the process.

Following this line, we are searching for a log-normal (or a Gaussian process) such that the median (or the mean) is a fractional Gompertz curve.

5.2. A Stochastic Model for the Fractional Gompertz Curve

Let us first obtain the stochastic model for the fractional Gompertz curve given in [21]. Recalling the construction of the classical stochastic Gompertz model, we are searching for a log-normal stochastic process whose median is a fractional Gompertz curve.

To do this, fix a time horizon $T > 0$ and a time window $J = [0, T]$. Let us consider the following stochastic linear fractional-integral equation:

$$Y_\nu(t) = \mathcal{I}_t^\nu(-\beta Y_\nu(t) + \alpha) + G(t) \tag{23}$$

for some $G \in \mathcal{G}_0(J)$ such that $y_\nu(t) := \mathbb{E}[Y_\nu(t)]$ is a $L^\infty(J)$ function. Equation (23) can be recognized as a stochastic version of Equation (3). Indeed, the latter can be written in integral form as

$$y_\nu(t) = \mathcal{I}_t^\nu(-\beta y_\nu(t) + \alpha)$$

and then Equation (23) follows by adding a noise $G(t)$. In particular, such equation follows as a generalization of Equation (22) by substituting the classical integral with the fractional one and the white noise with a general Gaussian one.

A natural choice for $G(t)$, if $\nu \in \left(\frac{1}{2}, 1\right)$, is the one given in Section 4.4 by Equation (19). In particular, if we set

$$G(t) = \frac{\sqrt{2\alpha}}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} dW(\tau),$$

Equation (23) can be formally seen as a fractional version of Equation (21).

Now, let us define the process $X_\nu(t) = X_0 e^{Y_\nu(t)}$ with $Y_\nu(t)$ defined in (23). We have the following result.

Proposition 4. *The process $X_\nu(t)$ is a log-normal process. Moreover, its median $x_\nu(t)$ is given by*

$$x_\nu(t) = x_0 \exp\left(\frac{\alpha}{\beta}(1 - E_\nu(-\beta t^\nu))\right),$$

i.e., is a fractional Gompertz curve.

Proof. The fact that $X_\nu(t)$ is a log-normal process follows from the fact that $Y_\nu \in \mathcal{G}(J)$. Moreover, since it is a log-normal process, we have

$$x_\nu(t) = x_0 e^{y_\nu(t)}$$

where $y_\nu(t) = \mathbb{E}[Y_\nu(t)]$. By Proposition 3, we know that $y_\nu(t)$ is solution of Equation (3) and we conclude the proof. □

5.3. A Stochastic Model for the Improved Fractional Gompertz Curve

Let us obtain a stochastic model for the improved fractional Gompertz curve. To do this, let us recall that the improved fractional Gompertz curve is solution of

$$\begin{cases} \left(e^{\beta t}\frac{d}{dt}\right)^{\nu} x_\nu(t) = \alpha x_\nu(t), \\ x_\nu(0) = x_0. \end{cases} \tag{24}$$

In this case, referring to Definition 6, let us choose Ψ as

$$\Psi(t) = \frac{1}{\beta}(1 - e^{-\beta t})$$

to rewrite Equation (24) as follows:

$$\begin{cases} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\nu} x_\nu(t) = \alpha x_\nu(t) \\ x_\nu(0) = x_0 \end{cases}$$

Now, let us consider Y_ν solution of

$$Y_\nu(t) = x_0 + \mathcal{I}_t^\nu(-\beta Y_\nu + \alpha) + G(t)$$

with $y_\nu(t) := \mathbb{E}[Y_\nu(t)]$ in $L^\infty(J)$. As before, for $\nu \in \left(\frac{1}{2}, 1\right)$, a natural choice for the noise is given by

$$G(t) = \frac{\sqrt{2\alpha}}{\Gamma(\nu)}\int_0^t (t-\tau)^{\nu-1} dW(\tau).$$

Now let us define $X_\nu(t) := Y_\nu(\Psi(t))$. We have the following result.

Proposition 5. *$X_\nu(t)$ is a Gaussian process whose mean $x_\nu(t) := \mathbb{E}[X_\nu(t)]$ is given by*

$$x_\nu(t) = x_0 E_\nu\left(\frac{\alpha}{\beta^\nu}(1 - e^{-\beta t})^\nu\right).$$

Proof. Recalling that $y_\nu(t) = \mathbb{E}[Y_\nu(t)]$, we have, from Proposition 3, that y_ν is solution of

$$\begin{cases} \frac{d^\nu y_\nu}{dt^\nu}(t) = -\beta y_\nu(t) + \alpha, \quad t \in J \\ y_\nu(0) = y_0. \end{cases}$$

Moreover, recalling also that $X_\nu(t) = Y_\nu(\Psi(t))$, we know that

$$x_\nu(t) = \mathbb{E}[X_\nu(t)] = \mathbb{E}[Y_\nu(\Psi(t))] = y_\nu(\Psi(t))$$

where $\Psi(t) = \frac{1}{\beta}(1 - e^{-\beta t})$. Let us observe that $x_\nu(0) = y_\nu(\Psi(0)) = x_0$. Finally, by Proposition 1,

$$\left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\nu} x_\nu(t) = \frac{d^\nu y_\nu}{dt^\nu}(\Psi(t)) = -\beta y_\nu(\Psi(t)) + \alpha = -\beta x_\nu(t) + \alpha,$$

concluding the proof. □

5.4. A New Fractional Model and Its Stochastic Counterpart

Now we want to give a fractional model that takes into account both the fractional Gompertz curve and the improved fractional Gompertz curve. To do this, we will suppose an a priori form for the growth rate. This has been done for instance in [44], with the introduction of a growth model that takes into account both the Gompertz and the Korf dynamics. In general, let us consider as a starting point a model of the form

$$\begin{cases} \frac{dx}{dt}(t) = r(t)x(t), \\ x(0) = x_0 \end{cases} \tag{25}$$

for some growth rate function $r(t)$. Equation (25) can be solved and solution is given by

$$x(t) = x_0 \exp\left(\int_0^t r(s)ds\right). \tag{26}$$

In our case, let us consider as growth rate the function

$$r_{\nu_1}(t) = \alpha t^{\nu_1 - 1} E_{\nu_1,\nu_1}(-\beta t^{\nu_1}),\ t > 0,$$

for some $\nu_1 \in (0,1)$, where $E_{\nu,\theta}(t)$ is the two-parameters Mittag-Leffler function (see, for instance, [45]) defined as

$$E_{\nu,\theta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(\nu k + \theta)},\ t \in \mathbb{R}.$$

Let us also denote with x_{ν_1} the solution of Equation (25) where $r_{\nu_1}(t)$ is used in place of $r(t)$. In this case, we can explicitly calculate the integral in Equation (26). Indeed, let us denote $R(t) = E_{\nu_1}(-t^{\nu_1})$. We have

$$R(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k t^{\nu_1 k}}{\Gamma(\nu_1 k + 1)}$$

thus, differentiating the series term by term, we have

$$\begin{aligned} R'(t) &= \sum_{k=1}^{+\infty} \frac{(-1)^k \nu_1 k t^{\nu_1 k - 1}}{\Gamma(\nu_1 k + 1)} \\ &= \sum_{k=1}^{+\infty} \frac{(-1)^k t^{\nu_1 k - 1}}{\Gamma(\nu_1 k)} = \sum_{k=0}^{+\infty} \frac{(-1)^{k+1} t^{\nu_1 k + \nu_1 - 1}}{\Gamma(\nu_1 k + \nu_1)} \\ &= -t^{\nu_1 - 1} \sum_{k=0}^{+\infty} \frac{(-1)^k t^{\nu_1 k}}{\Gamma(\nu_1 k + \nu_1)} = -t^{\nu_1 - 1} E_{\nu_1,\nu_1}(-t^{\nu_1}). \end{aligned} \tag{27}$$

Now let us consider

$$\Psi_{\nu_1}(t) = \frac{1}{\beta}(1 - E_{\nu_1}(-\beta t^{\nu_1})) = \frac{1}{\beta}(1 - R(\beta^{1/\nu_1} t));$$

by using Equation (27) we have

$$\Psi'_{\nu_1}(t) = -\frac{1}{\beta}\beta^{1/\nu_1} R'(\beta^{1/\nu_1} t) = t^{\nu_1 - 1} E_{\nu_1,\nu_1}(-\beta t^{\nu_1}) = \frac{r_{\nu_1}(t)}{\alpha}.$$

Thus, we have that

$$\int_0^t r_{\nu_1}(t)dt = \alpha \Psi_{\nu_1}(t).$$

By substituting this last integral in Equation (26) for r_{ν_1} and writing Ψ_{ν_1} explicitly, we achieve

$$x_{\nu_1}(t) = x_0 \exp\left(\frac{\alpha}{\beta}(1 - E_{\nu_1}(-\beta t^{\nu_1}))\right)$$

which is actually the fractional Gompertz curve given in [21]. Indeed, recalling that for the fractional Gompertz curve we had $x_{\nu_1}(t) = x_0 e^{y_{\nu_1}(t)}$ where

$$y_{\nu_1}(t) = \frac{\alpha}{\beta}(1 - E_{\nu_1}(-\beta t^{\nu})),$$

$x_{\nu_1}(t)$ was solution of the equation

$$\begin{cases} \frac{dx_{\nu_1}(t)}{dt} = y'_{\nu_1}(t) x_{\nu_1}(t) \\ x_{\nu_1}(0) = x_0. \end{cases}$$

Since $y_{\nu_1}(t) = \alpha \Psi_{\nu_1}(t)$, we have shown that $y'_{\nu_1}(t) = r_{\nu_1}(t)$ and we achieve Equation (25). Thus, we can conclude that equation

$$\begin{cases} \frac{1}{\Psi'_{\nu_1}(t)} \frac{dx_{\nu_1}}{dt}(t) = \alpha x_{\nu_1}(t), \\ x_{\nu_1}(0) = x_0 \end{cases}$$

defines the fractional Gompertz curve. Here we already introduced a first degree of fractionalization: now let us introduce another fractional timescale. To do this, let us work as in [22] and let us consider a fractional generalization obtained by introducing the fractional derivative with respect to Ψ_{ν_1}. So our new fractional Gompertz curve will be defined as the solution x_ν (where we denote $\nu = (\nu_1, \nu_2) \in (0,1)^2$) of the fractional Cauchy problem

$$\begin{cases} \left(\frac{1}{\Psi'_{\nu_1}(t)} \frac{d}{dt} \right)^{\nu_2} x_\nu(t) = \alpha x_\nu(t), \\ x_\nu(0) = x_0 \end{cases}$$

that, being a relaxation equation for the Caputo derivative with respect to Ψ, can be explicitly solved as

$$x_\nu(t) = x_0 E_{\nu_2}(\alpha \Psi(t)^{\nu_2}) = x_0 E_{\nu_2}\left(\frac{\alpha}{\beta^{\nu_2}} (1 - E_{\nu_1}(-\beta t^{\nu_1}))^{\nu_2} \right).$$

This new fractional Gompertz curve exhibits two degrees of fractionality: one given by the fact that we chose the growth function to be the one of the fractional Gompertz curve, the other from the fact that we introduced a fractional Caputo derivative (with respect to the integral of the growth rate) in the corresponding time in-homogeneous relaxation equation. This also leads to the possibility of considering two different fractional timescales: one for the population, the other for the growth rate. Concerning the stochastic model for such fractional Gompertz curve, fix $\nu = (\nu_1, \nu_2) \in (0,1)^2$ and let us consider Y_{ν_2} as the solution of

$$Y_{\nu_2}(t) = x_0 + \mathcal{I}_t^{\nu_2}(-\beta Y_{\nu_2} + \alpha) + G(t)$$

such that $y_{\nu_2}(t) = \mathbb{E}[Y_{\nu_2}(t)]$ is in $L^\infty(J)$. As we already stated, if $\nu_2 \in \left(\frac{1}{2}, 1\right)$, we could consider

$$G(t) = \frac{\sqrt{2\alpha}}{\Gamma(\nu_2)} \int_0^t (t - \tau)^{\nu_2 - 1} dW(\tau).$$

Finally, let us define $X_\nu(t) := Y_{\nu_2}(\Psi_{\nu_1}(t))$. We have the following result.

Proposition 6. *$X_\nu(t)$ is a Gaussian process whose mean $x_\nu(t) := \mathbb{E}[X_\nu(t)]$ is given by*

$$x_\nu(t) = x_0 E_{\nu_2}\left(\frac{\alpha}{\beta^{\nu_2}} (1 - E_{\nu_1}(-\beta t^{\nu_1}))^{\nu_2} \right).$$

The proof of such proposition is analogous to the one of Proposition 5.

6. Conclusions

In this paper, we have given some methods to construct stochastic fractional Gompertz models by using stochastic linear fractional equations with Gaussian driving processes. The choice of a Gaussian driving process is linked to the necessity (in the first class of models introduced in Section 5.2) to preserve the lognormality of the Gompertz model. In Sections 5.3 and 5.4 we obtain stochastic fractional Gompertz models with Gaussian one-dimensional law. These were actually only exemplifications. Indeed, one can use the construction method given in Sections 5.3 and 5.4 to obtain Gaussian stochastic models for general growth models of the form

$$\begin{cases} \left(\frac{1}{\Psi'(t)}\frac{d}{dt}\right)^{\nu} x_\nu(t) = \alpha x_\nu(t) \\ x_\nu(0) = x_0 \end{cases} \tag{28}$$

where $\Psi'(t) = \frac{r(t)}{\alpha}$ for some growth rate $r(t) > 0$.

One could also try to substitute the operator $\mathcal{I}_t^{\nu,\Psi}$ in place of $\mathcal{I}_t^{\nu}$ in Equation (12). In such a case one could show, by similar arguments, the existence and uniqueness of a Gaussian solution and then use the construction given in Section 5.2 to obtain a log-normal stochastic model for (28).

Concerning possible applications, it has been already observed in [21,22] that fractional Gompertz models are more appropriate than classical ones to describe some phenomena such as tumor growth (concerning the model in Section 5.2), dark fermentation and other fermentation phenomena (concerning the model in Section 5.3). In this paper we provided a method to introduce noise (due to eventual unpredictable variables in the environment) in such a way that a macroscopic observable function still preserves such laws. Concerning the choice of the noise, it depends on the autocorrelation one wants to introduce in the model. For instance, if one wants to introduce a long-range (or short-range) correlated noise, one could use a fractional Brownian motion as driving Gaussian process, while if a delta-correlated noise is needed one could use a classical Brownian motion as driving process.

Finally, we want to recall that our aim was to introduce some construction methods that could lead to log-normal or normal stochastic models for general fractional growth processes (as the ones in Equation (28)) with a general Gaussian noise, in order to provide a wide range of models that could be possibly useful in future applications.

Author Contributions: Conceptualization, G.A. and E.P.; methodology, G.A. and E.P.; writing—original draft preparation, G.A. and E.P.; writing—review and editing, G.A. and E.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research is partially supported by MIUR–PRIN 2017, project "Stochastic Models for Complex Systems", no. 2017JFFHSH.

Acknowledgments: We would like to thank the referees for their useful suggestions.

Conflicts of Interest: The authors declare no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

References

1. Debnath, L. A brief historical introduction to fractional calculus. *Int. J. Math. Educ. Sci. Technol.* **2004**, *35*, 487–501. [CrossRef]
2. Loverro, A. *Fractional Calculus: History, Definitions and Applications for the Engineer*; Rapport Technique; Univeristy of Notre Dame, Department of Aerospace and Mechanical Engineering: Notre Dame, IN, USA, 2004; pp. 1–28.
3. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]

4. Kilbas, A.A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier Science Limited: Amsterdam, The Netherlands, 2006; Volume 204.
5. Podlubny, I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Elsevier: Amsterdam, The Netherlands, 1998; Volume 198.
6. Yong, Z.; Jinrong, W.; Lu, Z. *Basic Theory of Fractional Differential Equations*; World Scientific: Singapore, 2016.
7. Caputo, M. Linear models of dissipation whose Q is almost frequency independent—II. *Geophys. J. Int.* **1967**, *13*, 529–539. [CrossRef]
8. Ameen, I.; Novati, P. The solution of fractional order epidemic model by implicit Adams methods. *Appl. Math. Model.* **2017**, *43*, 78–84. [CrossRef]
9. Belhamiti, O.; Absar, B. A numerical study of fractional order reverse osmosis desalination model using legendre wavelet approximation. *Iran. J. Math. Chem.* **2017**, *8*, 345–364.
10. Teka, W.W.; Upadhyay, R.K.; Mondal, A. Fractional-order leaky integrate-and-fire model with long-term memory and power law dynamics. *Neural Netw.* **2017**, *93*, 110–125. [CrossRef]
11. Bagley, R.L. Power law and fractional calculus model of viscoelasticity. *AIAA J.* **1989**, *27*, 1412–1417. [CrossRef]
12. Almeida, R.; Bastos, N.R.; Monteiro, M.T.T. Modeling some real phenomena by fractional differential equations. *Math. Methods Appl. Sci.* **2016**, *39*, 4846–4855. [CrossRef]
13. Almeida, R.; Bastos, N.R.; Monteiro, M.T.T. A fractional Malthusian growth model with variable order using an optimization approach. *Stat. Optim. Inf. Comput.* **2018**, *6*, 4–11. [CrossRef]
14. Laird, A.K. Dynamics of tumour growth. *Br. J. Cancer* **1964**, *18*, 490. [CrossRef]
15. Dominik, W.; Natalia, K. *Dynamics of Cancer: Mathematical Foundations of Oncology*; World Scientific: Singapore, 2014.
16. Afenya, E. Mathematical models of cancer and their relevant insights. In *Handbook of Cancer Models with Applications*; World Scientific: Singapore, 2008; pp. 173–223.
17. Albano, G.; Giorno, V. A stochastic model in tumor growth. *J. Theor. Biol.* **2006**, *242*, 329–336. [CrossRef] [PubMed]
18. Albano, G.; Giorno, V. On the first exit time problem for a Gompertz-type tumor growth. In Proceedings of the International Conference on Computer Aided Systems Theory, Las Palmas de Gran Canaria, Spain, 15–20 February 2009; pp. 113–120.
19. Giorno, V.; Román-Román, P.; Spina, S.; Torres-Ruiz, F. Estimating a non-homogeneous Gompertz process with jumps as model of tumor dynamics. *Comput. Stat. Data Anal.* **2017**, *107*, 18–31. [CrossRef]
20. Lisei, H.; Julitz, D. A stochastic model for the growth of cancer tumors. *Stud. Univ. Babes-Bolyai Math.* **2008**, *53*, 39–56.
21. Bolton, L.; Cloot, A.H.; Schoombie, S.W.; Slabbert, J.P. A proposed fractional-order Gompertz model and its application to tumour growth data. *Math. Med. Biol. A J. IMA* **2014**, *32*, 187–209. [CrossRef] [PubMed]
22. Frunzo, L.; Garra, R.; Giusti, A.; Luongo, V. Modeling biological systems with an improved fractional Gompertz law. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *74*, 260–267. [CrossRef]
23. Almeida, R. A Caputo fractional derivative of a function with respect to another function. *Commun. Nonlinear Sci. Numer. Simul.* **2017**, *44*, 460–481. [CrossRef]
24. Anh, P.; Doan, T.; Huong, P. A variation of constant formula for Caputo fractional stochastic differential equations. *Stat. Probab. Lett.* **2019**, *145*, 351–358. [CrossRef]
25. Son, D.T.; Huong, P.T.; Kloeden, P.E.; Tuan, H.T. Asymptotic separation between solutions of Caputo fractional stochastic differential equations. *Stoch. Anal. Appl.* **2018**, *36*, 654–664. [CrossRef]
26. Meerschaert, M.; Nane, E.; Vellaisamy, P. The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.* **2011**, *16*, 1600–1620. [CrossRef]
27. Orsingher, E.; Polito, F. On a fractional linear birth–death process. *Bernoulli* **2011**, *17*, 114–137. [CrossRef]
28. Leonenko, N.N.; Meerschaert, M.M.; Sikorskii, A. Fractional Pearson diffusions. *J. Math. Anal. Appl.* **2013**, *403*, 532–546. [CrossRef] [PubMed]
29. Ascione, G.; Leonenko, N.; Pirozzi, E. Fractional Immigration-Death Processes. *arXiv* **2019**, arXiv:1907.07588.
30. Li, C.; Qian, D.; Chen, Y. On Riemann-Liouville and Caputo derivatives. *Discret. Dyn. Nat. Soc.* **2011**, *2011*. [CrossRef]
31. Osler, T.J. The fractional derivative of a composite function. *SIAM J. Math. Anal.* **1970**, *1*, 288–293. [CrossRef]

32. Osler, T.J. Leibniz rule for fractional derivatives generalized and an application to infinite series. *SIAM J. Appl. Math.* **1970**, *18*, 658–674. [CrossRef]
33. Abundo, M.; Pirozzi, E. On the Fractional Riemann-Liouville Integral of Gauss-Markov processes and applications. *arXiv* **2019**, arXiv:1905.08167.
34. Khamsi, M.A.; Kirk, W.A.; Kirk, W. *An Introduction to Metric Spaces and Fixed Point Theory*; John Wiley & Sons: Hoboken, NJ, USA, 2001; Volume 53.
35. Halmos, P.R.; Sunder, V.S. *Bounded Integral Operators on L^2 Spaces*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012; Volume 96.
36. Mandelbrot, B.B.; Van Ness, J.W. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.* **1968**, *10*, 422–437. [CrossRef]
37. Biagini, F.; Hu, Y.; Øksendal, B.; Zhang, T. *Stochastic Calculus for Fractional Brownian Motion and Applications*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2008.
38. Mishura, Y. *Stochastic Calculus for Fractional Brownian Motion and Related Processes*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2008; Volume 1929.
39. Cheridito, P.; Kawaguchi, H.; Maejima, M. Fractional Ornstein-Uhlenbeck processes. *Electron. J. Probab.* **2003**, *8*. [CrossRef]
40. Ascione, G.; Mishura, Y.; Pirozzi, E. Fractional Ornstein-Uhlenbeck process with stochastic forcing, and its applications. *Methodol. Comput. Appl. Probab.* **2019**. [CrossRef]
41. Pirozzi, E. Colored noise and a stochastic fractional model for correlated inputs and adaptation in neuronal firing. *Biol. Cybern.* **2018**, *112*, 25–39. [CrossRef]
42. Ascione, G.; Pirozzi, E. On a stochastic neuronal model integrating correlated inputs. *Math. Biosci. Eng.* **2019**, *16*, 5206–5225. [CrossRef] [PubMed]
43. De Lauro, E.; De Martino, S.; De Siena, S.; Giorno, V. Stochastic roots of growth phenomena. *Phys. A Stat. Mech. Its Appl.* **2014**, *401*, 207–213. [CrossRef]
44. Di Crescenzo, A.; Spina, S. Analysis of a growth model inspired by Gompertz and Korf laws, and an analogous birth-death process. *Math. Biosci.* **2016**, *282*, 121–134. [CrossRef] [PubMed]
45. Haubold, H.J.; Mathai, A.M.; Saxena, R.K. Mittag-Leffler functions and their applications. *J. Appl. Math.* **2011**, *2011*. [CrossRef]