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Separation Axioms of Interval-Valued Fuzzy Soft Topology via Quasi-Neighborhood Structure

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Abstract: In this study, we present the concept of the interval-valued fuzzy soft point and then introduce the notions of its neighborhood and quasi-neighborhood in interval-valued fuzzy soft topological spaces. Separation axioms in an interval-valued fuzzy soft topology, so-called q-T_i for $i = 0, 1, 2, 3, 4$, are introduced, and some of their basic properties are also studied.

Keywords: interval-valued fuzzy soft set; interval-valued fuzzy soft topology; interval-valued fuzzy soft point; interval-valued fuzzy soft neighborhood; interval-valued fuzzy soft quasi-neighborhood; interval-valued fuzzy soft separation axioms

1. Introduction

In 1999, Molodtsov [1] proposed a new mathematical approach known as soft set theory for dealing with uncertainties and vagueness. Traditional tools such as fuzzy sets [2] and rough sets [3] cannot clearly define objects. Soft set theory is different from traditional tools for dealing with uncertainties. A soft set was defined by a collection of approximate descriptions of an object based on parameters by a given set-valued map. Maji et al. [4] initiated the research on both fuzzy set and soft set hybrid structures called fuzzy soft sets and presented a concept that was subsequently discussed by many researchers. Different extensions of the classical fuzzy soft sets were introduced, such as generalized fuzzy soft sets [5], intuitionist fuzzy soft sets [6,7], vague soft sets [8], interval-valued fuzzy soft sets [9], and interval-valued intuitive fuzzy soft sets [10]. In particular, to alleviate some disadvantages of fuzzy soft sets, interval-valued fuzzy soft sets were introduced where no objective procedure was available to select the crisp membership degree of elements in fuzzy soft sets. Tanya and Kandemir [11] started topological studies of fuzzy soft sets. They used the classical concept of topology to construct a topological space over a fuzzy soft set and named it the fuzzy soft topology. They also studied some fundamental topological properties for the fuzzy soft topology, such as interior, closure, and base. Later, Simsekler and Yuksel [12] studied the fuzzy soft topological space in the case of Tanay and Kandemir [11]. However, they established the concept of the fuzzy soft topology over a fuzzy soft set with a set of fixed parameters and considered some topological concepts for fuzzy soft topological spaces such as the base, subbase, neighborhood, and Q-neighborhood. Roy and Samanta [13] noted a new concept of the fuzzy soft topology. They suggested the notion of the fuzzy soft topology over an ordinary set by adding fuzzy soft subsets of it, where everywhere, the parameter set is supposed to be fixed. Then, in [14], they continued to study the fuzzy soft topology and established a fuzzy soft point definition and various neighborhood structures. Atmaca and Zorlutuna [15] considered the concept of soft quasi-coincidence for fuzzy soft sets. By applying this new concept, they also studied the basic topological notions such as interior and closure for fuzzy soft sets. The concept of the product fuzzy soft topology and the boundary fuzzy soft topology was introduced by Zahedi et al. [16,17], and they studied some of their properties. They also suggested a new definition for the fuzzy soft point and then different neighborhood structures. Separation axioms of the fuzzy topological space and fuzzy soft topological space were studied by many authors, see [18–23] and [24–27]. The aim of this work is to develop interval-valued fuzzy soft separation axioms. We start with preliminaries and then give the definition of the interval-valued fuzzy soft point as a generalization of the interval-valued fuzzy point and fuzzy soft point in order to create different neighborhood structures in the interval-valued fuzzy soft topological space in Sections 3 and 4. Finally, in Section 5, the notion of separation axioms q-$T_i, i = 0, 1, 2, 3, 4$ in the interval-valued fuzzy soft topology is introduced, and some of their basic properties are also studied.

2. Preliminaries

Throughout this paper, X is the set of objects and E is the set of parameters. The set of all subsets of X is denoted by $P(X)$ and $A \subset E$, showing a subset of E.

Definition 1 ([1]). *A pair (f, A) is called a soft set over X, if f is a mapping given by $f : A \to P(X)$. For any parameter $e \in A, f(e) \subset X$ may be considered as the set e-approximate elements of the soft set (f, A). In other words, the soft set is not a kind of set, but a parameterized family of subsets of the set X.*

Before introducing the notion of the interval-valued fuzzy soft sets, we give the concept of the interval-valued fuzzy set.

Definition 2 ([28]). *An interval-valued fuzzy (IVF) set over X is defined by the membership function $f : X \to int([0, 1])$, where $int([0, 1])$ denotes the set of all closed subintervals of $[0, 1]$. Suppose that $x \in X$. Then, $f(x) = [f^-(x), f^+(x)]$ is called the degree of membership of the element $x \in X$, where $f^-(x)$ and f^+ are the lower and upper degrees of the membership of x and $0 < f^-(x) < f^+(x) < 1$.*

Yang et al. [9] suggested the concept of interval-valued fuzzy soft set by combining the interval-valued fuzzy set and soft set as below.

Definition 3 ([9]). *An interval-valued fuzzy soft $(IVFS)$ set over X denoted by f_E or (f, E) is defined by the mapping $f : E \to \mathcal{IVF}(X)$, where $\mathcal{IVF}(X)$ is the set of all interval-valued fuzzy sets over X. For any $e \in E, f(e)$ can be written as an interval-valued fuzzy set such that $f(e) = \{\langle x, [f_e^-(x), f_e^+(x)]\rangle : x \in X\}$ where $f_e^-(x)$and$f_e^+(x)$ are the lower and upper degrees of the membership of x with respect to e, where $0 \le f_e^-(x) \le f_e^+(x) \le 1$.*

Note that $\mathcal{IVFS}(X, E)$ shows the set of all $IVFS$ sets over X.

Definition 4 ([9]). *Let f_A and g_B be two $IVFS$ sets over$X$. We say that:*

1. *f_A is an interval-valued fuzzy soft subset of g_B, denoted by $f_A \tilde{\le} g_B$, if and only if:*
   - *(i) $A \le B$,*
   - *(ii) For all $e \in A, f_e^-(x) \le g_e^-(x)$ and $f_e^+(x) \le g_e^+(x), \forall x \in X$.*
2. *$f_A = g_B$ if and only if $f_A \tilde{\le} g_B$ and $g_A \tilde{\le} f_B$.*
3. *The union of two $IVFS$ sets f_A and g_B, denoted by $f_A \tilde{\vee} g_B$, is the $IVFS$ set $(f \vee g, C)$, where $C = A \cup B$, and for all $e \in C$, we have:*

$$(f \vee g)_e(x) = \begin{cases} [f_e^-(x), f_e^+(x)], & e \in A - B \\ [g_e^-(x), g_e^+(x)], & e \in B - A \\ [\max(f_e^-(x), g_e^-(x), \max(f_e^+(x), g_e^+(x)] & e \in A \cap B, \end{cases}$$

for all $x \in X$.

4. *The intersection of two IVFS sets f_A and g_B, denoted by $f_A \tilde{\wedge} g_B$, is the IVFS set $(f \wedge g, C)$, where $C = A \cap B$, and for all $e \in C$, we have $(f \wedge g)_e(x) = [min f_e^-(x), g_e^-(x), min f_e^+(x), g_e^+(x)]$ for all $x \in X$.*
5. *The complement of the IVFS set f_A is denoted by $f_A^c(x)$ where for all $e \in A$, we have $f_e^c(x) = [1 - f_e^+(x), 1 - f_e^-(x)]$.*

Definition 5 ([9])**.** *Let f_E be an IVFS set. Then:*

1. *f_E is called the null interval-valued fuzzy soft set, denoted by $\emptyset_E$, if $f_e^-(x) = f_e^+(x) = 0$, for all $x \in X, e \in E$.*
2. *f_E is called the absolute interval-valued fuzzy soft set, denoted by X_E, if $f_e^-(x) = f_e^+(x) = 1$, for all $x \in X, e \in E$.*

Motivated by the definition of the soft mapping, discussed in [29], we define the concept of the $IVFS$ mapping as the following:

Definition 6. *Let f_A be an IVFS set over X_1 and g_B be an IVFS set over X_2, where $A \subseteq E_1$ and $B \subseteq E_2$. Let $\Phi_u : X_1 \to X_2$ and $\Phi_p : E_1 \to E_2$ be two mappings. Then:*

1. *The map $\Phi : \mathcal{IVFS}(X_1, E_1) \to \mathcal{IVFS}(X_2, E_2)$ is called an IVFS map from X_1 to X_2, and for any $y \in X_2$ and $\varepsilon \in B \subseteq E_2$, the lower image and the upper image of f_A under Φ is the IVFS$\Phi(f_A)$ over X_2, respectively, defined as below:*

$$[\Phi(f^-)](\varepsilon)(y) = \begin{cases} \sup_{x \in \Phi_{u^{-1}}(y)} [\sup_{e \in \Phi_{p^{-1} \cap A}} f^-(e)](x), & \text{if } \Phi_p^{-1}(\varepsilon) \cap A \neq \phi \text{ and } \Phi_u^{-1}(y) \neq \phi \\ 0, & \textit{otherwise}, \end{cases}$$

$$[\Phi(f^+)](\varepsilon)(y) = \begin{cases} \sup_{x \in \Phi_{u^{-1}}(y)} [\sup_{e \in \Phi_{p^{-1} \cap A}} f^+(e)](x), & \text{if } \Phi_p^{-1}(\varepsilon) \cap A \neq \phi \text{ and } \Phi_u^{-1}(y) \neq \phi \\ 0, & \textit{otherwise}. \end{cases}$$

2. *Let $\Phi : \mathcal{IVFS}(X_1, E_1) \to \mathcal{IVFS}(X_2, E_2)$ be an IVFS map from X_1 to X_2. The lower inverse image and the upper inverse image of IVFS g_B under Φ denoted by $\Phi^{-1}(g_B)$ is an IVFS over X_1, respectively, such that for all $x \in X_1$ and $e \in E_1$, it is defined as below:*

$$[\Phi^{-1}(g^-)](e)(x) = \begin{cases} g^-_{\Phi_{p(e)}} \Phi_u(x), & \text{if } \Phi_p(e) \in B \\ 0, & \textit{otherwise}, \end{cases}$$

$$[\Phi^{-1}(g^+)](e)(x) = \begin{cases} g^+_{\Phi_{p(e)}} \Phi_u(x), & \textit{if } \Phi_p(e) \in B \\ 0 & \textit{otherwise}. \end{cases}$$

Proposition 1. *Let $\Phi : \mathcal{IVFS}(X, E) \to \mathcal{IVFS}(Y, F)$ be an IVFS mapping between X and X, and let $\{f_{iA}\}_{i \in J} \subset \mathcal{IVFS}(X, E)$ and $\{g_{iB}\}_{i \in J} \subset \mathcal{IVFS}(Y, F)$ be two families of IVFS sets over X and Y, respectively, where $A \subseteq E$ and $B \subseteq F$, then the following properties hold.*

1. $[\Phi(f_{jA})]^c \tilde{\leq} \Phi(f_{jA})^c$ *for each $j \in J$.*
2. $[\Phi^{-1}(g_{jB})]^c = \Phi^{-1}(g_{jB})^c$ *for each $j \in J$.*
3. *If $g_{iB} \tilde{\leq} g_{jB}$, then $\Phi^{-1}(g_{iB}) \tilde{\leq} \Phi^{-1}(g_{jB})$ for each $i, j \in J$.*
4. *If $f_{iA} \tilde{\leq} f_{jA}$, then $\Phi(f_{iA}) \tilde{\leq} \Phi(f_{jA})$ for each $i, j \in J$.*
5. $\Phi[\tilde{\vee}_{j \in J} f_{jA}] = \tilde{\vee}_{j \in J} \Phi(f_{jA})$ *and* $\Phi^{-1}[\tilde{\vee}_{j \in J} g_{jB}] = \tilde{\vee}_{j \in J} \Phi^{-1}(g_{jB})$.
6. $\Phi[\tilde{\wedge}_{j \in J} f_{jA}] = \tilde{\wedge}_{j \in J} \Phi(f_{jA})$ *and* $\Phi^{-1}[\tilde{\wedge}_{j \in J} g_{jB}] = \tilde{\wedge}_{j \in J} \Phi^{-1}(g_{jB})$.

Proof. We only prove Part (5). The other parts follow a similar technique. For any $k \in F, y \in Y$, and $a \in A$, then:

$$\begin{aligned}
\Phi[\tilde{\vee}_{j\in J} f_{jA}](k)(y) &= \sup_{x\in\Phi_u^{-}(y)} \Big(\sup_{z\in\Phi_p^{-1}(k)} (\tilde{\vee}_{j\in J}) f_{jA})(z)(x) \\
&= \sup_{x\in\Phi_u^{-1}(y)} \Big(\sup_{z\in\Phi_p^{-1}(k)} (\max_{j\in J}([f_{ja}^{-}, f_{ja}^{+}])))(k)(y) \\
&= \sup_{x\in\Phi_u^{-1}(y)} (\max_{j\in J}(\sup_{z\in\Phi_p^{-1}(k)} [f_{ja}^{-}(k), f_{ja}^{+}(k)]))(y) \\
&= \max_{j\in J}(\sup_{x\in\Phi_u^{-1}(y)} (\sup_{z\in\Phi_p^{-1}(k)} [f_{ja}^{-}(k))(y), f_{ja}^{+}(k)]))(y)] \\
&= \max_{j\in J}(\sup_{x\in\Phi_u^{-}(y)} (\sup_{z\in\Phi_p^{-1}(k)} f_{jA}(k)(y))) \\
&= \max_{j\in J} \Phi(f_{jA})(k)(y) \\
&= \tilde{\vee}_{j\in J} \Phi(f_{jA})(k)(y).
\end{aligned}$$

Now, we prove that $\Phi^{-1}[\tilde{\vee}_{j\in J} g_{jB}] = \tilde{\vee}_{j\in J} \Phi^{-1}(g_{jB})$. For any $e \in E, x \in X$ and $b \in B$:

$$\begin{aligned}
\Phi^{-1}[\tilde{\vee}_{j\in J} g_{jB}](e)(x) &= (\tilde{\vee}_{j\in J}) g_{jB}(\Phi_p(e))(\Phi_u(x)) \\
&= [\max_{j\in J} g_{jb}^{-}, \max_{j\in J} g_{jb}^{+}](\Phi_p(e))(\Phi_u(x)) \\
&= [[\max_{j\in J} g_{jb}^{-}(\Phi_p(e))(\Phi_u(x)), \max_{j\in J} g_{jb}^{+}(\Phi_p(e))(\Phi_u(x))] \\
&= [\max_{j\in J} \Phi_u^{-1}(g_{jb}^{-})(e)(x), \max_{j\in J} \Phi_u^{-1}(g_{jb}^{+})(e)(x)] \\
&= \max_{j\in J} [\Phi_u^{-1}(g_{jb}^{-})(e)(x), \Phi_u^{-1}(g_{jb}^{+})(e)(x)] \\
&= \max_{j\in J} \Phi_u^{-1}(g_{jB})(e)(x) \\
&= \tilde{\vee}_{j\in J} \Phi_u^{-1}(g_{jB})(e)(x).
\end{aligned}$$

□

3. Interval-Valued Fuzzy Soft Topological Spaces

The interval-valued fuzzy topology $IVFT$ was discussed by Mondal and Samanta [30]. In this section, we recall their definition and then present different neighborhood structures in the interval-valued fuzzy soft topology $(IVFST)$.

Definition 7. *Let X be a non-empty set, and let τ be a collection of interval valued fuzzy soft sets over X with the following properties:*

(i) $\emptyset_E$, X_E *belong to* τ,
(ii) *If* f_{1E}, f_{2E} *are* $IVFS$ *sets belong to* τ, *then* $f_{1E} \tilde{\wedge} f_{2E}$ *belong to* τ,
(iii) *If the collection of* $IVFS$ *sets* $\{f_{jE} | j \in J\}$ *where* J *is an index set, belonging to* τ, *then* $\tilde{\vee}_{j\in J} f_{jE}$ *belong to* τ.

Then, τ is called the interval-valued fuzzy soft topology over X, and the triplet (X, E, τ) is called the interval-valued fuzzy soft topological space $(IVFST)$.

As the ordinary topologies, the indiscrete $IVFST$ over X contains only $\emptyset_E$ and X_E, while the discrete $IVFST$ over X contains all $IVFS$ sets. Every member of τ is called an interval-valued fuzzy soft open set ($IVFS$-open) in X. The complement of an $IVFS$-open set is called an $IVFS$-closed set.

Remark 1. *If $f_e^{-}(x) = f_e^{+}(x) = a \in [0, 1]$, then we put $[f_e^{-}(x), f_e^{+}(x)] = [a, a] = a$.*

Example 1. *Let $X = [0,1]$ and E be any subset of X. Consider the IVFS set f_E over X by the mapping:*

$$f : E \to \mathcal{IVF}([0,1])$$

such that for any $e \in E, x \in X$:

$$\tilde{f}_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Then, the collection $\tau = \{\Phi_E, X_E, f_E\}$ is an IVFST over X.

1. *Clearly $X_E, \emptyset_E \in \tau$.*
2. *Let $\{f_{jE}\}_{j \in J}$ be a sub-family of τ where for any $j \in J$ if $x \in X$ such that for all $e \in E$:*

$$f_{je}(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Since:

$$\vee_j f_{je}(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

then $\tilde{\vee}_j f_{jE} \in \tau$.

3. *Let $f_E, g_E \in \tau$, where:*

$$f_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and:

$$g_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Since:

$$f_e(x) \wedge g_e(x) = \begin{cases} 1 & 0 \leq x \leq e \\ 0 & e < x \leq 1. \end{cases}$$

Thus, $f_E \wedge g_E \in \tau$.

Example 2 ([23]). *Let $\mathbb{R}$ be the set of all real numbers with the usual topology τ_u where $\tau_u = \langle \{(a,b), a, b \in \mathbb{R}\} \rangle$ and E is a parameter set. Let $U = (a,b) \subset \mathbb{R}$ be an open interval in $\mathbb{R}$; we define IVFS $\tilde{U}_E$ over $\mathbb{R}$ by the mapping:*

$$\tilde{U} : E \to (Int[0,1])^{\mathbb{R}}$$

such that for all $x \in \mathbb{R}$:

$$\tilde{U}_e(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b). \end{cases}$$

Then, the family $\{\tilde{U}_E : (a,b) \subset \mathbb{R}, \forall a, b \in \mathbb{R}\}$ generates an IVFS over $\mathbb{R}$, and we denote it by $\tau_u^{(IVFS)}$:

1. *Clearly, $\mathbb{R}_E, \emptyset_E \in \tau_u^{(IVFS)}$ where for all $e \in E, k \in \mathbb{R}, \mathbb{R}_E(e)(k) = [1,1]$, and $\emptyset_e(k) = 0$*
2. *Let $\{\tilde{U}_{jE}\}_{j \in J}$ be a sub-family of $\tau_u^{(IVFS)}$ where for any $j \in J$ if $x \in (a_j, b_j)$ and interval (a_j, b_j) in $\mathbb{R}$ such that for all $e \in E$:*

$$\tilde{U}_{je}(x) = \begin{cases} 1 & x \in (a_j, b_j) \\ 0 & x \notin (a_j, b_j). \end{cases}$$

Since $\tilde{\vee}_j \tilde{U}_{jE} = (\widetilde{\cup_j U_j}, E)$ where $\cup_j U_{jE} \in \tau_u$, then $\tilde{\vee}_j \tilde{U}_{jE} \in \tau_u^{(IVFS)}$

3. *Let $\tilde{U}_E, \tilde{V}_E \in \tau_u^{(IVFS)}$, then $\tilde{U}_E \tilde{\wedge} \tilde{V}_E \in \tau_u^{(IVFS)}$ since $\tilde{U}_E \tilde{\wedge} \tilde{V}_E = (\widetilde{U \cap V}, E)$ where $U \cap V \in \tau_u$.*

Definition 8. *Let interval* $[\lambda_e^-, \lambda_e^+] \subseteq [0,1]$ *for all* $e \in E$*. Then,* $\tilde{x}_E$ *is called an interval-valued fuzzy soft point (IVFS point) with support* $x \in X$ *and e lower value* λ_e^- *and e upper value* λ_e^+*, if for each* $y \in X$*:*

$$\tilde{x}(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & y = x \\ 0 & otherwise. \end{cases}$$

Example 3. *Let* $X = [0,1]$ *and E be any subset of X. Consider IVFS point* $\tilde{x}_E$ *with support x, lower value zero, and upper value* 0.3*, we define IVFS point* $\tilde{x}_E$ *by:*

$$\tilde{x}(e)(c) = \begin{cases} [0, 0.3] & c = x \\ 0 & otherwise, \end{cases}$$

for any $e \in E$ *and* $c \in X$*.*

Definition 9. *The IVFS point* $\tilde{x}_E$ *belongs to IVFS set* f_E*, denoted by* $\tilde{x}_E \tilde{\in} f_E$*, whenever for all* $e \in E$*, we have* $\lambda_e^- \leq f_e^-(x)$ *and* $\lambda_e^+ \leq f_e^+(x)$*.*

Theorem 1. *Let* f_E *be an IVFS set, then* f_E *is the union of all its IVFS points,*
i.e., $f_E = \tilde{\bigtriangledown}_{\tilde{x}_E \tilde{\in} f_E} \tilde{x}_E$*.*

Proof. Let $x \in X$ be a fixed point, $y \in X$ and $e \in E$. Take all $\tilde{x}_E \tilde{\in} f_E$ with different *e* lower and *e* upper values $\lambda_{je}^-, \lambda_{je}^+$ where $j \in J$. Then, there exists $\lambda_{je}^- = f_e^-, \lambda_{je}^+ = f_e^+$ where:

$$\begin{aligned} \tilde{\bigtriangledown}_{\tilde{x}_E \in f_E} \tilde{x}_e(y) &= [\sup\ \tilde{x}_e^-(y), \sup\ \tilde{x}_e^+(y)] \\ &= [\sup_{\lambda_{je}^- \tilde{\leq} f^-(x)} \lambda_{je}^-, \sup_{\lambda_{je}^+ \tilde{\leq} f^+(x)} \lambda_{je}^+] \\ &= [f_e^-(x), f_e^+(x)]. \end{aligned}$$

□

Proposition 2. *Let* $\{f_{jE}\}_{j \in J}$ *be a family of IVFS sets over X, where J is an index set and* $\tilde{x}_E$ *is an IVFS point with support x, e lower value* λ_e^-*, and e upper value* λ_e^+*. If* $\tilde{x} \tilde{\in} \tilde{\wedge}_{j \in J}\{f_{jE}\}$*, then* $\tilde{x}_E \tilde{\in} \{f_{jE}\}$ *for each* $j \in J$*.*

Proof. Let $\tilde{x}_E$ be an *IVFS* point with support x, e lower value λ_e^-, and e upper value λ_e^+, and let $\tilde{x} \tilde{\in} \tilde{\wedge}_{j \in J}\{f_{jE}\}$. Then, $\lambda_e^- \leq \wedge_{j \in J}\{f_{je}^-\}(x) \leq \{f_{je}^-\}(x)$ for each $e \in E$, $x \in X$ and $\lambda_e^+ \leq \wedge_{j \in J}\{f_{je}^+\}(x) \leq \{f_{je}^+\}(x)$ for each $e \in E$, $x \in X$. Thus, $[\lambda_e^-, \lambda_e^+] \leq [\{f_{je}^-\}(x), \{f_{je}^+\}(x)]$, for each $e \in E$, $x \in X$. Hence, $\tilde{x}_E \tilde{\in} \{f_{jE}\}_{j \in J}$. □

Remark 2. *If* $\tilde{x}_E \tilde{\in} f_E \tilde{\bigtriangledown} g_E$ *does not imply* $\tilde{x}_E \tilde{\in} f_E$ *or* $\tilde{x}_E \tilde{\in} g_E$*.*

This is shown in the following example.

Example 4. *Let* τ *be an IVFST over X, where* $\tau = \{\emptyset_E, X_E, f_E, g_E, f_E \tilde{\wedge} g_E\}$*, and* $\tilde{x}_E$ *be the absolute IVFS point with support x, e lower value* λ_e^-*, and e upper value* λ_e^+*. If* f_E*and* g_E *are two IVFS sets in X defined as below:*

$$f : E \to \mathcal{IVF}([0,1])$$

and:

$$g : E \to \mathcal{IVF}([0,1])$$

such that for any $e \in E, x \in X$:

$$f_e(x) = \begin{cases} [1, 0.5] & 0 \le x \le e \\ 0 & e < x \le 1 \end{cases}$$

and:

$$g_e(x) = \begin{cases} [0.2, 1] & 0 \le x \le e \\ 0 & e < x \le 1. \end{cases}$$

Since:

$$f_e(x) \vee g_e(x) = \begin{cases} 1 & \text{if } 0 \le x \le e \\ 0 & \text{if } e < x \le 1, \end{cases}$$

then $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$, but $\tilde{x}_E \tilde{\notin} f_E$ and $\tilde{x}_E \tilde{\notin} g_E$.

Theorem 2. *Let $\tilde{x}_E$ be an IVFS point with support x, e lower value λ_e^-, and e upper value λ_e^+ and f_E and g_E be IVFS sets. If $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$, then there exists IVFS point $\tilde{x}_{1E} \tilde{\in} f_E$ and IVFS point $\tilde{x}_{2E} \tilde{\in} g_E$ such that $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.*

Proof. Let $\tilde{x}_E \tilde{\in} f_E \tilde{\vee} g_E$. Then, $\lambda_e^- \le f_e^-(x) \vee g_e^-(x)$ and $\lambda_e^+ \le f_e^+(x) \vee g_e^+(x)$, for each $e \in E$, $x \in X$. Let us choose

$E_1 = \{e \in E | \lambda_e^- \le f_e^-(x), \lambda_e^+ \le f_e^+(x) : x \in X\}$,

$E_2 = \{e \in E | \lambda_e^- \le g_E^-(x), \lambda_e^+ \le g_E^+(x) : x \in X\}$

and:

$$\tilde{x}_1(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+] & \text{if } y = x_1,\ e \in E_1 \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{x}_2(e)(y) = \begin{cases} [\lambda_e^-, \lambda_e^+], & \text{if } y = x_2, e \in E_2 \\ 0, & \text{otherwise.} \end{cases}$$

Since $x_{1e}^- \le f_{1e}^-(x)$ and $x_{1e}^+ \le f_{1e}^+(x)$ for each $e \in E_1, x \in X$, that implies $\tilde{x}_{1E} \tilde{\in} f_{1E}$ and also $x_{2e}^- \le f_{2e}^-(x)$, and $x_{2e}^+ \le f_{2e}^+(x)$ for each $e \in E_2, x \in X$, that implies $\tilde{x}_{2E} \tilde{\in} f_{2E}$. Consequently, $E_1 \tilde{\vee} E_2 = E$ and $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$. □

Definition 10. *Let (X, E, τ) be an IVFST space and $\tilde{x}_E$ be an IVFS point with support x, e lower value λ_e^-, and e upper value λ_e^+. The IVFS set g_E is called the interval-valued fuzzy soft neighborhood $(IVFSN)$ of IVFS point $\tilde{x}_E$, if there exists the IVFS-open set f_E in X such that $\tilde{x}_E \tilde{\in} f_E \tilde{<} g_E$. Therefore, the IVFS-open set f_E is an IVFSN of the IVFS point $\tilde{x}_E$ if $\forall e \in E, x \in X$ such that $\lambda_e^- < f_e^-(x)$ and $\lambda_e^+ < f_e^+(x)$.*

Definition 11. *Let (X, E, τ) be an IVFST space and $\tilde{x}_E$ be an IVFS point with support x, e lower value λ_e^-, and e upper value λ_e^+ and $\tilde{x}_E^\star$ be an IVFS point with support $x^\star$, e lower value ε_e^-, and e upper value ε_e^+. $\tilde{x}_E^\star$ is said to be compatible with λ_e^-, λ_e^+, if $\tilde{x}_E^\star$ provides that $0 \le \varepsilon_e^- \le \lambda_e^-$ and $0 \le \varepsilon_e^+ \le \lambda_e^+$ for each $e \in E$.*

Proposition 3.

1. *If f_E is an IVFSN of the IVFS point $\tilde{x}_E$ and $f_E \tilde{\le} h_E$, then h_E is also an IVFSN of $\tilde{x}_E$.*
2. *If f_E and g_E are two IVFSN of the IVFS point $\tilde{x}_E$, then $f_E \tilde{\wedge} g_E$ is also the IVFSN of $\tilde{x}_E$.*
3. *If f_E is an IVFSN of the IVFS point $\tilde{x}_E^\star$ with support $x^\star$, e lower value $\lambda_e^- - \varepsilon_e^-$, and e upper value $\lambda_e^+ - \varepsilon_e^+$, for all ε_e^- compatible with λ_e^- and ε_e^+ compatible with λ_e^+, then f_E is an IVFSN of the IVFS point $\tilde{x}_E$.*
4. *If f_E is an IVFSN of the IVFS point $\tilde{x}_{1E}$ and g_E is an IVFSN of the IVFS point $\tilde{x}_{2E}$, then $f_E \tilde{\vee} g_E$ is also an IVFSN of $\tilde{x}_{1E}$ and $\tilde{x}_{2E}$.*
5. *If f_E is an IVFSN of the IVFS point $\tilde{x}_E$, then there exists IVFSN g_E of $\tilde{x}_E$ such that $g_E \tilde{\le} f_E$ and g_E is IVFSN of IVFS point $\tilde{y}$ with support y, e lower value γ_e^-, and e upper value γ_e^+, for all $\tilde{y}_E \tilde{\in} g_E$.*

Proof.

1. Let f_E be an $IVFSN$ of the $IVFS$ point $\tilde{x}$. Then, there exists the $IVFS$-open set g_E in X such that $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$. Since $f_E \tilde{\leq} h_E$, $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E \tilde{\leq} h_E$. Thus, h_E is an $IVFSN$ of $\tilde{x}_E$.
2. Let f_E and g_E be two $IVFSN$ of the $IVFS$ point $\tilde{x}_E$. Then, there exists two $IVFS$-open sets h_E, k_E in X such that $\tilde{x}_E \tilde{\in} h_E \tilde{\leq} f_E$ and $\tilde{x}_E \tilde{\in} k_E \tilde{\leq} g_E$. Thus, $\tilde{x}_E \tilde{\in} h_E \tilde{\wedge} k_E \tilde{\leq} f_E \tilde{\wedge} g_E$. Since $h_E \tilde{\wedge} k_E$ is an $IVFS$-open set, $g_E \tilde{\wedge} f_E$ is an $IVFSN$ of $\tilde{x}_E$.
3. Let f_E be an $IVFSN$ of the $IVFS$ point $\tilde{x}^\star_E$ with support $x^\star$, e lower value $\lambda^-_e - \varepsilon^-_e$, and e upper value $\lambda^+_e - \varepsilon^+_e$, for all ε^-_e compatible with λ^-_e and ε^+_e compatible with λ^+_e. Then, there exists $IVFS$-open set $g^{x^\star}_E$ such that $\tilde{x}^\star_E \tilde{\in} g^{x^\star}_E \tilde{\leq} f_E$. Let $g_E = \tilde{\vee}_{x^\star} g^{x^\star}_E$, then g_E is $IVFS$-open in X and $g_E \tilde{\leq} f_E$. By Theorem 1 and since for all $e \in E$, $\tilde{\vee} \tilde{x}^\star_E = \tilde{x}_E \tilde{\leq} \tilde{\vee}_{x^\star} g^{x^\star}_E = g_E \tilde{\leq} f_E$. Hence, $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$, i.e., f_E is an $IVFSN$ of $\tilde{x}_E$.
4. Let f_E be an $IVFSN$ of the $IVFS$ point $\tilde{x}_{1E}$ with support x_1, e lower value λ^-_{1e}, and e upper value λ^+_{1e} and g_E be an $IVFSN$ of the $IVFS$ point $\tilde{x}_{2E}$ with support x_2, e lower value λ^-_{2e}, and e upper value λ^+_{2e}. Then, there exists $IVFS$-open sets h_{1E}, h_{2E} such that $\tilde{x}_{1E} \tilde{\in} h_{1E} \tilde{\leq} f_E$ and $\tilde{x}_{2E} \tilde{\in} h_{2E} \tilde{\leq} f_E$, respectively. Since $\tilde{x}_{1E} \tilde{\in} h_{1E}$, $\lambda^-_{1e} \leq h^-_{1e}(x), \lambda^+_{1e} \leq h^+_{1e}(x)$ for each $e \in E$ and $x \in X$. Since $\tilde{x}_{2E} \tilde{\in} h_{2E}$, $\lambda^-_{2e} \leq h^-_{2e}(x), \lambda^+_{2e} \leq h^+_{2e}(x)$ for each $e \in E$ and $x \in X$. Thus, we have:

$$\max\{[\lambda^-_{1e}, \lambda^+_{1e}], [\lambda^-_{2e}, \lambda^+_{2e}]\} \leq \max\{[h^-_{1e}(x), h^+_{1e}(x)], [h^-_{2e}(x), h^+_{2e}(x)]\}$$

   for each $e \in E$, $x \in X$. Therefore, $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E} \tilde{\in} h_{1E} \tilde{\vee} h_{2E}$, $h_{1E} \tilde{\vee} h_{2E} \in \tau$, and $h_{1E} \tilde{\vee} h_{2E} \tilde{\leq} f_E \tilde{\vee} g_E$. Consequently, $f_E \tilde{\vee} g_E$ is an $IVFSN$ of $x_{1E} \tilde{\vee} x_{2E}$.
5. Let f_E be an $IVFSN$ of the $IVFS$ point $\tilde{x}_E$, with support x, e lower value λ^-_e, and e upper value λ^+_e. Then, there exists $IVFS$-open set g_E such that $\tilde{x}_E \tilde{\in} g_E \tilde{\leq} f_E$. Since g_E is an $IVFS$-open set, g_E is a neighborhood of its points, i.e., g_E is an $IVFSN$ of $IVFS$ point $\tilde{y}_E$ with support y, e lower value γ^-_e, and e upper value γ^+_e, for all $e \in E$. Furthermore, g_E is an $IVFSN$ of $IVFS$ point $\tilde{x}_E$ since $\tilde{x}_E \tilde{\in} g_E$. Therefore, there exists g_E that is an $IVFSN$ of $\tilde{x}_E$ such that $g_E \tilde{\leq} f_E$ and g_E is an $IVFSN$ of $\tilde{y}_E$; since f_E is an $IVFSN$ of $\tilde{x}_E$.

□

Definition 12. *Let (X, E, τ) be an $IVFST$ space and f_E be an $IVFS$ set. The $IVFS$-closure of f_E denoted by Clf_E is the intersection of all $IVFS$-closed super sets of f_E. Clearly, Clf_E is the smallest $IVFS$-closed set over X that contains f_E.*

Example 5 ([23])**.** *Consider $IVFST$ τ^{IVFS}_u over $\mathbb{R}$ as introduced in Example 2, and if $\tilde{H}_E$ is an $IVFS$ over $\mathbb{R}$ related of the open interval $H = (a, b) \subset \mathbb{R}$ by mapping:*

$$\tilde{H} : E \to (Int[0,1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in (a,b) \\ 0 & x \notin (a,b), \end{cases}$$

where $e \in E$ and $x \in \mathbb{R}$, then the closure of $\tilde{H}_E$ is defined as:

$$Cl\tilde{H} : E \to (Int[0,1])^{\mathbb{R}}$$

$$\tilde{H}_e(x) = \begin{cases} 1 & x \in [a,b] \\ 0 & x \notin [a,b]. \end{cases}$$

Remark 3. *By replacing $\tilde{x}_E$ for f_E, the $IVFS$-closure of $\tilde{x}_E$ denoted by $Cl\tilde{x}_E$ is the intersection of all $IVFS$-closed super sets of $\tilde{x}_E$.*

Proposition 4. *Let (X, E, τ) be an IVFST space and f_E and g_E be two IVFSS over X. Then:*

1. $Cl\emptyset_E = \emptyset_E$ *and* $Cl\tilde{X}_E = \tilde{X}_E$,
2. $f_E \tilde{\leq} Clf_E$, *and* Clf_E *is the smallest IVFS-closed set containing* $IVFSf_E$,
3. $Cl(Clf_E) = Clf_E$,
4. *if* $f_E \tilde{\leq} g_E$, *then* $(Clf_E) \tilde{\leq} Clg_E$.
5. f_E *is an IVFS-closed set if and only if* $f_E = Clf_E$,
6. $Cl(f_E \tilde{\vee} g_E) = Clf_E \tilde{\vee} Clg_E$,
7. $Cl(f_E \tilde{\wedge} g_E) \tilde{\leq} Clf_E \tilde{\wedge} Clg_E$.

Proof. We only prove Part (6). A similar technique is used to show the other parts.

Since $f_E \tilde{\leq} f_E \tilde{\vee} g_E$ and $g_E \tilde{\leq} f_E \tilde{\vee} g_E$, by Part (4), we have $Clf_E \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$ and $Clg \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$. Then, $Clf_E \tilde{\vee} Clg_E \tilde{\leq} Cl(f_E \tilde{\vee} g_E)$.

Conversely, we have $f_E \tilde{\leq} Clf_E$ and $g_E \tilde{\leq} Clg_E$, by Part (2). Then, $f_E \tilde{\vee} g_E \tilde{\leq} Clf_E \tilde{\vee} Clg_E$ where $Clf_E \tilde{\vee} Clg_E$ is an $IVFS$-closed set. Thus, $Cl(f_E \tilde{\vee} g_E) \tilde{\leq} Clf_E \tilde{\vee} Clg_E$.

Therefore, $Cl(f_E \tilde{\vee} g_E) = Clf_E \tilde{\vee} Clg_E$. □

Definition 13. *Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two IVFSTS and:*

$$\Phi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$$

be an IVFS map. Then, Φ is called an:

1. *interval-valued fuzzy soft continuous $(IVFSC)$ map if and only if for each $g_{E_2} \in \tau_2$, we have $\Phi^{-1}(g_{E_2}) \in \tau_1$,*
2. *interval-valued fuzzy soft open $(IVFSO)$ map if and only if for each $f_E \in \tau_1$, we have $\Phi(f_{E_1}) \in \tau_2$.*

Theorem 3. *Let (X_1, E_1, τ_1) and (X_2, E_2, τ_2) be two IVFST and Φ be an IVFS mapping from X_1 to X_2, then the following statements are equivalent:*

1. *Φ is IVFC,*
2. *For each IVFS point $\tilde{x}_E$ on X_1, the inverse of every neighborhood of $\Phi(\tilde{x}_E)$ under Φ is a neighborhood of $\tilde{x}_E$,*
3. *For each IVFS point $\tilde{x}_E$ on X_1 and each neighborhood g_E of $\Phi(\tilde{x}_E)$, there exists a neighborhood f_E of $\tilde{x}_E$ such that $\Phi(f_E) \tilde{\leq} g_E$.*

Proof.

(1) ⇒ (2) Let g_E be an $IVFSN$ of $\Phi(\tilde{x}_E)$ in τ_2. Then, there exists an $IVFS$-open set f_E in τ_2 such that $\Phi(\tilde{x}_E) \tilde{\in} f_E \tilde{\leq} g_E$. Since Φ is $IVFSC$, $\Phi^{-1}(f_E)$ is an $IVFS$-open in τ_1, and we have $\tilde{x}_E \tilde{\in} \Phi^{-1}(f_E) \tilde{\leq} \Phi^{-1}(g_E)$.

(2) ⇒ (3) Let g_E be an $IVFSN$ of $\Phi(\tilde{x}_E)$. By the hypothesis, $\Phi^{-1}(g_E)$ is an $IVFSN$ of $\tilde{x}_E$. Consider $f_E = \Phi^{-1}(g_E)$ to be an $IVFSN$ of $\tilde{x}_E$. Then, we have $\Phi(f_E) = \Phi(\Phi^{-1}(g_E)) \tilde{\leq} g_E$.

(3) ⇒ (1) Let g_E be an $IVFS$-open set in τ_2. We must show that $\Phi^{-1}(g_E)$ is an $IVFS$-open set in τ_1. Now, let $\tilde{x}_E \tilde{\in} \Phi^{-1}(g_E)$. Then, $\Phi(\tilde{x}_E) \tilde{\in} g_E$. Since g_E is an $IVFS$-open set in τ_2, we get that g_E is an $IVFSN$ $\Phi(\tilde{x}_E)$ in τ_2. By the hypothesis, there exists $IVFS$-open set f_E that is an $IVFSN$ of $\tilde{x}_E$ such that $\Phi(f_E) \tilde{\leq} g_E$. Thus, $f_E \tilde{\leq} \Phi^{-1}[\Phi(f_E)] \tilde{\leq} \Phi^{-1}(g_E)$ for f_E is an $IVFSN$ of $\tilde{x}_E$. From here, $f_E \tilde{\leq} \Phi^{-1}(g_E)$, as f_E is an $IVFSN$ of $\tilde{x}_E$. Hence, $\Phi^{-1}(g_E) \tilde{\in} \tau_1$. □

4. Quasi-Coincident Neighborhood Structure of Interval-Valued Fuzzy Soft Topological Spaces

In this section, we present the quasi-coincident neighborhood structure in the interval-valued fuzzy soft topology $(IVFST)$ and its properties.

Definition 14. *The IVFS point* $\tilde{x}_E$ *is called soft quasi-coincident with IVFS* f_E*, denoted by* $\tilde{x}_E \tilde{q} f_E$*, if there exists* $e \in E$ *such that* $\lambda_e^- + f_e^-(x) > 1$ *and* $\lambda_e^+ + f_e^+(x) > 1$*. If* f_E *is not soft quasi-coincident with* f_E*, we write* $f_E \neg \tilde{q} g_E$*.*

Definition 15. *The IVFS set* f_E *is called soft quasi-coincident with IVFS* g_E*, denoted by* $f_E \tilde{q} g_E$*, if there exists* $e \in E$ *such that* $f_e^-(x) + g_e^-(x) > 1$ *and* $f_e^+(x) + g_e^+(x) > 1$*.*

Proposition 5. *Let* $\tilde{x}_E$ *be an IVFS point with support* x*, e lower value* λ_e^-*, and e upper value* λ_e^+ *and* f_E, g_E *two IVFS sets. Then:*

(i) $f_E \tilde{\leq} g_E \Leftrightarrow f_E \neg \tilde{q} g_E^c$*,*
(ii) $\tilde{x}_E \tilde{\in} f_E \Leftrightarrow \tilde{x}_E \neg \tilde{q} f_E^c$*.*

Proof. We just prove Part (i). A similar technique is used to show Part (ii). For two *IVFS* sets f_E, g_E, we have:

$$\begin{aligned} f_E \tilde{\leq} g_E &\Leftrightarrow \forall e \in E : [f_e^-(x), f_e^+(x)] \leq [g_e^-(x), g_e^+(x)], \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) \leq g_e^-(x) \text{ and } f_e^+(x) \leq g_e^+(x), \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) + 1 - g_e^-(x) \leq 1 \text{ and } f_e^+(x) + 1 - g_e^+(x) \leq 1, \forall x \in X \\ &\Leftrightarrow \forall e \in E : f_e^-(x) + {g^-}_e^c(x) \leq 1 \text{ and } f_e^+(x) + {g^+}_e^c(x) \leq 1, \forall x \in X \\ &\Leftrightarrow f_E \neg \tilde{q} g_E^c. \end{aligned}$$

□

Proposition 6. *Let* $\{f_{jE} : j \in J\}$ *be a family of IVFS sets over X and* $\tilde{x}_E$ *be an IVFS point with support* x*, e lower value* λ_e^-*, and e upper value* λ_e^+*. If* $\tilde{x}_E \tilde{q}(\tilde{\wedge} f_{jE})$*, then* $\tilde{x}_E \tilde{q} f_{jE}$ *for each* $j \in J$*.*

Proof. Let $\tilde{x}_E \tilde{q}(\tilde{\wedge} f_{jE})$. Then, $\lambda_e^- \tilde{q}(\tilde{\wedge}_j f_{je}^-)(x)$, $\lambda_e^+ \tilde{q}(\tilde{\wedge}_j f_{je}^+)(x)$ for $e \in E$, and $x \in X$. This implies that $\lambda_e^- > 1 - \wedge_j (f_{je}^-)(x)$ and $\lambda_e^+ > 1 - \wedge_j (f_{je}^+)(x)$, $x \in X$. Since $\wedge_j f_{je}^-(x) \leq f_{je}^-(x)$ and $\wedge_j f_{je}^+(x) \leq f_{je}^+(x)$, then $\lambda_e^- > 1 - \wedge_j (f_{je}^-)(x) > 1 - f_{je}^-(x)$ for each $e \in E, x \in X$ and $\lambda_e^+ > 1 - \wedge_j (f_{je}^+)(x) > 1 - f_{je}^+(x)$ for each $e \in E, x \in X$. Hence, $\lambda_e^- > 1 - f_{je}^-(x)$ and $\lambda_e^+ > 1 - f_{je}^+(x)$. Therefore, $[\lambda_e^-, \lambda_e^+] > [1,1] - [f_{je}^-(x), f_{je}^+(x)]$ implies that $\tilde{x}_E > 1 - f_{jE}^-$ and $\tilde{x}_E \tilde{q} f_{jE}$ for each $j \in J$. □

Remark 4. $\tilde{x}_E \tilde{q}(f_E \vee g_E)$ *does not imply* $\tilde{x}_E \tilde{q} f_E$ *or* $\tilde{x}_E \tilde{q} g_E$*. This is shown in the following example.*

Example 6. *Let us consider Example 4 where* $\tilde{x}_E \tilde{q}(f_E \tilde{\vee} g_E)$*, but* $\tilde{x}_E \neg \tilde{q} f_E$ *and* $\tilde{x}_E \neg \tilde{q} g_E$*.*

Theorem 4. *Let* $\tilde{x}_E$ *be an IVFS point* $\tilde{x}_E$ *with support* x*, e lower value* λ_e^-*, and e upper value* λ_e^+ *and* f_E, g_E *be IVFS sets over X. If* $\tilde{x}_E \tilde{q}(f_E \vee g_E)$*, then there exists* $\tilde{x}_{1E} \tilde{q} f_E$ *and* $\tilde{x}_{2E} \tilde{q} g_E$ *such that* $\tilde{x}_E = \tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$*.*

The proof is very similar to the proof of Theorem 2.

Definition 16. *Let* (X, E, τ) *be an IVFSTS and* $\tilde{x}_E$ *be an IVFS point with support* x*, e lower values* λ_e^-*, and e upper values* λ_e^+*. The IVFS set* g_E *is called a quasi-soft neighborhood* $(QIVFSN)$ *of IVFS point* $\tilde{x}_E$ *if there exists the IVFS-open set* f_E *in X such that* $\tilde{x}_E \tilde{q} f_E \tilde{\leq} g_E$*. Thus, the IVFS-open set* f_E *is a QIVFSN of the IVFS point* $\tilde{x}_E$ *if and only if* $\exists e \in E, x \in X$ *such that* $\lambda_e^- + f_e^-(x) > 1$ *and* $\lambda_e^+ + f_e^+(x) > 1$*.*

Remark 5. *A quasi-coincident soft neighborhood of an IVFS point generally does not contain the point itself. This is shown by the following:*

Example 7. *Let $X = [0,1]$ and E be any subset of X. Consider two IVFS sets f_E, g_E over X by the mapping $f : E \to \mathcal{IVF}([0,1])$ and $f : E \to \mathcal{IVF}([0,1])$ such that for any $e \in E, x \in X$:*

$$\tilde{f}_e(x) = \begin{cases} [0.4, 0.5] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and:

$$\tilde{g}_e(x) = \begin{cases} [0.6, 0.7] & 0 \leq x \leq e \\ 0 & e < x \leq 1, \end{cases}$$

and $\tilde{x}_E$ be any IVFS point defined by:

$$\tilde{x}_e(c) = \begin{cases} [0.4, 0.5] & c = x \\ 0 & c \neq x. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly, τ is an IVFST over X. Since $f_E \tilde{\leq} g_E$ and $\tilde{x}\tilde{q}f_E$, thus g_E is a QIVFSN of $\tilde{x}_E$. However, $\tilde{x}_E \notin g_E$.

Proposition 7.

(1) If $f_E \tilde{\leq} g_E$ and f_E is a QINVSN of $\tilde{x}_E$, then g_E is also a QINVSN of $\tilde{x}_E$,
(2) If f_E, g_E are QINVSN of $\tilde{x}_E$, then $f_E \tilde{\wedge} g_E$ is also a QINVSN of $\tilde{x}_E$.
(3) If f_E is a QINVSN of $\tilde{x}_{1E}$ and g_E is a QINVSN of $\tilde{x}_{2E}$, then $f_E \tilde{\vee} g_E$ is also a QINVSN of $\tilde{x}_{1E} \tilde{\vee} \tilde{x}_{2E}$.
(4) If f_E is a QINVSN of $\tilde{x}_E$, then there exists g_E that is a QINVSN of $\tilde{x}_E$, such that $g_E \tilde{\leq} f_E$, and g_E is a QINVSN of y_E, $\forall y_E \tilde{q} g_E$.

Proof. (1) and (2) are straightforward.

(3) Let f_E be a $QINVSN$ of $\tilde{x}_{1E}$ and g_E be a $QINVSN$ of $\tilde{x}_{2E}$. Then, there exists an $IVFS$-open set h_{1E} in X such that $\tilde{x}_{1E}\tilde{q}h_{1E}\tilde{\leq}f_E$ and g_E is a $QINVSN$ of $\tilde{x}_{2E}$. Thus, there exists an $IVFS$-open set h_{2E} in X such that $\tilde{x}_{2E}\tilde{q}h_{2E}\tilde{\leq}g_E$. Since $\tilde{x}_{1E}\tilde{q}h_{1E}$ for each $e \in E, x \in X, \lambda^-_{1e} + h^-_{1e} > 1, \lambda^+_{1e} + h^+_{1e} > 1$, this implies that $\lambda^-_{1e} > 1 - h^-_{1e}, \lambda^+_{1e} > 1 - h^+_{1e}$ for each $e \in E$. Since $\tilde{x}_{2E}\tilde{q}h_{2E}$, for each $e \in E, \lambda^-_{2e} + h^-_{2e} > 1$, $\lambda^+_{2e} + h^+_{2e} > 1$, this implies that $\lambda^-_{2e} > 1 - h^-_{2e}, \lambda^+_{2e} > 1 - h^+_{2e}$ for each $e \in E, x \in X$. From here,

$$\max(\lambda^-_{1e}, \lambda^-_{2e}) > \max(1 - h^-_{1e}(x)), (1 - h^-_{2e}(x)), \max(\lambda^+_{1e}, \lambda^+_{2e}) > \max(1 - h^+_{1e}(x)), (1 - h^+_{2e}(x)).$$

Therefore, $\tilde{x}_{1E}\tilde{\vee}\tilde{x}_{2E}\tilde{q}(h_{1E}\tilde{\vee}h_{2E})\tilde{\leq}f_E\tilde{\vee}g_E$. Consequently, $f_E\tilde{\vee}g_E$ is a $QINVSN$ of $\tilde{x}_{1E}\tilde{\vee}\tilde{x}_{2E}$.

(4) Let f_E be a $QINVSN$ of $\tilde{x}_E$. Then, there exists g_E that is a $QINVSN$ of $\tilde{x}_E$ such that $\tilde{x}_E\tilde{q}g_E\tilde{\leq}f_E$. Consider the $g_E = h_E$. Indeed, since $\tilde{x}_E\tilde{q}h_E$ and h_E is an $IVFS$-open set, h_E is a $QINVSN$ of $\tilde{x}_E$. Thus, we obtain h_E that is a $QINVSN$ of $\tilde{y}_E$.

□

Theorem 5. *In $IVFST(X, E, \tau)$, the IVFS point $\tilde{x}_E$ belongs to Clf_E if and only if each QIVFS of $\tilde{x}_E$ is soft quasi-coincident with f_E.*

Proof. Let $IVFS$ point $\tilde{x}_E$ with support x, e lower value λ^-_e, and e upper value λ^+_e belong to Clf_E, *i.e*, $\tilde{x}_E\tilde{\in}Clf_E$. For any $IVFS$-closed g_E containing f_E, $\tilde{x}_E\tilde{\in}g_E$, which implies that $\lambda^-_e \leq g^-_e(x)$ and $\lambda^+_e \leq g^+_e(x)$, for all $x \in X, e \in E$. Consider h_E to be an $QIVFN$ of the $IVFS$ point $\tilde{x}_E$ and $h_E\neg\tilde{q}f_E$. Then, for any $e \in E$ and $x \in X$, $h^-_e(x) + f^-_e(x) \leq 1, h^+_e(x) + f^+_e(x) \leq 1$, and so, $f_E\tilde{\leq}h^c_E$. Since h_E is a $QIVFSN$ of the $IVFS$ point $\tilde{x}_E$, by $\tilde{x}_E$, it does not belong to h^c_E. Therefore, we have that $\tilde{x}_E$ does not belong to Clf_E. This is a contradiction.

Conversely, let any $QIVFSN$ of the $IVFS$ point $\tilde{x}_E$ be soft quasi-coincident with f_E. Consider that $\tilde{x}_E$ doe not belong to $Clf_E, i.e, \tilde{x}_E \notin Clf_E$. Then, there exists an $IVFS$-closed set g_E, which contains f_E such that $\tilde{x}_E$ does not belong to g_E. We have $\tilde{x}_E \tilde{q} g^c_E$. Then, g^c_E is an $QIVFSN$ of the $IVFS$ point $\tilde{x}_E$ and $f_E \neg\tilde{q} g^c_E$. This is a contradiction with the hypothesis. □

5. IVFS Quasi-Separation Axioms

In this section, we develop the separation axioms to $IVFST$, so-called $IVFSQ$ separation axioms ($IVFSq$-T_i axioms) for $i = 0, 1, 2, 3, 4$, and consider some of their properties.

Definition 17. *Let (X, E, τ) be an $IVFST$ space. Let $\tilde{x}_E$ and $\tilde{y}_E$ be $IVFS$ points over X, where:*

$$\tilde{x}(e)(z) = \begin{cases} [\lambda^-_e, \lambda^+_e] & z = x \\ 0 & otherwise \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} [\gamma^-_e, \gamma^+_e] & z = y \\ 0 & otherwise, \end{cases}$$

then $\tilde{x}_E$ and $\tilde{y}_E$ are said to be distinct if and only if $\tilde{x}_E \tilde{\wedge} \tilde{y}_E = \emptyset_E$, which means $x \neq y$.

Definition 18. *Let (X, E, τ) be an $IVFST$ space. The $IVFS$ point $\tilde{x}_E$ is called a crisp $IVFS$ point $x^{[1,1]}_E$, if $\lambda^-_e = \lambda^+_e = 1$ for all $e \in E$.*

Definition 19. *Let (X, E, τ) be an $IVFST$ space and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points. If there exists $IVFS$ open sets f_E and g_E such that:*

- (a) *when $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ^-_e, λ^+_e and γ^-_e, γ^+_e, respectively, and f_E is an $IVFSN$ of the $IVFS$ point $\tilde{x}_E$ and $\tilde{y}_E \neg\tilde{q} f_E$ or g_E is an $IVFSN$ of the $IVFS$ point $\tilde{y}_E$ and $\tilde{x}_E \neg\tilde{q} g_E$,*
- (b) *when $\tilde{x}_E$ and $\tilde{y}_E$ are two $IVFS$ points with the same supports $x = y$, e value $\lambda^-_e < \gamma^-_e$, and e value $\lambda^+_e < \gamma^+_e$ and f_E is a $QIVFSN$ of the $IVFS$ point $\tilde{y}_E$ such that $\tilde{x}_E \neg\tilde{q} f_E$,*

then (X, E, τ) is an interval-valued fuzzy soft quasi-T_0 space ($IVFSq$-T_0 space).

Example 8. *Consider the $IVFS$ set defined in Example 3.1 and $\tilde{x}_E, \tilde{y}_E$ to be any two distinct $IVFS$ points in X defined by:*

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} 0 & if\, z = y \\ 1 & if\, z \neq y. \end{cases}$$

Then, f_E is an $IVFSN$ of $\tilde{x}_E$ and $\tilde{y}_E \neg\tilde{q} f_E$. Thus, X is an $IVFSq$-T_0 space.

Theorem 6. *(X, E, τ) is an $IVFSq$-T_0 space if and only if for every two $IVFS$ points $\tilde{x}_E$, $\tilde{y}_E$ and $\tilde{x}_E \notin Cl\tilde{y}_E$ or $\tilde{y}_E \notin Cl\tilde{x}_E$.*

Proof. Let (X, E, τ) be an $IVFS$q-T_0 space and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points in X.

First consider that $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ^-_e, γ^-_e and λ^+_e, γ^+_e, respectively. Then, a crisp $IVFS$ point $\tilde{x}^{[1,1]}_E$ has an $IVFSN$ f_E such that $\tilde{y}_E \neg\tilde{q} f_E$ or a crisp $IVFS$ point $\tilde{y}^{[1,1]}_E$ has an $IVFSN$ g_E such that $\tilde{x}_E \neg\tilde{q} f_E$. Consider that the crisp $IVFS$ point $\tilde{x}^{[1,1]}_E$ has an $IVFSN$ f_E such that $\tilde{y}_E \neg\tilde{q} f_E$. Moreover, f_E is an $QINFSN$ of $\tilde{x}_E$

and $\tilde{y}_E \neg \tilde{q} f_E$. Hence, $\tilde{x_E} \notin Cl\tilde{y_E}$. Next, we consider the case $\tilde{x}_E$ and $\tilde{y}_E$ to be two $IVFS$ points with the same supports $x = y$, e lower value $\lambda_e^- < \gamma_e^-$, and e upper value $\lambda_e^+ < \gamma_e^+$. Then, $\tilde{y}_E$ has a $QIVFSN$ that is not quasi-coincident with $\tilde{x}_E$, and so, by Theorem 5, $\tilde{x_E} \notin Cl\tilde{y_E}$.

Conversely, let $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points in X. Consider without loss of generality that $\tilde{x_E} \notin Cl\tilde{y_E}$. First, consider that $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^-, respectively, since $\tilde{x}_E \notin Cl\tilde{y}_E$ for any $e \in E$, $f_e^-(y) = f_e^+(y) = 0$ and $f_e^-(x) = f_e^+(x) = 1$. Then, $Cl(\tilde{y}_E)^c$ is an $IVFSN$ of $\tilde{x}_E$ such that $Cl(\tilde{y}_E)^c \neg \tilde{q} \tilde{y}_E$. Next, let $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points with the same supports $x = y$, and we must have e lower value $\lambda_e^- > \gamma_e^-$ and e upper value $\lambda_e^+ > \gamma_e^+$, then $\tilde{x}_E$ has a $QIVFSN$ that is not quasi-coincident with $\tilde{y}_E$. □

Definition 20. *Let (X, E, τ) be an $IVFST$ and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points, if there exists $IVFS$ open sets f_E and g_E such that:*

(a) when $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, f_E is an $IVFSN$ of $IVFS$ points $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$, and g_E is an $IVFSN$ of $IVFS$ points $\tilde{y}_E$ and $\tilde{x}_E \neg \tilde{q} g_E$,

(b) when $\tilde{x}_E$ and $\tilde{y}_E$ are two $IVFS$ points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$, f_E is an $QIVFSN$ of the $IVFS$ point $\tilde{y}_E$ such that $\tilde{x}_E \neg \tilde{q} f_E$,

then (X, E, τ) is an interval-valued fuzzy soft quasi-T_1 space ($IVFSq$-T_1 space).

Theorem 7. *(X, E, τ) is an $IVFSq$-T_1 space if and only if any $IVFS$ point $\tilde{x_E}$ in X is an $IVFS$-closed set.*

Proof. Suppose that each $IVFS$ point $\tilde{x_E}$ in X is an $IVFS$-closed set, i.e., $g_E = \tilde{x}_E^c$. Then, g_E is an $IVFS$-open set. Let x_E and y_E be two $IVFS$ points as follows: First, consider that $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively. Then, g_E is an $IVFS$-open set such that g_E is an $IVFSN$ of $IVFS$ point $\tilde{y}_E$ and $\tilde{x}_E \neg \tilde{q} g_E$. Similarly, $f_E = \tilde{y}_E^c$ is an $IVFS$-open set and f_E is an $IVFSN$ of the $IVFS$ points $\tilde{x}_E$ and $\tilde{y}_E \neg \tilde{q} f_E$. Next, we consider the case $\tilde{x}_E$ and $\tilde{y}_E$ to be two $IVFS$ points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$. Then, $\tilde{y}_E$ has a $QIVFSN$ g_E, which is not quasi-coincident with $\tilde{x}_E$. Thus, X is an $IVFS$q-T_1 space.

Conversely, Let (X, E, τ) be an $IVFS$q-T_1 space. Suppose that any $IVFS$ point $\tilde{x}_E$ is not an $IVFS$-closet set in X, i.e., $f_E \doteq \tilde{x}_E^c$. Then, $\tilde{f}_E \neq Cl\tilde{f}_E$, and there exists $\tilde{y}_E \tilde{\in} Cl\tilde{f}_E$ such that $\tilde{x}_E \neq \tilde{y}_E$.

First, consider that $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively. Suppose that e lower value $\lambda_e^- \leq 0.5$ and e upper value $\lambda_e^+ \leq 0.5$. Since $\tilde{y}_E \tilde{\in} Clf_E$, by Theorem 4.1, any f_E is a $QIVFSN$ of $\tilde{y}_E$ and $\tilde{x}_E \tilde{q} f_E$. Then, there exists $IVFS$-open set h_E such that $\tilde{y} \tilde{q} h_E \tilde{\leq} f_E$. Hence, $h_e^-(y) + \gamma_e^- > 1$. Next, let $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$. Since $y_E \tilde{\in} Clx_E$, by Theorem 5, each f_E is a $QIVFSN$ of $IVFS$ points $\tilde{y}_E$, $\tilde{x}_E \tilde{q} f_E$. This is a contradiction. □

Definition 21. *Let (X, E, τ) be an $IVFST$ and $\tilde{x}_E$ and $\tilde{y}_E$ be two $IVFS$ points, if there exists $IVFS$ open sets f_E and g_E such that:*

(a) when $\tilde{x}_E$ and $\tilde{y}_E$ are two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, f_E is an $IVFSN$ of the $IVFS$ point $\tilde{x}_E$ and g_E is an $IVFSN$ of the $IVFS$ point $\tilde{y}_E$, such that $f_E \neg \tilde{q} g_E$,

(b) when $\tilde{x}_E$ and $\tilde{y}_E$ are two $IVFS$ points with the same supports $x = y$, e value $\lambda_e^- < \gamma_e^-$, and e value $\lambda_e^+ < \gamma_e^+$, f_E is an $IVFSN$ of $IVFS$ point $\tilde{x}_E$ and g_E is a $QIVFSN$ of $IVFS$ point $\tilde{y}_E$,

then (X, E, τ) is an interval-valued fuzzy soft quasi-T_2 space ($IVFS$ q-T_2 space).

Example 9. *Suppose that $X = [0,1]$ and E are any proper ($E \subset X$). Consider IVFS sets f_E and g_E over X defined as below: $f : E \to \mathcal{IVF}([0,1])$ and $g : E \to \mathcal{IVF}([0,1])$, such that for any $e \in E, x \in X$:*

$$f(e)(x) = \begin{cases} 1 & 0 \le x \le e \\ 0 & e < x \le 1 \end{cases}$$

and:

$$g(e)(x) = \begin{cases} 0 & 0 \le x \le e \\ 1 & e \le x \le 1. \end{cases}$$

Let $\tau = \{\emptyset_E, X_E, f_E, g_E\}$. Then clearly, τ is an IVFST over X. Therefore, for any two absolute distinct IVFS points $\tilde{x}_E, \tilde{y}_E$ in X defined by:

$$\tilde{x}(e)(z) = \begin{cases} 1 & z = x \\ 0 & z \neq x \end{cases}$$

and:

$$\tilde{y}(e)(z) = \begin{cases} 0 & \text{if } z = y \\ 1 & \text{if } z \neq y. \end{cases}$$

Then, f_E is an IVFSN of $\tilde{x}_E$, and g_E is an IVFSN of $\tilde{y}_E$, such that $f_E \neg\tilde{q} g_E$. Then, X is an IVFS q-T_2 space.

Theorem 8. *$IVFST(X, E, \tau)$ is an IVFSq-T_2 space if and only if for any $x \in X$, we have $\tilde{x}_E = \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN \text{ of } \tilde{x}_E\}$.*

Proof. Let (X, E, τ) be a crisp $IVFS$q-T_2 space and $\tilde{x}_E$ be an $IVFS$ point with support x, e lower value λ_e^-, and e upper value γ_e^+. Let y_E be a crisp $IVFS$ point with support y, e lower value γ_e^-, and e upper value λ_e^+. If $\tilde{x}_E$ and $\tilde{y}_E$ are two $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, γ_e^- and λ_e^+, γ_e^+, respectively, then there exist two $IVFS$-open sets f_E and g_E containing $IVFS$ points $\tilde{y}_E$ and $\tilde{x}_E$, respectively, such that $f_E \neg\tilde{q} g_E$. Then, g_E is an $IVFSN$ of $IVFS$ point $\tilde{x}_E$ and f_E is a $QIVFSN$ of $\tilde{y}_E$ such that $f_E \neg\tilde{q} g_E$. Hence, $\tilde{y}_E \notin Clg_E$. If $\tilde{x}_E$ and $\tilde{y}_E$ are two $IVFS$ points with the same supports $x = y$, then $\gamma_e^- > \lambda_e^-$and $\gamma_e^- > \lambda_e^+$. Thus, there are $QIVFSN$ f_E of $IVFS$ points $\tilde{y}_E$ and $IVFSN$ g_E such that $f_E \neg\tilde{q} g_E$. Hence, $\tilde{y}_E \notin Clg_E$.

Conversely, let $\tilde{x}_E$ and $\tilde{y}_E$ be two distinct $IVFS$ points with different supports x and y, e lower values, and e upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively. Since:

$$\tilde{x}_E = \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN \text{ of } \tilde{x}_E\}, \text{ and } \tilde{\bigwedge}\{Cl([f_e^-, f_e^+])(y) : f_E \in IVFSN \text{ of } \tilde{x}_E\} = 0.$$

Thus, $\tilde{y}_E \neg\tilde{q} \tilde{\bigwedge}\{Clf_E : f_E \in IVFSN \text{ of } \tilde{x}_E\}$. Therefore, there exists f_E that is an $IVFSN$ of $\tilde{x}$ and $\tilde{y}_E \neg\tilde{q} Clf_E$. Take two τ-$IVFS$-open sets f_E and $(Clf_E)^c$. Therefore, f_E is an $IVFSN$ of $IVFS$ point $\tilde{x}_E$, $(Clf_E)^c$ an $IVFSN$ of $IVFS$ point $\tilde{y}_E$, and $f_E \neg\tilde{q}(Clf_E)^c$. □

Definition 22. *Let (X, E, τ) be an IVFST. If for any IVFS point $\tilde{x}_E$ with support x, e lower values λ_e^-, and e upper values λ_e^+ and any IVFS-closed set f_E in X such that $\tilde{x}_E \neg\tilde{q} f_E$, there exists two IVFS-open sets h_E and k_E such that $\tilde{x}_E \tilde{\in} h_E$ and $f_E \tilde{\le} k_E, h_E \neg\tilde{q} k_E$, then (X, E, τ) is called an interval-valued fuzzy soft quasi regular space (IVFS q-regular space).*

(X, E, τ) is called an interval-valued fuzzy soft quasi-T_3 space, if it is an $IVFS$ q-regular space and an $IVFS$ q-T_1 space.

Theorem 9. *$IVFST(X, E, \tau)$ is an IVFS q-T_3 space if and only if for any IVFSN g_E of IVFS point $\tilde{x}_E$ there exists an IVFS-open set f_E in X such that $\tilde{x}_E \tilde{\in} f_E \tilde{\le} clf_E \tilde{\le} g_E$.*

Proof. Let g_E be an $IVFS$ set in X and $\tilde{x}_E$ be an $IVFS$ point with support x, e lower value λ_e^-, and e upper value λ_e^+ such that $\tilde{x}_E \tilde{\in} g_E$. Then, clearly, g_E^c is an $IVFS$-closed set. Since X is an $IVFS$ q-T_3 space, there exist two $IVFS$-open sets f_E, h_E such that $\tilde{x}_E \tilde{\in} f_E, g_E^c \tilde{\leq} h_E, h_E$ and $f_E \neg \tilde{q} h_E$. Thus, $f_E^c \tilde{\leq} h_E^c$. Therefore, $Clf_E \tilde{\leq} h_E^c$ implies $Clf_E \tilde{\leq} g_E$. Hence, $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} Clf_E \tilde{\leq} g_E$.

Conversely, let $\tilde{x}_E$ be an $IVFS$ point with different support x, e lower value λ_e^-, and e upper value λ_e^+, and let g_E be an $IVFS$-closed set such that $\tilde{x}_E \neg \tilde{q} g_E$. Then, g_E^c is an $IVFS$-open set containing the $IVFS$ point $\tilde{x}_E$, i.e., $\tilde{x}_E \tilde{\in} g_E^c$. Thus, there exists an $IVFS$-open set f_E containing $\tilde{x}_E$ such that $\tilde{x}_E \tilde{\in} f_E \tilde{\leq} Clf_E \tilde{\leq} g_E$ $g_E \tilde{\leq} (Clf_E)^c$. Therefore, clearly, $(Clf_E)^c$ is an $IVFS$-open set containing g_E and $f_E \neg \tilde{q} (Clf_E)^c$. Hence, X is an $IVFS$ q-T_3 space. □

Definition 23. *Let (X, E, τ) be an $IVFST$. If for any two IVFS-closed sets f_E and g_E such that $f_E \neg \tilde{q} g_E$, there exists two IVFS-open sets h_E and k_E such that $f_E \tilde{\leq} h_E$ and $g_E \tilde{\leq} k_E$, then (X, E, τ) is called an interval-valued fuzzy soft quasi-normal space ($IVFS$ q-normal space).*

(X, E, τ) is called an interval-valued fuzzy soft quasi T_4 space if it is an $IVFS$ q-normal space and an $IVFS$q-T_1 space.

Theorem 10. *$IVFST(X, E, \tau)$ is an $IVFS$ q-T_4 space if and only if for any $IVFS$-closed set f_E and $IVFS$-open set containing f_E, there exists an $IVFS$-open set h_E in X such that $f_E \tilde{\leq} h_E \tilde{\leq} clh_E \tilde{\leq} g_E$.*

Proof. Let f_E be an $IVFS$-closed set in X and g_E be an $IVFS$-open set in X containing f_E, i.e., $f_E \tilde{\leq} g_E$. Then, g_E^c is an $IVFS$-closed set such that $f_E \neg \tilde{q} g_E^c$.

Since X is an $IVFS$ q-T_4 space, there exist two $IVFS$-open sets h_E, k_E such that $f_E \tilde{\leq} h_E, g_E^c \tilde{\leq} k_E$, and $h_E \neg \tilde{q} k_E$. Thus, $h_E \tilde{\leq} k_E^c$, but $Clh_E \tilde{\leq} Clk_E^c = k_E$. Furthermore, $g_E^c \tilde{\leq} k_E$ implies $k^c \tilde{\leq} g_E$. That is an $IVFS$-closed set over X. Therefore, $Clh_E \tilde{\leq} k_E^c$. Hence, we have $f_E \tilde{\leq} h_E \tilde{\leq} Clh_E \tilde{\leq} g_E$.

Conversely, let $\tilde{f}_E$ and g_E be any $IVFS$-closed sets such that $f_E \neg \tilde{q} g_E$. Then, $f_E \tilde{\leq} g_E^c$. Thus, there exists an $IVFS$-open set h_E such that $f_E \tilde{\leq} h_E \tilde{\leq} Clh_E \tilde{\leq} g_E$. Therefore, there are two $IVFS$-open sets h_E and $(Clh_E)^c$ such that $f_E \tilde{\leq} h_E$, $g_E \tilde{\leq} (Clh_E)^c$. This shows that X is an $IVFS$ q-T_4 space. □

Theorem 11. *If $\Phi : (X_1, E_1, \tau_1) \to (X_2, E_2, \tau_2)$ is an IVFSC and IVFSO map where $\Phi_u X_1 \to X_2$ and $\Phi_p E_1 \to E_2$ are two ordinary bijections, then X_1 is an $IVFS$q-T_i space if and only if X_2 is an $IVFS$q-T_i space for $i = 0, 1, 2, 3, 4$.*

Proof. We just prove when $i = 2$. The other parts are similar.

Suppose that we have two $IVFS$ points $\tilde{k}_{E_2}$ and $\tilde{s}_{E_2}$ with different supports k and s, e lowers value, and e upper values λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively, for any $e \in E_2$. Then, the inverse lower and upper image of $IVFS$ point $\tilde{k}_{E_2}$ under the $IVFSO$ map Φ is an $IVFS$ point in X_1 with different support $\Phi^{-1}(k)$ as below:

$$\Phi^{-1}(\tilde{k}^-)(e)(x) = \tilde{k}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{k}^+)(e)(x) = \tilde{k}^+(\Phi_p(e))(\Phi_u(x)).$$

Furthermore, the inverse lower and upper image of $IVFS$ point $\tilde{s}_{E_2}$ under the $IVFSO$ map Φ is an $IVFS$ point in X_1 with different support $\Phi^{-1}(s)$ as below:

$$\Phi^{-1}(\tilde{s}^-)(e)(x) = \tilde{s}^-(\Phi_p(e))(\Phi_u(x)) \text{ and } \Phi^{-1}(\tilde{s}^+)(e)(x) = \tilde{s}^+(\Phi_p(e))(\Phi_u(x)).$$

Since (X_1, E_1, τ_1) is an $IVFS$q-T_2 space, there exist two $IVFS$-open sets f_E and g_E in X_1 such that $\Phi^{-1}(\tilde{k}_{E_2}) \tilde{\in} f_E$, $\Phi^{-1}(\tilde{s}_{E_2}) \tilde{\in} g_E$, and $f_E \neg \tilde{q} g_E$. Thus, $\tilde{k}_{E_2} \tilde{\in} f_E$ and $\tilde{s}_{E_2} \tilde{\in} g_E$, while $\Phi(f_E) \neg \tilde{q} \Phi(g_E)$. Therefore, (X_2, E_2, τ_2) is an $IVFS$q-T_2 space.

Conversely, suppose that we have two $IVFS$ points $\tilde{x}_E$ and $\tilde{y}_E$ with different supports $x, y \in X_1$, e lower value, and e upper value λ_e^-, λ_e^+ and γ_e^-, γ_e^+, respectively. Then, the lower and upper image of an $IVFS$ point $\tilde{x}_E$ under the $IVFSC$ map Φ is an $IVFS$ point in X_2 with different support $\Phi_u(x)$ as below:

$$\begin{aligned}\Phi(\tilde{x}^-)(\varepsilon)(k) &= \sup_{z\in\Phi^{-1}(k)}\Big[\sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{x}^-)(e)\Big](z)\\ &= \begin{cases}\lambda_e^- & \text{if } k=\Phi_u(x)\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

and:

$$\begin{aligned}\Phi(\tilde{x}^+)(\varepsilon)(k) &= \sup_{z\in\Phi^{-1}(k)}\Big[\sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{x}^+)(e)\Big](z)\\ &= \begin{cases}\lambda_e^+ & \text{if } k=\Phi_u(x)\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and the lower and upper image of an $IVFS$ point $\tilde{y}_E$ under the $IVFSC$ map Φ is an $IVFS$ point in X_2 with different support $\Phi_u(y)$ as below:

$$\begin{aligned}\Phi(\tilde{y}^-)(\varepsilon)(k) &= \sup_{z\in\Phi^{-1}(k)}\Big[\sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{y}^-)(e)\Big](z)\\ &= \begin{cases}\gamma_e^- & \text{if } k=\Phi_u(y)\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

and:

$$\begin{aligned}\Phi(\tilde{y}^+)(\varepsilon)(k) &= \sup_{z\in\Phi^{-1}(k)}\Big[\sup_{e\in\Phi_p^{-1}(\varepsilon)}(\tilde{y}^+)(e)\Big](z)\\ &= \begin{cases}\gamma_e^+ & \text{if } k=\Phi_u(y)\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since (X_2, E_2, τ_2) is an $IVFS$q-T_2 space, there exist two $IVFS$-open sets f_{E_2} and g_{E_2} in X_2 such that $\Phi(\tilde{x})\tilde{\in} f_{E_2}$, $\Phi(\tilde{y})\tilde{\in} g_{E_2}$, and $f_{E_2}\neg\tilde{q} g_{E_2}$. Clearly, $\tilde{x}_E\tilde{\in}\Phi^{-1}(f_{E_2}), \tilde{y}_E\tilde{\in}\Phi^{-1}(g_{E_2})$ and $\Phi^{-1}(f_{E_2})\neg\tilde{q}\Phi^{-1}(g_{E_2})$. Then, (X_1, E_1, τ_1) is an $IVFS$q-T_2 space. □

6. Conclusions

The aim of this study was to develop the interval-valued fuzzy soft separation axioms in order to build a framework that will provide a method for object ranking. Thus, in this paper, we introduced a new definition of the interval-valued fuzzy soft point and then considered some of its properties, and different types of neighborhoods of the $IVFS$ point were studied in interval-valued fuzzy soft topological spaces. The separation axioms of interval-valued fuzzy soft topological spaces were presented, and furthermore, the basic properties were also studied.

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