

Article

A Note on Extended Z-Contraction

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Abstract: In this article, we aim to evaluate and merge the as-scattered-as-possible results in fixed point theory from a general framework. In particular, we considered a common fixed point theorem via extended Z-contraction with respect to ψ -simulation function over an auxiliary function ζ in the setting of b -metric space. We investigated both the existence and uniqueness of common fixed points of such mappings. We used an example to illustrate the main result observed. Our main results cover several existing results in the corresponding literature.

Keywords: Z-contraction; simulation functions; b -metric

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1. Introduction and Preliminaries

The concept of fixed point first appeared in articles where solutions of differential equations were discussed, especially solutions of initial value problems. Among all such pioneer results in differential equations, we can mention and underline the renowned paper of Liouville [1], and the distinguished paper of Picard [2]. In these papers fixed point approaches were used implicitly under the name of the method of successive approximation. The first fixed point theorem, in the setting of complete normed space, which can be described as the abstraction of the method of the "successive approximation," was announced in Banach's thesis in 1922 [3]. Although not very accurate, the Banach fixed point theorem in most sources is given as follows: each contraction in a complete metric spaces admits a unique fixed point. Indeed, this is a characterization of Banach's original result, in the context of metric space, was given by Caccioppoli [4]. Therefore, this result would be more accurately called Banach-Caccioppoli. In addition to the magnificence of the expression of the theorem, its proof also has special significance. What makes this result very useful and interesting is that it not only guarantees the existence of the fixed point but also shows how to find this desired fixed point. Roughly speaking, for any contraction mapping $T : M \rightarrow M$ in a complete metric space (M, δ) , each recursive sequence $\{T^n p\}$ (for an arbitrarily chosen initial point $p \in M$) converges to $p^* \in M$ and this limit forms a unique fixed point for T .

On the other hand, a fixed problem can be considered a simple equation $Tp = p$. In almost all scientific disciplines, most of the problems can be converted into fixed point equations. This explains why fixed point theory has wide application capacity, as well as why fixed point theory has been hard work. As a result, too many results have been reported in this regard. Naturally, the emergence of so many results makes it very difficult to follow, process and functionalize these results. Under these circumstances, the best thing to do is to evaluate the current results as widely as possible and to combine these existing results as much as possible. Recently, for that purpose, some interesting papers started to

appear, such as, [5–19]. Among all these approaches, in this article, we focus on the notion of simulation function. The main idea of the simulation function is very simple, but also very useful and effective: For a self-mapping T on a metric space, contraction inequality $d(Tx, Ty) \leq kd(x, y)$ can be expressed as $0 \leq kd(x, y) - d(Tx, Ty) = \zeta(d(x, y), d(Tx, Ty))$, where $k \in [0, 1)$ and $\zeta : [0, \infty) \times [0, \infty) \rightarrow (-\infty, \infty)$. By letting $d(x, y) = u$ and $d(Tx, Ty) = r$, the corresponding simulation function for Banach’s fixed point theorem is $\zeta(u, r) = ku - r$. It is clear that for many other well-known results (Rakotch, Geraghty, Boyd-Wong, etc.), one can find a corresponding simulation function; see e.g., [5,7–12,20]. In other words, simulation function can be considered a generator of different contraction type inequalities.

Inspired from the results in [5], very recently, Joonaghany et al. [6] proposed a new notion, the ψ -simulation function, and with the help of it, the \mathcal{Z}_ψ -contraction in the setting of the standard metric space. The notion of the \mathcal{Z}_ψ -contraction covers several distinct types of contraction, including the \mathcal{Z} -contraction that was defined in [5].

From now on, we presume that \mathcal{B} and any considered sets are nonempty. Moreover, we shall fix the symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{R}_0^+, \mathbb{R}^+$ to indicate the set of positive integers, non-negative integers, real numbers, non-negative reals and positive reals.

$$\Psi := \{\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \psi \text{ is continuous and nondecreasing, and } \psi(r) = 0 \Leftrightarrow r = 0.\}$$

Definition 1 ([6]). We say that $\xi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is a ψ -simulation function, if there exists $\psi \in \Psi$ such that:

(ξ_1) $\xi(p, q) < \psi(q) - \psi(p)$ for all $p, q > 0$,

(ξ_2) if $\{p_n\}, \{q_n\} \subset \mathbb{R}^+$ so that

$$\lim_{n \rightarrow \infty} p_n = \lim_{n \rightarrow \infty} q_n > 0 \text{ implies } \lim_{n \rightarrow \infty} \sup \xi(t_n, s_n) < 0.$$

Let \mathcal{Z}_ψ be the set of all ψ -simulation functions. Note that if we take ψ as an identity mapping, then " ψ -simulation function" becomes "simulation function" in the sense of [5].

Example 1 ([6]). Let $\psi \in \Psi$.

(i) $\xi_1(p, q) = k\psi(q) - \psi(p)$ for all $p, q \in [0, \infty)$, where $k \in [0, 1)$.

(ii) $\xi_2(p, q) = \phi(\psi(q)) - \psi(p)$ for all $p, q \in [0, \infty)$, where ϕ is a self-mapping from $[0, +\infty)$ to $[0, +\infty)$ so that $\phi(0) = 0$ and for each $q > 0, \phi(q) < q$,

$$\limsup_{t \rightarrow q} \phi(p) < q.$$

(iii) $\xi_3(p, q) = \psi(q) - \phi(q) - \psi(p)$ for all $p, q \in [0, \infty)$, where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a mapping such that, for each $q > 0$,

$$\liminf_{p \rightarrow q} \phi(p) > 0.$$

It is clear that $\xi_1, \xi_2, \xi_3 \in \mathcal{Z}_\psi$.

Remark 1. Each simulation function forms a ψ -simulation function. The contrary of the statement is false [6].

Definition 2 (See [21,22]). Let $s \in [1, \infty)$ be a given real number and \mathcal{B} be a set. A function $d : \mathcal{B} \times \mathcal{B} \rightarrow \mathbb{R}_0^+$ is called b -metric with constant s , if the following are fulfilled:

1. $d(p, q) = d(q, p)$ (symmetric);
2. $p = q$ if and only if $d(p, q) = 0$ (zero self-distance);
3. $d(p, q) \leq s[d(p, r) + d(r, q)]$ (s -weighted triangle inequality)

for all $p, q, r \in \mathcal{B}$.

Throughout the paper, the triple (\mathcal{B}, d, s) denotes a b -metric space with $s \in [1, \infty)$. Further, the triple (\mathcal{B}^*, d, s) indicates that the corresponding b -metric space is complete.

Definition 3. For (\mathcal{B}, d, s) , we have

- A sequence $\{v_n\}$ in \mathcal{B} is called b -convergent if there exists $v \in \mathcal{B}$ such that $d(v_n, x) \rightarrow 0$ as $n \rightarrow \infty$. In this case, we write $\lim_{n \rightarrow \infty} v_n = v$.
- A sequence $\{v_n\}$ in \mathcal{B} is called b -Cauchy if $d(v_n, v_m) \rightarrow 0$ as $n, m \rightarrow \infty$.
- A b -metric space (\mathcal{B}, d) is said to be complete b -metric if every b -Cauchy sequence in \mathcal{B} is b -convergent.

Lemma 1 ([23]). Suppose a sequence $\{v_n\}$ in (\mathcal{B}, d, s) provides that $d(v_n, v_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{v_n\}$ is not a b -Cauchy sequence then there exists $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(v_{m_k}, v_{n_k}) \geq \varepsilon$, $d(v_{m_k}, v_{n_k-1}) < \varepsilon$ and

- $\varepsilon \leq \liminf_{k \rightarrow \infty} d(v_{m_k}, v_{n_k}) \leq \limsup_{k \rightarrow \infty} d(v_{m_k}, v_{n_k}) \leq s\varepsilon$
- $\frac{\varepsilon}{s} \leq \liminf_{k \rightarrow \infty} d(v_{m_k+1}, v_{n_k}) \leq \limsup_{k \rightarrow \infty} d(v_{m_k+1}, v_{n_k}) \leq s^2\varepsilon$
- $\frac{\varepsilon}{s} \leq \liminf_{k \rightarrow \infty} d(v_{m_k}, v_{n_k+1}) \leq \limsup_{k \rightarrow \infty} d(v_{m_k}, v_{n_k+1}) \leq s^2\varepsilon$
- $\frac{\varepsilon}{s^2} \leq \liminf_{k \rightarrow \infty} d(v_{m_k+1}, v_{n_k+1}) \leq \limsup_{k \rightarrow \infty} d(v_{m_k+1}, v_{n_k+1}) \leq s^3\varepsilon$.

The aim of this paper is to combine and unify several existing fixed point results by using the extended simulations function in the setting of b -metric spaces.

2. Main Results

Theorem 1. Let T, S be two self-mappings on (\mathcal{B}^*, d, s) Suppose that there exists $\xi \in \mathcal{Z}_\psi$ such that

$$\frac{1}{2s} \min \{d(v, Tv), d(\omega, S\omega)\} \leq d(v, \omega) \text{ implies } \xi \left(s^4 d(Tv, S\omega), M(v, \omega) \right) \geq 0, \tag{1}$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2s} \right\}.$$

Then T and S have a unique common fixed point, provided either T or S is continuous.

Proof. Let

$$K = \left\{ (v, \omega) \in \mathcal{B} \times \mathcal{B} : \frac{1}{2s} \min \{d(v, Tv), d(\omega, S\omega)\} \leq d(v, \omega) \right\}.$$

Since $(v, Tv) \in K$, for each $v \in \mathcal{B}$, then $K \neq \emptyset$.

At first, we choose any $v_0 \in \mathcal{B}$ and define the sequence $\{v_n\}$ by

$$v_{2n+1} = Tv_{2n} \text{ and } v_{2n+2} = Sv_{2n+1}.$$

Without loss of generality, we presume that successive terms in the recursive sequence $\{v_n\}$ are distinct. Indeed, if there exists $n_0 \in \mathbb{N}$ with $v_{2n_0} = v_{2n_0+1}$, then there is nothing to prove. More precisely, from our assumption, $v_{2n_0} = v_{2n_0+1} = Tv_{2n_0}$, we deduce that v_{2n_0} is a fixed point of T . In case of $M(v_{2n_0}, v_{2n_0+1}) = 0$, then we find $v_{2n_0+1} = v_{2n_0+2}$. Consequently, we have $v_{2n_0+1} = v_{2n_0+2} = Sv_{2n_0+1}$. So, v_{2n_0+1} is a fixed point of S . Since, we presumed that $v_{2n_0} = v_{2n_0+1}$, we conclude that $v_{2n_0} = v_{2n_0+1}$

is a common fixed point of S and T . On the other hand, to complete the discussion we need to address the other case: we suppose that $M(v_{2n_0}, v_{2n_0+1}) \neq 0$. Note that we get

$$\begin{aligned} \frac{1}{2s} \min \{d(v_{2n_0}, Tv_{2n_0}), d(v_{2n_0+1}, Sv_{2n_0+1})\} &= \frac{1}{2s} \min \{d(v_{2n_0}, v_{2n_0+1}), d(v_{2n_0+1}, v_{2n_0+2})\} \\ &\leq d(v_{2n_0}, v_{2n_0+1}), \end{aligned}$$

which implies that

$$\zeta \left(s^4 d(v_{2n_0+1}, v_{2n_0+2}), M(v_{2n_0}, v_{2n_0+1}) \right) \geq 0.$$

Taking the property $(\zeta 1)$ into account, we derive

$$\psi \left(s^4 d(v_{2n_0+1}, v_{2n_0+2}) \right) < \psi \left(M(v_{2n_0}, v_{2n_0+1}) \right).$$

On account of the definition of ψ , we deduce that

$$s^4 d(v_{2n_0+1}, v_{2n_0+2}) < \psi \left(M(v_{2n_0}, v_{2n_0+1}) \right) < M(v_{2n_0}, v_{2n_0+1}),$$

where

$$\begin{aligned} M(v_{2n_0}, v_{2n_0+1}) &= \max \left\{ d(v_{2n_0}, v_{2n_0+1}), d(v_{2n_0}, Tv_{2n_0}), d(v_{2n_0+1}, Sv_{2n_0+1}), \right. \\ &\quad \left. \frac{d(v_{2n_0}, Sv_{2n_0+1}) + d(v_{2n_0+1}, Tv_{2n_0})}{2s} \right\} \\ &= \max \left\{ d(v_{2n_0+1}, v_{2n_0+2}), \frac{d(v_{2n_0}, v_{2n_0+2})}{2s} \right\} \\ &= d(v_{2n_0+1}, v_{2n_0+2}), \end{aligned}$$

a contradiction.

Attendantly, we conclude that successive terms in the recursive sequence $\{v_n\}$ are distinct. Hence, $d(v_n, v_{n+1}) > 0$, and $M(v_n, v_{n+1}) \neq 0$ for all $n \in \mathbb{N}_0$,

Now, we subdivide the rest of the proof into four steps:

Step 1: We show that $\lim_{k \rightarrow \infty} d(v_k, v_{k+1}) = 0$. To prove that, let $k = 2n$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{2s} \min \{d(v_{2n}, Tv_{2n}), d(v_{2n+1}, Sv_{2n+1})\} &= \frac{1}{2s} \min \{d(v_{2n}, v_{2n+1}), d(v_{2n+1}, v_{2n+2})\} \\ &\leq d(v_{2n}, v_{2n+1}), \end{aligned}$$

which implies

$$\zeta \left(s^4 d(v_{2n+1}, v_{2n+2}), M(v_{2n}, v_{2n+1}) \right) \geq 0.$$

By $(\zeta 1)$, we have

$$\psi \left(s^4 d(v_{2n+1}, v_{2n+2}) \right) < \psi \left(M(v_{2n}, v_{2n+1}) \right).$$

Hence,

$$s^4 d(v_{2n+1}, v_{2n+2}) < M(v_{2n}, v_{2n+1}), \tag{2}$$

where,

$$\begin{aligned}
 M(v_{2n}, v_{2n+1}) &= \max \left\{ d(v_{2n}, v_{2n+1}), d(v_{2n}, Tv_{2n}), d(v_{2n+1}, Sv_{2n+1}), \right. \\
 &\quad \left. \frac{d(v_{2n}, Sv_{2n+1}) + d(v_{2n+1}, Tv_{2n})}{2s} \right\} \\
 &= \max \left\{ d(v_{2n}, v_{2n+1}), d(v_{2n+1}, v_{2n+2}), \frac{d(v_{2n}, v_{2n+2})}{2s} \right\} \\
 &\leq \max \left\{ d(v_{2n}, v_{2n+1}), d(v_{2n+1}, v_{2n+2}), \frac{s[d(v_{2n}, v_{2n+1}) + d(v_{2n+1}, v_{2n+2})]}{2s} \right\} \\
 &= \max \{ d(v_{2n}, v_{2n+1}), d(v_{2n+1}, v_{2n+2}) \}.
 \end{aligned}$$

If $d(v_{2n}, v_{2n+1}) < d(v_{2n+1}, v_{2n+2})$ for some $n \in \mathbb{N}$, then

$$M(v_{2n}, v_{2n+1}) \leq d(v_{2n+1}, v_{2n+2}),$$

which contradicts (2). Therefore, for each $n \in \mathbb{N}$,

$$d(v_{2n+1}, v_{2n+2}) < d(v_{2n}, v_{2n+1}),$$

and

$$M(v_{2n}, v_{2n+1}) \leq d(v_{2n}, v_{2n+1}).$$

Consequently, $d(v_{k+1}, v_{k+2}) < d(v_k, v_{k+1})$ for all even number $k \geq 0$. Also, we can prove the same argument for all odd number $k \geq 0$. Hence

$$d(v_{n+1}, v_{n+2}) < d(v_n, v_{n+1}) \quad \text{for all } n.$$

Therefore, the sequence $\{d(v_n, v_{n+1})\}$ is a non-increasing and bounded below. Then it is convergent and there exists a real number $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(v_n, v_{n+1}) = \lim_{n \rightarrow \infty} M(v_n, v_{n+1}) = r. \tag{3}$$

To prove that $r = 0$, suppose $r > 0$. For $n \geq 0$, we consider

$$\begin{aligned}
 \frac{1}{2s} \min \{ d(v_{2n}, Tv_{2n}), d(v_{2n+1}, Sv_{2n+1}) \} &= \frac{1}{2s} \min \{ d(v_{2n}, v_{2n+1}), d(v_{2n+1}, v_{2n+2}) \} \\
 &\leq d(v_{2n}, v_{2n+1}).
 \end{aligned}$$

Thus $(v_{2n}, v_{2n+1}) \in K$ for each $n \geq 0$. Moreover, we have

$$\xi \left(s^4 d(v_{2n+1}, v_{2n+2}), M(v_{2n}, v_{2n+1}) \right) \geq 0.$$

So,

$$\limsup_{n \rightarrow \infty} \xi \left(s^4 d(v_{2n+1}, v_{2n+2}), M(v_{2n}, v_{2n+1}) \right) \geq 0.$$

From (3), we have,

$$\lim_{n \rightarrow \infty} d(v_{2n+1}, v_{2n+2}) = \lim_{n \rightarrow \infty} M(v_{2n}, v_{2n+1}) = r > 0.$$

By using (ξ 2) with $t_n = d(v_{2n+1}, v_{2n+2})$ and $s_n = M(v_{2n}, v_{2n+1})$, we have

$$\limsup_{n \rightarrow \infty} \xi \left(s^4 d(v_{2n+1}, v_{2n+2}), M(v_{2n}, v_{2n+1}) \right) < 0,$$

which is a contradiction. Therefore, the claim is proven; i.e.,

$$\lim_{n \rightarrow \infty} d(v_n, v_{n+1}) = \lim_{n \rightarrow \infty} M(v_n, v_{n+1}) = 0. \tag{4}$$

Step 2: The step is to show the obtained recursive sequence $\{v_n\}$ forms a b -Cauchy. By (4) it is sufficient to show that the subsequence $\{v_{2n}\}$ is a b -Cauchy sequence in X .

On contrary, suppose that the sequence $\{v_{2n}\}$ does not form a b -Cauchy. Attendantly, there are $\varepsilon > 0$ and sequences of integers $\{2m_k\}$ and $\{2n_k\}$ with $n_k > m_k \geq k \geq 1$ such that

$$d(v_{2m_k}, v_{2n_k}) \geq \varepsilon \text{ and } d(v_{2m_k}, v_{2n_k-2}) < \varepsilon,$$

and

$$\begin{aligned} \varepsilon &\leq \liminf_{k \rightarrow \infty} d(v_{2m_k}, v_{2n_k}) \leq \limsup_{k \rightarrow \infty} d(v_{2m_k}, v_{2n_k}) \leq s\varepsilon \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} d(v_{2m_k+1}, v_{2n_k}) \leq \limsup_{k \rightarrow \infty} d(v_{2m_k+1}, v_{2n_k}) \leq s^2\varepsilon \\ \frac{\varepsilon}{s} &\leq \liminf_{k \rightarrow \infty} d(v_{2m_k}, v_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} d(v_{2m_k}, v_{2n_k+1}) \leq s^2\varepsilon \\ \frac{\varepsilon}{s^2} &\leq \liminf_{k \rightarrow \infty} d(v_{2m_k+1}, v_{2n_k+1}) \leq \limsup_{k \rightarrow \infty} d(v_{2m_k+1}, v_{2n_k+1}) \leq s^3\varepsilon. \end{aligned}$$

Now, from the definition of $M(v, \omega)$, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} M(v_{2n_k}, v_{2m_k-1}) &= \lim_{k \rightarrow \infty} \max \left\{ d(v_{2n_k}, v_{2m_k-1}), d(v_{2n_k}, Tv_{2n_k}), d(v_{2m_k-1}, Sv_{2m_k-1}), \right. \\ &\quad \left. \frac{d(v_{2n_k}, Sv_{2m_k-1}) + d(v_{2m_k-1}, Tv_{2n_k})}{2s} \right\} \\ &= \lim_{k \rightarrow \infty} \max \left\{ d(v_{2n_k}, v_{2m_k-1}), d(v_{2n_k}, v_{2n_k+1}) + d(v_{2m_k-1}, v_{2m_k}), \right. \\ &\quad \left. \frac{d(v_{2n_k}, v_{2m_k}) + d(v_{2m_k-1}, v_{2n_k+1})}{2s} \right\} \\ &= \max \left\{ s^2\varepsilon, 0, 0, \frac{s\varepsilon + s^3\varepsilon}{2s} \right\} \\ &= s^2\varepsilon. \end{aligned}$$

On the other hand, by taking k sufficiently large with $n_k > m_k > k$ and since $\{d(v_n, v_{n+1})\}$ is a non-increasing, we have

$$\begin{aligned} d(v_{2n_k}, Tv_{2n_k}) &= d(v_{2n_k}, v_{2n_k+1}) \\ &\leq d(v_{2m_k}, v_{2m_k+1}) \\ &\leq d(v_{2m_k-1}, v_{2m_k}) \\ &= d(v_{2m_k-1}, Sv_{2m_k-1}). \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{2s} \min \{d(v_{2n_k}, Tv_{2n_k}), d(v_{2m_k-1}, Sv_{2m_k-1})\} &= \frac{1}{2s} d(v_{2n_k}, Tv_{2n_k}) \\ &= \frac{1}{2s} d(v_{2n_k}, v_{2n_k+1}). \end{aligned} \tag{5}$$

By using (4), there exists $k_1 \in \mathbb{N}$ such that for any $k > k_1$,

$$d(v_{2n_k}, v_{2n_k+1}) < \frac{1}{2s}\epsilon.$$

Also, there exists $k_2 \in \mathbb{N}$ such that for any $k > k_2$,

$$d(v_{2m_k-1}, v_{2m_k}) < \frac{1}{2s}\epsilon.$$

Thus, for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$, we have

$$\begin{aligned} \epsilon &\leq d(v_{2m_k}, v_{2n_k}) \\ &\leq s [d(v_{2m_k}, v_{2m_k-1}) + d(v_{2m_k-1}, v_{2n_k})] \\ &\leq \frac{\epsilon}{2} + sd(v_{2m_k-1}, v_{2n_k}), \end{aligned}$$

which implies that

$$\frac{\epsilon}{2} \leq d(v_{2m_k-1}, v_{2n_k}).$$

Thus we obtain that for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$,

$$\begin{aligned} d(v_{2n_k}, v_{2n_k+1}) &< \frac{1}{2s}\epsilon \\ &< \frac{1}{2}\epsilon \\ &\leq d(v_{2m_k-1}, v_{2n_k}). \end{aligned}$$

Hence, from (5), we have

$$\begin{aligned} \frac{1}{2s} \min \{d(v_{2n_k}, Tv_{2n_k}), d(v_{2m_k-1}, Sv_{2m_k-1})\} &= \frac{1}{2s}d(v_{2n_k}, v_{2n_k+1}) \\ &\leq d(v_{2m_k-1}, v_{2n_k}). \end{aligned}$$

Therefore, by (11) and (ξ_1)

$$\begin{aligned} 0 &\leq \zeta \left(s^4d(Tv_{2n_k}, Sv_{2m_k-1}), M(v_{2n_k}, v_{2m_k-1}) \right) \\ &< \psi \left(M(v_{2n_k}, v_{2m_k-1}) \right) - \psi \left(s^4d(Tv_{2n_k}, Sv_{2m_k-1}) \right), \end{aligned}$$

implies

$$\psi \left(s^4d(Tv_{2n_k}, Sv_{2m_k-1}) \right) < \psi \left(M(v_{2n_k}, v_{2m_k-1}) \right).$$

Since ψ is non-decreasing, then

$$s^4d(Tv_{2n_k}, Sv_{2m_k-1}) < M(v_{2n_k}, v_{2m_k-1}).$$

So

$$\limsup_{k \rightarrow \infty} s^4d(v_{2n_k+1}, v_{2m_k}) < \limsup_{k \rightarrow \infty} M(v_{2n_k}, v_{2m_k-1}),$$

and then $s^4(s^2\epsilon) < s^2\epsilon$, which is a contradiction. Therefore, $\{v_n\}$ is a b -Cauchy sequence in X . Since \mathcal{B} is b -complete, $\{v_n\}$ is b -convergent to some point $u \in \mathcal{B}$.

Step 3: In this step, we shall indicate that u is a common fixed point of T and S . Notice that

$$\begin{aligned} d(u, Su) &\leq M(v_{2n}, u) \\ &= \max\{d(v_{2n}, u), (v_{2n}, v_{2n+1}), d(u, Su), \frac{d(v_{2n}, Su) + d(u, v_{2n+1})}{2s}\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\begin{aligned} d(u, Su) &\leq \lim_{n \rightarrow \infty} M(v_{2n}, u) \\ &\leq \max\left\{0, 0, d(u, Su), \frac{d(u, Su) + 0}{2s}\right\} \\ &= d(u, Su). \end{aligned}$$

Thus

$$\lim_{n \rightarrow \infty} M(u, v_{2n}) = d(Su, u). \tag{6}$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} M(u, v_{2n+1}) = d(Tu, u). \tag{7}$$

Now, we claim that for each $n \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2s}d(v_{2n}, v_{2n+1}) \leq d(v_{2n}, u), \tag{8}$$

or

$$\frac{1}{2s}d(v_{2n+1}, v_{2n+2}) \leq d(v_{2n+1}, u). \tag{9}$$

If we suppose for some $n_0 \geq 0$ that both (8) and (9) are false, then we get

$$\begin{aligned} d(v_{2n_0}, v_{2n_0+1}) &\leq s [d(v_{2n_0}, u) + d(u, v_{2n_0+1})] \\ &< s \left[\frac{1}{2s}d(v_{2n_0}, v_{2n_0+1}) + \frac{1}{2s}d(v_{2n_0+1}, v_{2n_0+2}) \right] \\ &= \frac{1}{2}d(v_{2n_0}, v_{2n_0+1}) + \frac{1}{2}d(v_{2n_0+1}, v_{2n_0+2}) \\ &\leq \frac{1}{2}d(v_{2n_0}, v_{2n_0+1}) + \frac{1}{2}d(v_{2n_0}, v_{2n_0+1}) \\ &= d(v_{2n_0}, v_{2n_0+1}), \end{aligned}$$

a contradiction. So the claim is proven. Now, we shall examine the following two cases:

Case 1: The inequality (8) is true for infinitely many $n \geq 0$. In this case, we have

$$\begin{aligned} \frac{1}{2s} \min\{d(v_{2n}, Tv_{2n}), d(u, Su)\} &= \frac{1}{2s} \min\{d(v_{2n}, v_{2n+1}), d(u, Su)\} \\ &\leq \frac{1}{2s}d(v_{2n}, v_{2n+1}) \\ &\leq d(v_{2n}, u), \end{aligned}$$

which implies that

$$\xi(s^4d(Tv_{2n}, Su), M(v_{2n}, u)) \geq 0.$$

Thus,

$$\limsup_{k \rightarrow \infty} \xi(s^4d(Tv_{2n}, Su), M(v_{2n}, u)) \geq 0.$$

Now, we prove that $d(Su, u) = 0$. On the contrary, suppose that $d(Su, u) \neq 0$. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} d(Tv_{2n}, Su) &= \lim_{n \rightarrow \infty} M(v_{2n}, u) \\ &= d(Su, u) \\ &> 0, \end{aligned}$$

by (ξ2) and Remark 2, we have

$$\limsup_{k \rightarrow \infty} \left(s^4 d(Tv_{2n}, Su), M(v_{2n}, u) \right) < 0,$$

a contradiction. Hence, $d(Su, u) = 0$; i.e., $Su = u$.

On the other hand,

$$\begin{aligned} M(u, u) &= \max \left\{ d(u, u), d(u, Tu), d(u, Su), \frac{d(u, Su) + d(u, Tu)}{2s} \right\} \\ &= \max \left\{ 0, d(u, Tu), 0, \frac{d(u, Tu)}{2s} \right\} \\ &= d(u, Tu). \end{aligned}$$

Then,

$$M(u, u) = d(u, Tu). \tag{10}$$

Moreover, suppose $d(u, Tu) > 0$. We have

$$\begin{aligned} \frac{1}{2s} \min \{ d(u, Tu), d(u, Su) \} &= \frac{1}{2s} \min \{ d(u, Tu), 0 \} \\ &= 0 \\ &\leq d(u, u). \end{aligned}$$

The above implies, with (11)

$$\xi \left(s^4 d(Tu, Su), M(u, u) \right) \geq 0.$$

By (ξ1),

$$s^4 d(Tu, Su) < M(u, u);$$

i.e.,

$$s^4 d(Tu, u) < M(u, u),$$

which contradicts (10). Then, $d(u, Tu) = 0$; i.e., $Tu = u$. Hence $Tu = Su = u$.

The inequality (8) is true only for infinitely many $n \geq 0$. In this case, there is $n_0 \geq 0$ such that inequality (9) is true for any $n \geq n_0$. Similarly to Case 1, we can prove that $Tu = Su = u$. Therefore, by Case 1 and Case 2, u is a common fixed point of T and S .

Step 4. We prove that u is a unique common fixed point of S and T . Let u and v be two common fixed points of S and T such that $d(u, v) > 0$. We have

$$0 = \frac{1}{2s} \min \{ d(u, Tu), d(v, Sv) \} < d(u, v).$$

This implies

$$\xi \left(s^4 d(Tu, Sv), M(u, v) \right) \geq 0,$$

where $M(u, v) \neq 0$ since $d(u, v) \neq 0$. By $(\zeta 1)$, we have

$$s^4 d(Tu, Sv) < M(u, v);$$

i.e.,

$$s^4 d(u, v) < M(u, v).$$

But

$$\begin{aligned} M(u, v) &= \max \left\{ d(u, v), d(u, Tu), d(v, Sv), \frac{d(u, Sv) + d(v, Tu)}{2s} \right\} \\ &= \max \left\{ d(u, v), \frac{1}{s} d(u, v) \right\} \\ &= d(u, v). \end{aligned}$$

Hence

$$s^4 d(u, v) < d(u, v),$$

which is a contradiction. We conclude that $d(u, v) = 0$; i.e., $u = v$ and the theorem is proven. \square

Putting $S = T$ in Theorem 1 we have:

Corollary 1. Let T, S be two self-mappings on (\mathcal{B}^*, d, s) Suppose that there exists $\zeta \in \mathcal{Z}_\psi$ such that

$$\frac{1}{2s} d(v, Tv) \leq d(v, \omega)$$

implies that

$$\zeta \left(s^4 d(Tv, T\omega), M(v, \omega) \right) \geq 0,$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2s} \right\}.$$

Then T has a unique fixed point, provided that T is b -continuous.

Corollary 2. Let T, S be two self-mappings on (\mathcal{B}^*, d, s) Suppose that there exists $\zeta \in \mathcal{Z}_\psi$ such that

$$\zeta \left(s^4 d(Tv, S\omega), M(v, \omega) \right) \geq 0,$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2s} \right\}.$$

Then T and S have a unique common fixed point, provided either T or S is continuous.

Immediate Consequences in the Standard Metric Space

In this part, the pairs (\mathcal{M}, d) and (\mathcal{M}^*, d) denote the metric space and complete metric spaces, respectively.

Theorem 2. Let T, S be two self-mappings on (\mathcal{M}^*, d) Suppose that there exists $\zeta \in \mathcal{Z}_\psi$ such that

$$\frac{1}{2} \min \{d(v, Tv), d(\omega, S\omega)\} \leq d(v, \omega) \text{ implies } \zeta \left(d(Tv, S\omega), M(v, \omega) \right) \geq 0, \tag{11}$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2} \right\}.$$

Then T and S have a unique common fixed point, provided either T or S is continuous.

Putting $S = T$ in Theorem 1 we have:

Corollary 3. Let T, S be two self-mappings on (\mathcal{M}^*, d) Suppose that there exists $\xi \in \mathcal{Z}_\psi$ such that

$$\frac{1}{2s} d(v, Tv) \leq d(v, \omega)$$

implies that

$$\xi \left(s^4 d(Tv, T\omega), M(v, \omega) \right) \geq 0,$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2s} \right\}.$$

Then T has a unique fixed point, provided that T is b -continuous.

Corollary 4. Let T, S be two self-mappings on (\mathcal{M}^*, d) Suppose that there exists $\xi \in \mathcal{Z}_\psi$ such that

$$\xi \left(d(Tv, S\omega), M(v, \omega) \right) \geq 0,$$

for all $v, \omega \in \mathcal{B}$, where

$$M(v, \omega) = \max \left\{ d(v, \omega), d(v, Tv), d(\omega, S\omega), \frac{d(v, S\omega) + d(\omega, Tv)}{2} \right\}.$$

Then T and S have a unique common fixed point, provided either T or S is continuous.

Remark 2. It is evident that we can list more corollaries from the our main results in several aspects: For example, by substituting the example of ψ -simulation function, letting the ψ be an identity and using the example of simulation function introduced in [7–9,11,12].

Example 2. Let $\mathcal{B} = [0, \infty)$ and let $d : X \times X \rightarrow \mathbb{R}_0^+$ be defined by $d(v, \omega) = |v - \omega|^2$. Hence $(\mathcal{B}, d, \frac{3}{2})$ forms a complete b -metric. We define $T, S : \mathcal{B} \rightarrow \mathcal{B}$ by

$$Tv = \begin{cases} (\frac{v}{2})^2, & v \in [0, 1) \\ \frac{3v}{4} + \frac{1}{4}, & v \in [1, \infty) \end{cases}$$

and

$$S\omega = \begin{cases} (\frac{\omega}{2})^2, & y \in [0, 1) \\ \frac{\omega}{4} + \frac{1}{4}, & \omega \in [1, \infty) \end{cases}.$$

We now verify the inequality (11). For this purpose we define $\xi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$ by choosing $\xi(p, q) = q - p$, $\psi(t) = \frac{t}{2}$, $s = \frac{3}{2}$. Now we have the following cases.

Case 1: $v, \omega \in [0, 1)$

In this case $Tv = (\frac{v}{2})^2, S\omega = (\frac{\omega}{2})^2$, then we have

$$\begin{aligned} \zeta \left(s^4 d(Tv, S\omega), M(v, \omega) \right) &\leq \zeta \left(s^4 d(Tv, S\omega), d(v, \omega) \right) \\ &= \psi(d(v, \omega)) - \psi(s^4(d(Tv, S\omega))) \\ &= \frac{d(v, \omega)}{2} - s^4 \frac{d(Tv, S\omega)}{2} \\ &= \frac{d(v, \omega)}{2} - \frac{1}{2} \left(\frac{3}{2} \right)^4 d \left(\left(\frac{v}{2} \right)^2, \left(\frac{\omega}{2} \right)^2 \right) \\ &= \frac{1}{2} |v - \omega|^2 - \frac{81}{32} \left| \left(\frac{v}{2} \right)^2 - \left(\frac{\omega}{2} \right)^2 \right|^2 \\ &= \frac{1}{2} |v - \omega|^2 - \frac{81}{512} |v - \omega|^2 |x + y|^2 \\ &= \frac{1}{2} |v - \omega|^2 \left(1 - \frac{81}{256} |x + y|^2 \right) \geq 0. \end{aligned}$$

Case 2: $v, \omega \in [1, \infty)$

In this case $Tx = \frac{3v}{4} + \frac{1}{4}, Sy = \frac{\omega}{4} + \frac{1}{4}$, then we have

$$\begin{aligned} \zeta \left(s^4 d(Tv, S\omega), M(v, \omega) \right) &\leq \zeta \left(s^4 d(Tv, S\omega), d(v, \omega) \right) \\ &= \psi(d(v, \omega)) - \psi(s^4(d(Tv, S\omega))) \\ &= \frac{d(v, \omega)}{2} - s^4 \frac{d(Tv, S\omega)}{2} \\ &= \frac{d(v, \omega)}{2} - \frac{1}{2} \left(\frac{3}{2} \right)^4 d \left(\frac{3v}{4} + \frac{1}{4}, \frac{\omega}{4} + \frac{1}{4} \right) \\ &= \frac{1}{2} |v - \omega|^2 - \frac{81}{32} \left| \frac{3v}{4} + \frac{1}{4} - \frac{\omega}{4} - \frac{1}{4} \right|^2 \\ &= \frac{1}{2} |v - \omega|^2 - \frac{81}{512} |3v - \omega|^2 \\ &\geq \frac{1}{2} |v - \omega|^2 - \frac{81}{512} |v - \omega|^2 \\ &= \frac{1}{2} |v - \omega|^2 \left(1 - \frac{81}{256} \right) \geq 0. \end{aligned}$$

Hence, it is readily verified that ζ is a Ψ -simulation function where ψ is the identity function on $[0, \infty)$ and all the hypothesis of Theorem 1 are verified.

3. Nomenclature

The symbols $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{R}_0^+, \mathbb{R}^+$ indicate the set of positive integers, non-negative integers, real numbers, non-negative reals and positive reals.

$$\Psi := \{ \psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+ \mid \psi \text{ is continuous and nondecreasing, and } \psi(r) = 0 \Leftrightarrow r = 0. \}$$

\mathcal{B} always show the non-empty set.

The triple (\mathcal{B}, d, s) denotes a b -metric space with $s \in [1, \infty)$.

Further, the triple (\mathcal{B}^*, d, s) indicates that the corresponding b -metric space is complete.

4. Conclusions

In this paper, we conclude that several existing fixed point results can be unified and merged by using the extended simulation function, in the framework of b -metric spaces. The main result produces

several consequences by considering distinct extended simulation functions. Further, all observed results can be derived in the framework of standard metric space, by let $s = 1$. Notice also that our results covers several existing results, such as the results in [5,7,18]. Regarding the richness of the simulation function, as it is done in [19], one can derive several well-known fixed point results from our main theorem. Indeed, explicitly writing the consequences/corollaries of our main result cannot easily fit on several pages.

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