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On a New Extended Hardy–Hilbert's Inequality with Parameters

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Abstract: In this paper, by introducing parameters and weight functions, with the help of the Euler–Maclaurin summation formula, we establish the extension of Hardy–Hilbert's inequality and its equivalent forms. The equivalent statements of the best possible constant factor related to several parameters are provided. The operator expressions and some particular cases are also discussed.

Keywords: weight coefficient; Hardy–Hilbert's inequality; equivalent statement; parameter; operator expression

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1. Introduction

Let

$$p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0,\ 0 < \sum\nolimits_{m=1}^{\infty} a_m^p < \infty \ and\ 0 < \sum\nolimits_{n=1}^{\infty} b_n^q < \infty.$$

We have the following classical Hardy–Hilbert's inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\Big(\sum_{m=1}^{\infty} a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty} b_n^q\Big)^{\frac{1}{q}}. \tag{1}$$

In 2006, by introducing parameters $\lambda_i \in (0,2](i=1,2), \lambda_1 + \lambda_2 = \lambda \in (0,4]$, an extension of (1) was provided in [2], as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\Big[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\Big]^{\frac{1}{p}}\Big[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\Big]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible and

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}dt\,(u,v>0)$$

is the beta function. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, Inequality (2) reduces to (1); for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$, Inequality (2) reduces to Yang's inequalities in [3].

Recently, in virtue of (2), a new inequality with the kernel $\frac{1}{(m+n)^{\lambda}}$ involving partial sums was presented in [4].

If $f(x), g(y) \geq 0, 0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(y)dy < \infty$, then we still have the following Hardy–Hilbert's integral Inequality (cf. [1], Theorem 316):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{3}$$

where the constant factor $\pi/\sin(\frac{\pi}{p})$ is the best possible.

Inequalities (1) and (3) with their extensions play an important role in analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was put forwarded (cf. [1], Theorem 351): If $K(t)(t > 0)$ is a decreasing function, $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \phi(s) = \int_0^\infty K(t)t^{s-1}dt < \infty$, $a_n \geq 0$, $0 < \sum_{n=1}^\infty a_n^p < \infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^\infty K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) can be found in [16–19].

In 2016, by means of the technique of real analysis, Hong [20] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. For some similar works on the extensions of (3) and (4), we refer the reader to [21–25].

In a recent paper [26], Yang, Wu, and Wang gave a reverse half-discrete Hardy–Hilbert's inequality and its equivalent forms and dealt with their equivalent statements of the best possible constant factor related to several parameters.

Following the way of [20,26], in this paper, by the idea of introducing weight functions and parameters and using Euler–Maclaurin's summation formula, we give an extension of Hardy–Hilbert's inequality and its equivalent forms. The equivalent statements of the best possible constant factor related to several parameters are provided. We also discuss the operator expressions and some particular cases of these types of inequalities.

2. Some Lemmas

In what follows, we assume that $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda \in (0, \frac{5}{2}], \lambda_i \in (0, \frac{5}{4}] \cap (0, \lambda)$ $(i = 1, 2)$, $a_m, b_n \geq 0$, $m, n \in \mathrm{N} = \{1, 2, \cdots\}$ such that

$$0 < \sum_{m=1}^\infty m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p < \infty \text{ and } 0 < \sum_{n=1}^\infty n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q < \infty.$$

Lemma 1. (cf. [5], (2.2.13)). *If $g(t)$ is a positive decreasing function in $[m, \infty)(m \in \mathrm{N})$ with $g(\infty) = 0, P_i(t), B_i(i \in \mathrm{N})$ are the Bernoulli functions and the Bernoulli numbers of i-order, then we have*

$$\int_m^\infty P_{2q-1}(t)g(t)dt = \varepsilon\frac{B_{2q}}{q}\left(\frac{1}{2^{2q}} - 1\right)g(m)(0 < \varepsilon < 1, q = 1, 2, \cdots), \tag{5}$$

in particular, for $q = 1$, in view of $B_2 = \frac{1}{6}$, we have

$$-\frac{1}{8}g(m) < \int_m^\infty P_1(t)g(t)dt < 0. \tag{6}$$

Lemma 2. *For $\lambda \in (0, \frac{5}{2}]$,$\lambda_2 \in (0, \frac{5}{4}] \cap (0, \lambda)$, we define the following weight coefficient*

$$\varpi(\lambda_2, m) := m^{\lambda-\lambda_2} \sum_{n=1}^{\infty} \frac{n^{\lambda_2-1}}{m^\lambda + n^\lambda} (m \in \mathrm{N}). \tag{7}$$

Then, we have the following inequalities

$$\frac{\pi}{\lambda \sin(\pi\lambda_2/\lambda)}(1 - \theta_m(\lambda_2)) < \varpi(\lambda_2, m) < k_\lambda(\lambda_2) := \frac{\pi}{\lambda \sin(\pi\lambda_2/\lambda)} (m \in \mathrm{N}), \tag{8}$$

where $\theta_m(\lambda_2)$ is indicated by

$$\theta_m(\lambda_2) := \frac{\sin(\pi\lambda_2/\lambda)}{\pi} \int_0^{\frac{1}{m^\lambda}} \frac{u^{(\lambda_2/\lambda)-1}}{1+u} du = O(\frac{1}{m^{\lambda_2}}) \in (0, 1) (m \in \mathrm{N}). \tag{9}$$

Proof. For fixed $m \in \mathrm{N}$, we set function $g(m, t) := \frac{t^{\lambda_2-1}}{m^\lambda+t^\lambda} (t > 0)$. By using the Euler–Maclaurin summation formula (cf. [2,3]), we have

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^\infty g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt \\ &= \int_1^\infty g(m,t)dt - h(m), \\ h(m) &:= \int_0^1 g(m,t)dt - \tfrac{1}{2}g(m,1) - \int_1^\infty P_1(t)g'(m,t)dt. \end{aligned}$$

It is easy to find that $\frac{1}{2}g(m,1) = \frac{1}{2(m^\lambda+1)}$. Integration by parts, it follows that

$$\begin{aligned} \int_0^1 g(m,t)dt &= \int_0^1 \tfrac{t^{\lambda_2-1}}{m^\lambda+t^\lambda}dt = \tfrac{1}{\lambda_2}\int_0^1 \tfrac{dt^{\lambda_2}}{m^\lambda+t^\lambda} \\ &= \tfrac{1}{\lambda_2}\tfrac{t^{\lambda_2}}{m^\lambda+t^\lambda}|_0^1 + \tfrac{\lambda}{\lambda_2}\int_0^1 \tfrac{t^{\lambda+\lambda_2-1}}{(m^\lambda+t^\lambda)^2}dt > \tfrac{1}{\lambda_2}\tfrac{1}{m^\lambda+1}. \end{aligned}$$

We obtain

$$\begin{aligned} g'(m,t) &= \tfrac{(\lambda_2-1)t^{\lambda_2-2}}{m^\lambda+t^\lambda} - \tfrac{\lambda t^{\lambda+\lambda_2-2}}{(m^\lambda+t^\lambda)^2} = -\tfrac{(1-\lambda_2)t^{\lambda_2-2}}{m^\lambda+t^\lambda} - \tfrac{\lambda(m^\lambda+t^\lambda-m^\lambda)t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2} \\ &= -\tfrac{(\lambda+1-\lambda_2)t^{\lambda_2-2}}{m^\lambda+t^\lambda} + \tfrac{\lambda m^\lambda t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2}, \end{aligned}$$

and for $0 < \lambda_2 \le \frac{5}{4}, \lambda_2 < \lambda \le \frac{5}{2}$, we have

$$\frac{d}{dt}[\frac{t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^i}] < 0 (i = 1, 2).$$

By using the Euler–Maclaurin summation formula and (6), we obtain

$$\begin{aligned} (\lambda+1-\lambda_2)\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{m^\lambda+t^\lambda}dt > -\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)}, \\ -m^\lambda\lambda\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{(m^\lambda+t^\lambda)^2}dt > 0. \end{aligned}$$

Hence, we find

$$-\int_1^\infty P_1(t)g'(m,t)dt > -\frac{\lambda+1-\lambda_2}{8(m^\lambda+1)},$$

and then one has

$$\begin{aligned} h(m) &> \tfrac{1}{\lambda_2}\tfrac{1}{m^\lambda+1} - \tfrac{1}{2(m^\lambda+1)} - \tfrac{\lambda+1-\lambda_2}{8(m^\lambda+1)} = \tfrac{8-(5+\lambda)\lambda_2+\lambda_2^2}{8\lambda_2(m^\lambda+1)} \\ &\geq \tfrac{8-(5+\frac{5}{2})\lambda_2+\lambda_2^2}{8\lambda_2(m^\lambda+1)} = \tfrac{16-15\lambda_2+2\lambda_2^2}{16\lambda_2(m^\lambda+1)}. \end{aligned}$$

Since $\lambda_2 \in (0, \frac{5}{4}]$, we obtain

$$\frac{d}{d\lambda_2}(6 - 15\lambda_2 + 2\lambda_2^2) = -15 + 4\lambda_2^2 < 0,$$

it follows that

$$h(m) > \frac{16 - 15(\frac{5}{4}) + 2(\frac{5}{4})^2}{16\lambda_2(m^\lambda + 1)} = \frac{3}{128\lambda_2(m^\lambda + 1)} > 0.$$

Setting $t = mu^{1/\lambda}$, we find

$$\begin{aligned} \varpi(\lambda_2, m) &= m^{\lambda-\lambda_2}\sum_{n=1}^{\infty} g(m,n) < m^{\lambda-\lambda_2}\int_0^\infty g(m,t)dt \\ &= m^{\lambda-\lambda_2}\int_0^\infty \tfrac{t^{\lambda_2-1}}{m^\lambda+t^\lambda}dt = \tfrac{1}{\lambda}\int_0^\infty \tfrac{u^{(\lambda_2/\lambda)-1}}{1+u}du = \tfrac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}, \end{aligned}$$

in the above expressions, the last equality follows from the properties of Beta function and Gamma function, i.e.,

$$\begin{aligned} &\int_0^\infty \tfrac{u^{(\lambda_2/\lambda)-1}}{1+u}du = \int_0^1 \quad (1-x)^{(\lambda_2/\lambda)-1}x^{-(\lambda_2/\lambda)}dx \\ &= B(\lambda_2/\lambda, 1-(\lambda_2/\lambda)) \\ &= \Gamma(\lambda_2/\lambda)\Gamma(1-(\lambda_2/\lambda)) = \tfrac{\pi}{\sin(\pi\lambda_2/\lambda)}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^\infty g(m,t)dt + \tfrac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt \\ &= \int_1^\infty g(m,t)dt + H(m), \\ H(m): &= \tfrac{1}{2}g(m,1) + \int_1^\infty P_1(t)g'(m,t)dt. \end{aligned}$$

Since $\frac{1}{2}g(m,1) = \frac{1}{2(m^\lambda+1)}$ and

$$g'(m,t) = -\frac{(\lambda+1-\lambda_2)t^{\lambda_2-2}}{m^\lambda + t^\lambda} + \frac{\lambda m^\lambda t^{\lambda_2-2}}{(m^\lambda + t^\lambda)^2},$$

in view of (6), we obtain

$$-(\lambda + 1 - \lambda_2)\int_1^\infty P_1(t)\frac{t^{\lambda_2-2}}{m^\lambda + t^\lambda}dt > 0,$$

and

$$\lambda m^\lambda \int_1^\infty P_1(t)\frac{t^\lambda 2^{-2}}{(m^\lambda + t^\lambda)^2}dt > -\frac{\lambda m^\lambda}{8(m^\lambda + 1)^2}.$$

Hence, we have

$$H(m) > \frac{1}{2(m^\lambda + 1)} - \frac{\lambda m^\lambda}{8(m^\lambda + 1)^2} > \frac{4}{8(m^\lambda + 1)} - \frac{5/2}{8(m^\lambda + 1)} > 0,$$

and then setting $t = mu^{1/\lambda}$, we obtain

$$\begin{aligned} & m^{\lambda-\lambda_2}\sum_{n=1}^{\infty} g(m,n) > m^{\lambda-\lambda_2}\int_1^{\infty} g(m,t)dt \\ = & \ m^{\lambda-\lambda_2}\int_0^{\infty} g(m,t)dt - m^{\lambda-\lambda_2}\int_0^1 g(m,t)dt \\ = & \ \frac{\pi}{\lambda sin(\pi\lambda_2/\lambda)}[1-\frac{\lambda sin(\pi\lambda_2/\lambda)}{\pi}m^{\lambda-\lambda_2}\int_0^1 \frac{t^{\lambda_2-1}}{m^{\lambda}+t^{\lambda}}dt] \\ = & \ \frac{\pi}{\lambda sin(\pi\lambda_2/\lambda)}(1-\theta_m(\lambda_2)) > 0, \end{aligned}$$

where $\theta_m(\lambda_2)$ is indicated by (9). Since we find

$$0 < \int_0^{\frac{1}{m^{\lambda}}} \frac{u^{(\lambda_2/\lambda)-1}}{1+u}du < \int_0^{\frac{1}{m^{\lambda}}} u^{(\lambda_2/\lambda)-1}du = \frac{\lambda}{\lambda_2 m^{\lambda_2}},$$

namely, $\theta_m(\lambda_2) = O(\frac{1}{m^{\lambda_2}}) \in (0,1)(m \in \mathrm{N})$. Therefore, (8) and (9) follow. This completes the proof of Lemma 2. □

Lemma 3. *The following extended Hardy–Hilbert's inequality holds true:*

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\}^{\frac{1}{q}}. \quad (10)$$

Proof. In the same way as the proof of Lemma 2, for $n \in N$, $\lambda \in (0,\frac{5}{2}]$,$\lambda_1 \in (0,\frac{5}{4}] \cap (0,\lambda)$, we have the following inequality for the weight coefficient:

$$\begin{aligned} \omega(\lambda_1,n) := n^{\lambda-\lambda_1}\sum_{m=1}^{\infty}\frac{m^{\lambda_1-1}}{m^{\lambda}+n^{\lambda}}(n \in \mathrm{N}), \\ \frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}(1-\theta_n(\lambda_1)) < \omega(\lambda_1,n) < k_{\lambda}(\lambda_1) = \frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}, \end{aligned} \quad (11)$$

$$\theta_n(\lambda_1) = \frac{\sin(\pi\lambda_1/\lambda)}{\pi}\int_0^{\frac{1}{n^{\lambda}}}\frac{u^{(\lambda_1/\lambda)-1}}{1+u}du = O(\frac{1}{n^{\lambda_1}}) \in (0,1)(n \in \mathrm{N}). \quad (12)$$

By Hölder's inequality with weight (cf. [27]), we obtain

$$\begin{aligned} I & = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}[\frac{n^{(\lambda_2-1)/p}}{m^{(\lambda_1-1)/q}}a_m][\frac{m^{(\lambda_1-1)/q}}{n^{(\lambda_2-1)/p}}b_n] \\ & \le [\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\frac{n^{\lambda_2-1}}{m^{(\lambda_1-1)(p-1)}}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}\frac{m^{\lambda_1-1}}{n^{(\lambda_2-1)(q-1)}}b_n^q]^{\frac{1}{q}} \\ & = \{\sum_{m=1}^{\infty}\varpi(\lambda_2,m)m^{p[1-(\lambda-\lambda_2 p+\lambda_1 q)]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\omega(\lambda_1,n)n^{q[1-(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\}^{\frac{1}{q}}. \end{aligned}$$

Then by (8) and (11), we get (10). Lemma 3 is proved. □

Remark 1. *By (10), for* $\lambda_1+\lambda_2 = \lambda \in (0,\frac{5}{2}], 0 < \lambda_i \le \frac{5}{4}(i = 1,2)$, *one has*

$$0 < \sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p < \infty \text{ and } 0 < \sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q < \infty,$$

and the following inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}} < k_{\lambda}(\lambda_1)[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q]^{\frac{1}{q}}. \quad (13)$$

Lemma 4. *For $\lambda_1+\lambda_2=\lambda \in (0,\frac{5}{2}]$, the constant factor $k_{\lambda}(\lambda_1)$ in (13) is the best possible.*

Proof. For any $0<\varepsilon<p\lambda_1$, we set

$$\widetilde{a}_m := m^{\lambda_1-\frac{\varepsilon}{p}-1}, \widetilde{b}_n := n^{\lambda_2-\frac{\varepsilon}{q}-1} (m,n \in \mathrm{N}).$$

If the constant factor $k_{\lambda}(\lambda_1)$ in (13) is not the best possible, then there exists a positive constant $M < k_{\lambda}(\lambda_1)$, such that (13) is valid when replacing $k_{\lambda}(\lambda_1)$ by M. In particular, by substitution of $a_m=\widetilde{a}_m$ *and* $b_n=\widetilde{b}_n$ in (13), we have

$$\widetilde{I} := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{m^{\lambda}+n^{\lambda}} < M[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}\widetilde{a}_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}\widetilde{b}_n^q]^{\frac{1}{q}}. \quad (14)$$

By (14) and the decreasingness property, we obtain

$$\begin{aligned}\widetilde{I} &< M[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}m^{p\lambda_1-\varepsilon-p}]^{\frac{p}{1}}[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}n^{q\lambda_2-\varepsilon-q}]^{\frac{1}{q}} \\ &= M(1+\sum_{m=2}^{\infty} m^{-\varepsilon-1})^{\frac{1}{p}}(1+\sum_{n=2}^{\infty} n^{-\varepsilon-1})^{\frac{1}{q}} \\ &< M(1+\int_1^{\infty} x^{-\varepsilon-1}dx)^{\frac{1}{p}}(1+\int_1^{\infty} y^{-\varepsilon-1}dy)^{\frac{1}{q}} = \tfrac{M}{\varepsilon}(\varepsilon+1).\end{aligned}$$

By (11) and (12), setting $\hat{\lambda}_1=\lambda_1-\frac{\varepsilon}{p}\in(0,\frac{5}{4})\cap(0,\lambda)(0<\hat{\lambda}_2=\lambda_2+\frac{\varepsilon}{p}=\lambda-\hat{\lambda}_1<\lambda)$, we find

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}[n^{(\lambda_2+\frac{\varepsilon}{p})}\sum_{m=1}^{\infty}\tfrac{1}{m^{\lambda}+n^{\lambda}}m^{(\lambda_1-\frac{\varepsilon}{p})-1}]n^{-\varepsilon-1} \\ &= \sum_{n=1}^{\infty}\omega(\hat{\lambda}_1,n)n^{-\varepsilon-1} > k_{\lambda}(\hat{\lambda}_1)\sum_{n=1}^{\infty}(1-O(\tfrac{1}{n^{\hat{\lambda}_1}}))n^{-\varepsilon-1} \\ &= k_{\lambda}(\hat{\lambda}_1)(\sum_{n=1}^{\infty}n^{-\varepsilon-1}-\sum_{n=1}^{\infty}\tfrac{1}{O(n^{\lambda_1+\frac{\varepsilon}{q}+1})}) > k_{\lambda}(\hat{\lambda}_1)(\int_1^{\infty}x^{-\varepsilon-1}dx-O(1)) \\ &= \tfrac{1}{\varepsilon}k_{\lambda}(\hat{\lambda}_1)(1-\varepsilon O(1)).\end{aligned}$$

In view of the above results, we have

$$k_{\lambda}(\lambda_1-\frac{\varepsilon}{p})(1-\varepsilon O(1)) < \varepsilon\widetilde{I} < M(\varepsilon+1).$$

For $\varepsilon\to 0^+$, we find $k_{\lambda}(\lambda_1)\le M$, which is a contradiction. Hence, $M=k_{\lambda}(\lambda_1)$ is the best possible constant factor of (13). Lemma 4 is proved. □

Remark 2. *Setting*

$$\widetilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widetilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p},$$

we find

$$\widetilde{\lambda}_1+\widetilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p} = \frac{\lambda}{p}+\frac{\lambda}{q} = \lambda,$$

and we can rewrite (10) as follows:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^\lambda + n^\lambda} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)[\sum_{m=1}^{\infty} m^{p(1-\widetilde{\lambda}_1)-1}a_m^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} n^{q(1-\widetilde{\lambda}_2)-1}b_n^q]^{\frac{1}{q}}. \quad (15)$$

Lemma 5. *If Inequality (15) exists with the best possible constant factor $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$, then we have $\lambda = \lambda_1 + \lambda_2$.*

Proof. Note that

$$\widetilde{\lambda}_1 = \tfrac{\lambda-\lambda_2}{p} + \tfrac{\lambda_1}{q} > 0, \widetilde{\lambda}_1 < \tfrac{\lambda}{p} + \tfrac{\lambda}{q} = \lambda,$$
$$0 < \widetilde{\lambda}_2 = \lambda - \widetilde{\lambda}_1 < \lambda.$$

Hence, we have

$$k_\lambda(\widetilde{\lambda}_1) = \frac{\pi}{\lambda \sin(\pi\widetilde{\lambda}_1/\lambda)} \in \mathrm{R}_+ = (0,\infty).$$

If the constant factor $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (15) is the best possible, then in view of (13), the unique best possible constant factor must be $k_\lambda(\widetilde{\lambda}_1)(\in \mathrm{R}_+)$, namely,

$$k_\lambda(\widetilde{\lambda}_1) = k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1).$$

Recalling the Hölder's integral inequality with weight (cf. [27]):

$$\int_0^\infty \omega(x)f(x)g(x)dx \le (\int_0^\infty \omega(x)f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty \omega(x)g^q(x)dx)^{\frac{1}{q}},$$

where

$$p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1, \omega(x) > 0, f(x) > 0, g(x) > 0,$$

with equality holding if and only if there exist constants A and B (not all zero) such that $Af^p(x) = Bg^q(x)$ *a.e.* in R_+.

By using Hölder's integral inequality, one has

$$\begin{aligned} & k_\lambda(\widetilde{\lambda}_1) = k_\lambda(\tfrac{\lambda-\lambda_2}{p} + \tfrac{\lambda_1}{q}) \\ = & \int_0^\infty \tfrac{1}{1+u^\lambda}u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du \\ = & \int_0^\infty (\tfrac{1}{1+u^\lambda})^{\frac{1}{p}+\frac{1}{q}}(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du \\ \le & (\int_0^\infty \tfrac{1}{1+u^\lambda}u^{\lambda-\lambda_2-1}du)^{\frac{1}{p}}(\int_0^\infty \tfrac{1}{1+u^\lambda}u^{\lambda_1-1}du)^{\frac{1}{q}} \\ = & (\int_0^\infty \tfrac{1}{1+v^\lambda}v^{\lambda_2-1}dv)^{\frac{1}{p}}(\int_0^\infty \tfrac{1}{1+u^\lambda}u^{\lambda_1-1}du)^{\frac{1}{q}} \\ = & k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1). \end{aligned} \quad (16)$$

We observe that (16) keeps the form of equality if and only if there exist constants A and B (not all zero) such that

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } R_+.$$

Assuming that $A \neq 0$, we have

$$u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A} \text{ a.e. in } R_+.$$

and then $\lambda - \lambda_2 - \lambda_1 = 0$, namely, $\lambda = \lambda_1 + \lambda_2$. This completes the proof of Lemma 5. □

3. Main Results

Theorem 1. *Inequality (10) is equivalent to the following inequality:*

$$J := [\sum_{n=1}^{\infty} n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}(\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}a_m)^p]^{\frac{1}{p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}. \quad (17)$$

If the constant factor in (10) is the best possible, then so is the constant factor in (17).

Proof. Suppose that (17) is valid. By Hölder's inequality (cf. [27]), we have

$$I = \sum_{n=1}^{\infty}[n^{\frac{-1}{p}+(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}\sum_{m=1}^{\infty}\frac{1}{m^\lambda, n^\lambda}a_m][n^{\frac{1}{p}-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})}b_n] \le J\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}}. \quad (18)$$

Then by (17), we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1}(\sum_{m=1}^{\infty}\frac{1}{m^\lambda+n^\lambda}a_m)^{p-1}, n \in \mathrm{N}.$$

If $J = 0$, then (17) is naturally valid; if $J = \infty$, then it is impossible to make (17) valid, namely,$J < \infty$. Suppose that $0 < J < \infty$. By (10), we have

$$\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q = J^p = I < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$$
$$\times\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{q}},$$
$$J = \{\sum_{n=1}^{\infty} n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\}^{\frac{1}{p}} < k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\{\sum_{m=1}^{\infty} m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\}^{\frac{1}{p}},$$

namely, (17) follows which is equivalent to (10).

If the constant factor in (10) is the best possible, then so is the constant factor in (17). Otherwise, by (18), we would reach a contradiction that the constant factor in (10) is not the best possible. The proof of Theorem 1 is complete. □

Theorem 2. *The following statements (i), (ii), (iii), and (iv) are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is independent of* p, q;
(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is expressible as a single integral;*
(iii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in (10) is the best possible constant factor;*
(iv) $\lambda = \lambda_1 + \lambda_2$.

If the statement (iv) follows, namely, $\lambda = \lambda_1 + \lambda_2$, then we have (13) and the following equivalent inequalities with the best possible constant factor $k_\lambda(\lambda_1)$:

$$[\sum_{n=1}^{\infty} n^{p\lambda_2-1}(\sum_{m=1}^{\infty} \frac{1}{m^\lambda + n^\lambda} a_m)^p]^{\frac{1}{p}} < k_\lambda(\lambda_1)[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p]^{\frac{1}{p}}. \quad (19)$$

Proof. (i)⇒(ii). By (i), we have

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to 1^+}\lim_{q\to\infty} k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_2),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral

$$k_\lambda(\lambda_2) = \int_0^\infty \frac{1}{1+u^\lambda} u^{\lambda_2 - 1} du.$$

(ii)⇒(iv). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a convergent single integral $k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$, then (16) keeps the form of equality. In view of Lemma 5, it follows that $\lambda = \lambda_1 + \lambda_2$.

(iv)⇒(i). If $\lambda = \lambda_1 + \lambda_2$, then $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1)$, which is independent of p, q. Hence, it follows that (i)⇔(ii)⇔(iv).

(iii)⇒(iv). By Lemma 5, we have $\lambda = \lambda_1 + \lambda_2$.

(iv)⇒(iii). By Lemma 4, for $\lambda = \lambda_1 + \lambda_2$, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)(= k_\lambda(\lambda_1))$ is the best possible constant factor of (10). Therefore, we have (iii)⇔(iv).

Hence, the statements (i), (ii), (iii), and (iv) are equivalent. This completes the proof of Theorem 2. □

4. Operator Expressions and Some Particular Cases

We define the functions:

$$\phi(m) := m^{p[1-(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}, \psi(n) := n^{q[1-(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1},$$

where from,

$$\psi^{1-p}(n) = n^{p(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})-1} (m, n \in \mathrm{N}).$$

Define the following real normed spaces:

$$l_{p,\phi} := \{a = \{a_m\}_{m=1}^\infty; ||a||_{p,\phi} := (\sum_{m=1}^{\infty} \phi(m)\big|a_m|^p)^{\frac{1}{p}} < \infty\},$$

$$l_{q,\psi} := \{b = \{b_n\}_{n=1}^\infty; ||b||_{q,\psi} := (\sum_{n=1}^{\infty} \psi(n)\big|b_n|^q)^{\frac{1}{q}} < \infty\},$$

$$l_{p,\psi^{1-p}} := \{c = \{c_n\}_{n=1}^\infty; ||c||_{p,\psi^{1-p}} := (\sum_{n=1}^{\infty} \psi^{1-p}(n)\big|b_n|^p)^{\frac{1}{p}} < \infty\}.$$

Assuming that $a \in l_{p,\phi}$, setting

$$c = \{c_n\}_{n=1}^\infty, c_n := \sum_{m=1}^{\infty} \frac{1}{m^\lambda + n^\lambda} a_m, n \in \mathrm{N},$$

we can rewrite (17) as follows

$$||c||_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)||a||_{p,\phi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 1. *Define an extended Hardy–Hilbert's operator $T : l_{p,\phi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\phi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:*

$$(Ta,b) := \sum_{n=1}^{\infty}(\sum_{m=1}^{\infty}\frac{1}{m^{\lambda}+n^{\lambda}}a_m)b_n,$$
$$||T|| := \sup_{a(\neq\theta)\in l_{p,\phi}}\frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\phi}}.$$

By Theorems 1 and 2, we have the following result:

Theorem 3. *If $a \in l_{p,\phi}, b \in l_{q,\psi}, ||a||_{p,\phi}, ||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta,b) < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{20}$$

$$||Ta||_{p,\psi^{1-p}} < k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)||a||_{p,\phi}. \tag{21}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $k_{\lambda}^{\frac{1}{p}}(\lambda_2)k_{\lambda}^{\frac{1}{q}}(\lambda_1)$ in (20) and (21) is the best possible, namely.

$$||T|| = k_{\lambda}(\lambda_1) = \frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}. \tag{22}$$

Remark 3. *(i) For $\lambda = 1, \lambda_1 = \frac{1}{r}, \lambda_2 = \frac{1}{s}(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ in (13) and (19), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{\sin(\pi/r)}$:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/r)}(\sum_{m=1}^{\infty}m^{\frac{p}{s}-1}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}n^{\frac{q}{r}-1}b_n^q)^{\frac{1}{q}}, \tag{23}$$

$$[\sum_{n=1}^{\infty}n^{\frac{p}{s}-1}(\sum_{m=1}^{\infty}\frac{1}{m+n}a_m)^p]^{\frac{1}{p}} < \frac{\pi}{\sin(\pi/r)}(\sum_{m=1}^{\infty}m^{\frac{p}{s}-1}a_m^p)^{\frac{1}{p}}. \tag{24}$$

For $r = q$, $s = p$, (23) reduces to (1).

(ii) For $\lambda = 2, \lambda_1 = \lambda_2 = 1$ in (13) and (19), we have the following equivalent inequalities with the best possible constant factor $\frac{\pi}{2}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^2+n^2} < \frac{\pi}{2}(\sum_{m=1}^{\infty}\frac{1}{m}a_m^p)^{\frac{1}{p}}(\sum_{n=1}^{\infty}\frac{1}{n}b_n^q)^{\frac{1}{q}}, \tag{25}$$

$$[\sum_{n=1}^{\infty}n^{p-1}(\sum_{m=1}^{\infty}\frac{1}{m^2+n^2}a_m)^p]^{\frac{1}{p}} < \frac{\pi}{2}(\sum_{m=1}^{\infty}\frac{1}{m}a_m^p)^{\frac{1}{p}}. \tag{26}$$

(iii) For $\lambda = \frac{5}{2}, \lambda_1 = \lambda_2 = \frac{5}{4}$ in (13) and (19), we have the following equivalent inequalities with the best possible constant factor $\frac{2\pi}{5}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^{5/2}+n^{5/2}} < \frac{2\pi}{5}\Big(\sum_{m=1}^{\infty}\frac{1}{m^{1+p/4}}a_m^p\Big)^{\frac{1}{p}}\Big(\sum_{n=1}^{\infty}\frac{1}{n^{1+q/4}}b_n^q\Big)^{\frac{1}{q}}, \tag{27}$$

$$\Big[\sum_{n=1}^{\infty} n^{(5p/4)-1}\Big(\sum_{m=1}^{\infty}\frac{1}{m^{5/2}+n^{5/2}}a_m\Big)^p\Big]^{\frac{1}{p}} < \frac{2\pi}{5}\Big(\sum_{m=1}^{\infty}\frac{1}{m^{1+p/4}}a_m^p\Big)^{\frac{1}{p}}. \tag{28}$$

5. Conclusions

We described the advancements compared to existing technologies and results like Inequalities (1), (2), and (4) in the introduction section. The first result obtained in the present paper, Inequality (10) asserted by Lemma 3, is the extended Hardy–Hilbert's inequality. The subsequent results are the equivalent forms of Inequality (10) and the equivalent statements of the best possible constant factor related to several parameters; these meaningful results are stated in Theorems 1 and 2, which have significant applications in the theory of inequalities. The operator expressions of the extended Hardy–Hilbert's inequality and its equivalent forms have wide applications in the theory of functional analysis. The idea and method presented in this paper can be spread for general use to investigate more inequalities involving infinite series or infinite integrals.

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