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A Novel Decay Rate for a Coupled System of Nonlinear Viscoelastic Wave Equations

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Abstract: The main goal of the present paper is to study the existence, uniqueness and behavior of a solution for a coupled system of nonlinear viscoelastic wave equations with the presence of weak and strong damping terms. Owing to the Faedo-Galerkin method combined with the contraction mapping theorem, we established a local existence in $[0, T]$. The local solution was made global in time by using appropriate a priori energy estimates. The key to obtaining a novel decay rate is the convexity of the function χ, under the special condition of the initial energy $E(0)$. The condition of the weights of weak and strong damping has a fundamental role in the proof. The existence of both three different damping mechanisms and strong nonlinear sources make the paper very interesting from a mathematics point of view, especially when it comes to unbounded spaces such as $\mathbb{R}^n$.

Keywords: viscoelastic wave equation; coupled system; global solution; Faedo–Galerkin approximation; decay rate

1. Introduction

In this paper we investigate the coupled system

$$\begin{cases} \left(|u_t|^{\kappa-2}u_t\right)_t + au_t = \Theta(x)\Delta\left(u + \omega u_t - \int_0^t \varpi_1(t-\tau)u(\tau)\,d\tau\right) + f_1(u,v), \\ \left(|v_t|^{\kappa-2}v_t\right)_t + av_t = \Theta(x)\Delta\left(v + \omega v_t - \int_0^t \varpi_2(t-\tau)v(\tau)\,d\tau\right) + f_2(u,v), \\ \\ u(x,0) = u_0(x), v(x,0) = v_0(x), \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{1}$$

where $t > 0, x \in \mathbb{R}^n, a \in \mathbb{R}$, $\omega > 0$, and $n \geq 3, \kappa \geq 2$; and the functions $f_j : \mathbb{R}^2 \to \mathbb{R}, j = 1,2$, are given by

$$f_1(\xi_1, \xi_2) = (p+1)\left[|\xi_1 + \xi_2|^{(p-1)}(\xi_1 + \xi_2) + |\xi_1|^{(p-3)/2}\xi_1|\xi_2|^{(p+1)/2}\right], \tag{2}$$

and

$$f_2(\xi_1, \xi_2) = (p+1)\left[|\xi_1 + \xi_2|^{(p-1)}(\xi_1 + \xi_2) + |\xi_2|^{(p-3)/2}\xi_2|\xi_1|^{(p+1)/2}\right], \tag{3}$$

respectively, $p > 3$. The viscoelastic term is $\int_0^t \varpi_j(t-s)u(s)\,ds$. The function $\Theta(x) > 0$, $x \in \mathbb{R}^n$, is a density and $(\Theta)^{-1}(x) = 1/\Theta(x) \equiv \theta(x)$, satisfies

$$\theta \in L^\tau(\mathbb{R}^n) \quad \text{with} \quad \tau = \frac{2n}{2n - rn + 2r} \quad \text{for} \quad 2 \leq r \leq \frac{2n}{n-2}. \tag{4}$$

There exists a function $\mathcal{F} \in C^1(\mathbb{R}^2, \mathbb{R})$ such that

$$\xi_1 f_1(\xi_1, \xi_2) + \xi_2 f_2(\xi_1, \xi_2) = (p+1)\mathcal{F}(\xi_1, \xi_2),\ \forall(\xi_1, \xi_2) \in \mathbb{R}^2, \tag{5}$$

and

$$(p+1)\mathcal{F}(\xi_1, \xi_2) = |\xi_1 + \xi_2|^{p+1} + 2|\xi_1 \xi_2|^{(p+1)/2}, \tag{6}$$

where $f_1 = \frac{\partial \mathcal{F}}{\partial u}, f_2 = \frac{\partial \mathcal{F}}{\partial v}$. For more details see [1–3].

From qualitative and quantitative point of view, we recall some previous works regarding the nonlinear coupled systems of wave equations. We start with the single wave equation treated in [4], where the following problem is considered

$$\begin{cases} u_{tt} + \mu u_t - \Delta u - \omega \Delta u_t = u \ln |u|,\ (x,t) \in \Omega \times (0, \infty) \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x). \end{cases} \tag{7}$$

$x \in \Omega$, where Ω is bounded domain of $\mathbb{R}^n,\ n \geq 1$ with smooth boundary $\partial\Omega$. The authors proved the local and global existence of a weak solution by using the contraction mapping principle, and they established a decay rate and infinite time blow up of solution with condition on initial energy.

In the case of unbounded domain, we mention the paper recently published by T. Miyasita and Kh. Zennir in [5], where the following equation is studied:

$$u_{tt} + a u_t - \phi(x)\Delta\left(u + \omega u_t - \int_0^t g(t-s)u(s)\,ds\right) = u|u|^{p-1}, \tag{8}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x). \end{cases} \tag{9}$$

The authors established the existence of local and global solution. For results related to decay rate of solutions for this type of problems, we refer the reader to [6–10] and references therein.

B. Feng et al. considered in [1] a coupled system for viscoelastic wave equations with nonlinear sources in a bounded domain with smooth boundary

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + u_t = f_1(u,v) \\ \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + v_t = f_2(u,v). \end{cases} \tag{10}$$

The authors discussed (10) in $\mathbb{R}^n (n = 1,2,3)$. Under appropriate hypotheses, they established a general decay result by multiplication techniques to extend some existing results for a single equation to case of coupled system.

The IBVP for system of nonlinear viscoelastic wave equations in bounded domain was considered in the problem

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds + (|u|^k + |v|^q)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)\,ds + (|v|^\theta + |u|^\rho)|v_t|^{r-1}v_t = f_2(u,v), \quad t > 0, \quad x \in \Omega, \\ u(x,t) = v(x,t) = 0, x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x) \\ u_t(x,0) = u_1(x), v_t(x,0) = v_1(x), \end{cases} \tag{11}$$

where Ω is bounded domain with smooth boundary. In [11], under certain conditions on the kernel functions, degenerate damping and nonlinear source terms, a decay rate of the energy function with some initial data is established.

Concerning the nonexistence of solutions for more degenerate cases of a coupled system of wave equations with different damping, we refer the reader to the papers [2,3,12,13] and references therein.

The novelty of our work lies primarily in the use of a new condition between the weights of weak and strong damping to get an estimates in Lemma 3, which is useful in the calculation, where we outlined the effects of damping terms. The constant λ_1 being the first eigenvalue of the operator—Δ. We also proposed a more general nonlinearities in sources and used classical arguments (Holder, Young and Minkowski's inequalities) to estimate them. The more complected case was considered, where we took a nonlinearity in the second derivative in time (first term in both equations) to get a more general case and obtain the first derivative in time for the variable in very large spaces ($\|u_t\|_{L^{\kappa}_{\theta}(\mathbb{R}^n)}, \|v_t\|_{L^{\kappa}_{\theta}(\mathbb{R}^n)}$, this result is not classical). These nonlinearities make the problem very interesting from the application point of view. In order to compensate the lack of classical Poincaré's inequality in $\mathbb{R}^n$, we used the weighted function to use the generalized poincaré's one.

The main contribution is located in Theorem 3, where we obtained a novel decay rate under a very general assumption of the kernels (related with a convex function) by taking in account two kernels. Of course, we were obliged to show the global existence of unique solution.

2. Preliminaries

We define the function spaces $\mathcal{H}$ as the closure of $C_0^{\infty}(\mathbb{R}^n)$ as follows:

$$\mathcal{H} = \{v \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \mid \nabla v \in L^2((\mathbb{R}^n))^n\},$$

with respect to the norm $\|v\|_{\mathcal{H}} = (v,v)_{\mathcal{H}}^{1/2}$ and the inner product

$$(v,w)_{\mathcal{H}} = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx.$$

The space $L^2_{\theta}(\mathbb{R}^n)$ is endowed with the norm $\|v\|_{L^2_{\theta}} = (v,v)_{L^2_{\theta}}^{1/2}$ for

$$(v,w)_{L^2_{\theta}} = \int_{\mathbb{R}^n} \theta v w \, dx.$$

For $r \in [1,+\infty)$, denote by

$$\|v\|_{L^r_{\theta}} = \left(\int_{\mathbb{R}^n} \theta \, |v|^r \, dx \right)^{\frac{1}{r}},$$

the norm of weighted space $L^r_{\theta}(\mathbb{R}^n)$.

We introduce a useful Sobolev embedding and generalized Poincaré's inequalities.

Lemma 1 ([5]). *Let θ satisfy (4). For positive constants $C_{\tau}, C_P > 0$ depending only on θ and n, we have*

$$\|v\|_{\frac{2n}{n-2}} \leq C_{\tau} \, \|v\|_{\mathcal{H}},$$

and

$$\|v\|_{L^2_{\theta}} \leq C_P \, \|v\|_{\mathcal{H}},$$

for $v \in \mathcal{H}$.

Lemma 2 ([14]). *Let θ satisfy (4). Then the estimates*

$$\|v\|_{L^r_{\theta}} \leq C_r \, \|v\|_{\mathcal{H}},$$

and

$$C_r = C_\tau \, \|\theta\|_\tau^{\frac{1}{r}},$$

hold for $v \in \mathcal{H}$. *Here* $\tau = 2n/(2n - rn + 2r)$ *for* $1 \le r \le 2n/(n-2)$.

We assume that the kernel functions $\varpi_1, \varpi_2 \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$1 - \overline{\varpi_1} = l > 0 \quad \text{for} \quad \overline{\varpi_1} = \int_0^{+\infty} \varpi_1(\tau)\, d\tau,\ \varpi_1'(t) \le 0, \tag{12}$$

$$1 - \overline{\varpi_2} = m > 0 \quad \text{for} \quad \overline{\varpi_2} = \int_0^{+\infty} \varpi_2(\tau)\, d\tau,\ \varpi_2'(t) \le 0. \tag{13}$$

With $\mathbb{R}^+$, we denote the set $\{\tau \mid \tau \ge 0\}$. Denote

$$\mu(t) = \max_{t \ge 0} \Big\{ \varpi_1(t), \varpi_2(t) \Big\}, \tag{14}$$

and

$$\mu_0(t) = \min_{t \ge 0} \Big\{ \int_0^t \varpi_1(\tau) d\tau, \int_0^t \varpi_2(\tau) d\tau \Big\}. \tag{15}$$

We assume that there is a convex function $\chi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\varpi_i'(t) + \chi(\varpi_i(t)) \le 0, \quad \chi(0) = 0, \quad \chi'(0) > 0 \quad \text{and} \quad \chi''(\xi) \ge 0, i = 1, 2, \tag{16}$$

for any $\xi \ge 0$.

Holder and Young's inequalities give

$$\begin{aligned} \|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} &\le \left(\|u\|_{L_\theta^{(p+1)}}^2 + \|v\|_{L_\theta^{(p+1)}}^2 \right)^{(p+1)/2} \\ &\le \left(l\|u\|_{\mathcal{H}}^2 + m\|v\|_{\mathcal{H}}^2 \right)^{(p+1)/2}. \end{aligned}$$

Thanks to Minkowski's inequality, we have

$$\begin{aligned} \|u+v\|_{L_\theta^{(p+1)}}^{(p+1)} &\le 2^{(p+1)/2} \left(\|u\|_{L_\theta^{(p+1)}}^2 + \|v\|_{L_\theta^{(p+1)}}^2 \right)^{(p+1)/2} \\ &\le c \left(\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 \right)^{(p+1)/2}. \end{aligned}$$

Then there exists $\eta > 0$ such that

$$\|u+v\|_{L_\theta^{(p+1)}}^{(p+1)} + 2\, \|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} \le \eta \left(l\|u\|_{\mathcal{H}}^2 + m\|v\|_{\mathcal{H}}^2 \right)^{(p+1)/2}. \tag{17}$$

Define positive constants λ_0 and $\mathcal{E}_0$ by

$$\lambda_0 \equiv \eta^{-1/(p-1)} \quad \text{and} \quad \mathcal{E}_0 = \Big(\frac{1}{2} - \frac{1}{p+1} \Big) \eta^{-2/(p-1)}. \tag{18}$$

With $\{(\lambda_i, e_i)\}_{i \in \mathbb{N}} \subset \mathbb{R} \times \mathcal{H}$ we denote an eigenpair of

$$-\Theta(x) \Delta e_i = \lambda_i e_i \quad x \in \mathbb{R}^n,$$

for any $i \in \mathbb{N}$. Then

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_i \le \cdots \uparrow +\infty,$$

holds and $\{e_i\}$ is a complete orthonormal system in $\mathcal{H}$.
(A1). The exponent p satisfies

$$1 < p \le \frac{n+2}{n-2}, \quad n \ge 3. \tag{19}$$

(A2). Assume that the weights of damping terms satisfy

$$a + \lambda_1 \omega > 0. \tag{20}$$

(A3). Assume that there exists a constant γ such that

$$\gamma = \eta\Big(\frac{2(p+1)}{p-1}\mathcal{E}(0)\Big)^{(p-1)/2} < 1. \tag{21}$$

Remark 1. *The constant λ_1, introduced in (20), is the first eigenvalue of the operator—Δ.*

We introduce an inner product as follows:

$$(v, w)_* = \omega \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx + a \int_{\mathbb{R}^n} \theta v w \, dx.$$

Then the associated norm is given by

$$\|v\|_* = \sqrt{(v, v)_*}, \quad v, w \in \mathcal{H}.$$

By (20), we get

$$(v, v)_* = \omega \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + a \int_{\mathbb{R}^n} \theta v^2 \, dx \ge (\omega\lambda_1 + a) \int_{\mathbb{R}^n} \theta v^2 \, dx \ge 0.$$

Lemma 3. *Let θ satisfy (4). Under the condition (20), we have*

$$\sqrt{\omega}\, \|v\|_{\mathcal{H}} \le \|v\|_* \le \sqrt{\omega + C_P^2}\, \|v\|_{\mathcal{H}}, \quad v \in \mathcal{H}.$$

Definition 1. *The pair (u, v) is said to be a weak solution to (1) on $[0, T]$ if it satisfies, for $x \in \mathbb{R}^n$,*

$$\begin{cases} \int_{\mathbb{R}^n} \left(|u_t|^{\kappa-2} u_t\right)_t \varphi dx + a \int_{\mathbb{R}^n} u_t \varphi dx = \int_{\mathbb{R}^n} \Theta(x) \Delta \left(u + \omega u_t - \int_0^t \varpi_1(t-\tau) u(\tau)\, d\tau\right) \varphi dx \\ \qquad + \int_{\mathbb{R}^n} f_1(u, v) \varphi dx, \\ \int_{\mathbb{R}^n} \left(|v_t|^{\kappa-2} v_t\right)_t \psi dx + a \int_{\mathbb{R}^n} v_t \psi dx = \int_{\mathbb{R}^n} \Theta(x) \Delta \left(v + \omega v_t - \int_0^t \varpi_2(t-\tau) v(\tau)\, d\tau\right) \psi dx \\ \qquad + \int_{\mathbb{R}^n} f_2(u, v) \psi dx, \end{cases} \tag{22}$$

for all test functions $\varphi, \psi \in \mathcal{H}, t \in [0, T]$.

3. Main Results

We start this section with the local existence result.

Theorem 1. *(Local existence.) Assume that (A1) and (A2) hold. Let $(u_0, v_0) \in \mathcal{H}^2$ and $(u_1, v_1) \in L^{\kappa}_{\theta}(\mathbb{R}^n) \times L^{\kappa}_{\theta}(\mathbb{R}^n)$. Under the assumptions (4)–(6) and (12)–(16), the problem (1) admits a unique local solution (u, v) such that*

$$(u, v) \in \mathcal{X}_T^2,\ \mathcal{X}_T \equiv C\big([0, T]; \mathcal{H}\big) \cap C^1\big([0, T]; L^{\kappa}_{\theta}(\mathbb{R}^n)\big),$$

for sufficiently small $T > 0$.

In order to establish the global solution, we introduce a potential energy $J : \mathcal{H} \to \mathbb{R}$ as follows

$$J(u, v) = \left(1 - \int_0^t \varpi_1(\tau)\, d\tau\right) \|u\|^2_{\mathcal{H}} + (\varpi_1 \circ u) + \left(1 - \int_0^t \varpi_2(\tau)\, d\tau\right) \|v\|^2_{\mathcal{H}} + (\varpi_2 \circ v)\,. \tag{23}$$

The modified energy is defined by

$$\mathcal{E}(t) = \frac{\kappa - 1}{\kappa}\left(\|u_t\|^{\kappa}_{L^{\kappa}_{\theta}} + \|v_t\|^{\kappa}_{L^{\kappa}_{\theta}} \right) + \frac{1}{2} J(u, v) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u, v) dx, \tag{24}$$

where

$$\left(\varpi_j \circ w\right)(t) = \int_0^t \varpi_j(t - s)\, \|w(t) - w(s)\|^2_{\mathcal{H}}\, ds,$$

for any $w \in L^2(\mathbb{R}^n)$.

Theorem 2. *(Global existence.) Assume that (A1) and (A2) hold. Under (4)–(6) and (12)–(16) and for sufficiently small $(u_0, u_1), (v_0, v_1) \in \mathcal{H} \times L^{\kappa}_{\theta}(\mathbb{R}^n)$, the problem (1) admits a unique global solution (u, v) such that*

$$(u, v) \in \mathcal{X}^2,\ \mathcal{X} \equiv C\big([0, +\infty); \mathcal{H}\big) \cap C^1\big([0, +\infty); L^{\kappa}_{\theta}(\mathbb{R}^n)\big). \tag{25}$$

The nonclassical decay rate for the solution is given in the next Theorem.

Theorem 3. *(Decay of solution.) Assume that (A1), (A2), and (A3) hold. Under (4)–(6) and (12)–(16), there exists $t_0 > 0$, depending on ϖ_1, ϖ_2, a, ω, λ_1 and $H'(0)$, such that*

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(- \int_{t_0}^t \frac{\mu(\tau)}{1 - \mu_0(t)} d\tau \right) d\tau, \tag{26}$$

for all $t \geq t_0$.

In particular, by the positivity of μ in (14), we have, as in [15],

$$0 \leq \mathcal{E}(t) < \mathcal{E}(t_0) \exp\left(- \int_{t_0}^t \mu(\tau)\, d\tau \right),$$

for a single wave equation. The next Lemma will play an important role in the sequel.

Lemma 4. *For $(u, v) \in \mathcal{X}_T^2$, the functional $\mathcal{E}(t)$ associated with the problem (1) is non-increasing.*

Proof. For $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned} &\mathcal{E}(t_2) - \mathcal{E}(t_1) \\ &= \int_{t_1}^{t_2} \frac{d}{dt} E(t)\, dt \\ &= -\int_{t_1}^{t_2} \left(a \|u_t\|^2_{L^2_{\theta}} + \omega \|u_t\|^2_{\mathcal{H}} + \frac{1}{2}\varpi_1(t) \|u\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi_1' \circ u) \right) dt \end{aligned}$$

$$
\begin{aligned}
& - \int_{t_1}^{t_2} \left(a \|v_t\|_{L_\theta^2}^2 + \omega \|v_t\|_{\mathcal{H}}^2 + \frac{1}{2}\varpi_2(t) \|v\|_{\mathcal{H}}^2 - \frac{1}{2}(\varpi_2' \circ v) \right) dt \\
\leq \; & 0.
\end{aligned}
$$

This completes the proof. □

4. Proofs

We outline the proof for the local existence of a solution by a standard procedure (See [6,10]).

Proof. *(Of Theorem 1.)* Let $(u_0, u_1), (v_0, v_1) \in \mathcal{H} \times L_\theta^\kappa(\mathbb{R}^n)$. For any $(u, v) \in \mathcal{X}_T^2$, we can obtain weak solution of related system

$$
\begin{cases}
\left(|z_t|^{\kappa-2} z_t\right)_t + a z_t - \Theta(x)\Delta\left(z + \omega z_t\right) = -\Theta(x)\Delta \int_0^t \varpi_1(t-\tau) u(\tau)\, d\tau + f_1(u, v) \\
\left(|y_t|^{\kappa-2} y_t\right)_t + a y_t - \Theta(x)\Delta\left(y + \omega y_t\right) = -\Theta(x)\Delta \int_0^t \varpi_2(t-\tau) v(\tau)\, d\tau + f_2(u, v) \\
z(x, 0) = u_0(x), y(x, 0) = v_0(x) \\
z_t(x, 0) = u_1(x), y_t(x, 0) = v_1(x).
\end{cases}
\tag{27}
$$

We reduce the problem (27) to the Cauchy problem for a system of ODE by using the Faedo-Galerkin approximations. Then we find a solution map $\top : (u, v) \mapsto (z, y)$ from $\mathcal{X}_T^2$ to $\mathcal{X}_T^2$. We then show that $\top$ is a contraction mapping in an appropriate subset of $\mathcal{X}_T^2$ for a small $T > 0$. Hence, $\top$ has a fixed point (u, v). This gives a unique solution in $\mathcal{X}_T^2$, which completes the proof. □

We are now ready to show the global solution. By using conditions on ϖ_1, ϖ_2, we obtain

$$
\begin{aligned}
\mathcal{E}(t) \;\; & \geq \;\; \frac{1}{2} J(u, v) - \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u, v) dx \\
& \geq \;\; \frac{1}{2} J(u, v) - \frac{1}{p+1} \|u + v\|_{L_\theta^{(p+1)}}^{(p+1)} - \frac{2}{p+1} \|uv\|_{L_\theta^{(p+1)/2}}^{(p+1)/2} \\
& \geq \;\; \frac{1}{2} J(u, v) - \frac{\eta}{p+1} \left[l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 \right]^{(p+1)/2} \\
& \geq \;\; \frac{1}{2} J(u, v) - \frac{\eta}{p+1} \Big(J(u, v) \Big)^{(p+1)/2} \\
& = \;\; G(\beta),
\end{aligned}
\tag{28}
$$

where $\beta^2 = J(u, v)$, $t \in [0, T)$ and

$$
G(\xi) = \frac{1}{2}\xi^2 - \frac{\eta}{p+1}\xi^{(p+1)}.
$$

Note that $\mathcal{E}_0 = G(\lambda_0)$ is given in (18). Then

$$
\begin{cases}
G(\xi) > 0 & in \;\; \xi \in [0, \lambda_0] \\
G(\xi) < 0 & in \;\; \xi > \lambda_0.
\end{cases}
\tag{29}
$$

Moreover, $\lim\limits_{\xi \to +\infty} G(\xi) \to -\infty$.

Lemma 5. *Let* $0 \leq \mathcal{E}(0) < \mathcal{E}_0$.
(i) If $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 < \lambda_0^2$*, then the local solution of (1) satisfies*

$$
J(u, v) < \lambda_0^2, \; \forall t \in [0, T).
$$

(ii) If $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 > \lambda_0^2$*, then the local solution of (1) satisfies*

$$\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 > \lambda_1^2,\ \forall t \in [0,T), \lambda_1 > \lambda_0.$$

Proof. Since $0 \le \mathcal{E}(0) < \mathcal{E}_0 = G(\lambda_0)$, there exist ξ_1 and ξ_2 such that $G(\xi_1) = G(\xi_2) = \mathcal{E}(0)$ with $0 < \xi_1 < \lambda_0 < \xi_2$.
The case (i)**.** By (28), we have

$$G(J(u_0, v_0)) \le \mathcal{E}(0) = G(\xi_1),$$

which implies that $J(u_0, v_0) \le \xi_1^2$. Then we claim that $J(u,v) \le \xi_1^2$, $\forall t \in [0,T)$. Moreover, there exists $t_0 \in (0,T)$ such that

$$\xi_1^2 < J(u(t_0), v(t_0)) < \xi_2^2.$$

Then

$$G(J(u(t_0), v(t_0))) > \mathcal{E}(0) \ge \mathcal{E}(t_0),$$

which contradicts with (28). Hence, we have

$$J(u,v) \le \xi_1^2 < \lambda_0^2,\ \forall t \in [0,T).$$

The case (ii)**.** We can now show that $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 \ge \xi_2^2$ and $\|u\|_{\mathcal{H}}^2 + \|v\|_{\mathcal{H}}^2 \ge \xi_2^2 > \lambda_0^2$ in the same way as (i). □

Proof. (*Of Theorem 2.*) Let $(u_0, u_1), (v_0, v_1) \in \mathcal{H} \times L_\theta^\kappa(\mathbb{R}^n)$ satisfy $0 \le \mathcal{E}(0) < \mathcal{E}_0$ and $\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 < \lambda_0^2$. By Lemma 4 and Lemma 5, we have

$$\begin{aligned}
&\frac{2(\kappa-1)}{\kappa}\left(\|u_t\|_{L_\theta^\kappa}^\kappa + \|v_t\|_{L_\theta^\kappa}^\kappa\right) + l\,\|u\|_{\mathcal{H}}^2 + m\,\|v\|_{\mathcal{H}}^2 \\
&\le \frac{2(\kappa-1)}{\kappa}\left(\|u_t\|_{L_\theta^\kappa}^\kappa + \|v_t\|_{L_\theta^\kappa}^\kappa\right) + \left(1 - \int_0^t \varpi_1(\tau)\,d\tau\right)\|u\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) \\
&+ \left(1 - \int_0^t \varpi_2(\tau)\,d\tau\right)\|u\|_{\mathcal{H}}^2 + (\varpi_2 \circ v) \\
&\le 2\mathcal{E}(t) + \frac{2\eta}{p+1}\left[l\,\|u\|_{\mathcal{H}}^2 + m\,\|u\|_{\mathcal{H}}^2\right]^{(p+1)/2} \\
&\le 2\mathcal{E}(0) + \frac{2\eta}{p+1}\Big(J(u,v)\Big)^{(p+1)/2} \\
&\le 2\mathcal{E}_0 + \frac{2\eta}{p+1}\lambda_0^{p+1} \\
&= \eta^{-2/(p-1)}.
\end{aligned} \tag{30}$$

This completes the proof. □

Let

$$\begin{aligned}
\Lambda(u,v) &= \frac{1}{2}\left(1 - \int_0^t \varpi_1(\tau)\,d\tau\right)\|u\|_{\mathcal{H}}^2 + \frac{1}{2}(\varpi_1 \circ u) \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_2(\tau)\,d\tau\right)\|v\|_{\mathcal{H}}^2 + \frac{1}{2}(\varpi_2 \circ v) - \int_{\mathbb{R}^n} \theta(x)\mathcal{F}(u,v)dx,
\end{aligned} \tag{31}$$

$$\Pi(u,v) = \left(1 - \int_0^t \varpi_1(\tau)\,d\tau\right)\|u\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) \tag{32}$$

$$+ \quad \left(1 - \int_0^t \varpi_2(\tau)\,d\tau\right) \|v\|_{\mathcal{H}}^2 + (\varpi_2 \circ v) - (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u,v) dx.$$

Lemma 6. *Let (u, v) be the solution of problem (1) and*

$$\|u_0\|_{\mathcal{H}}^2 + \|v_0\|_{\mathcal{H}}^2 - (p+1) \int_{\mathbb{R}^n} \theta(x) \mathcal{F}(u_0, v_0) dx > 0. \tag{33}$$

Then, under condition (21), we have $\Pi(u,v) > 0,\ \forall t > 0.$

Proof. By (33) and the continuity, it follows that there exists a time $t_1 > 0$ such that

$$\Pi(u,v) \geq 0, \forall t < t_1.$$

Let

$$Y = \{(u,v) \mid \Pi(u(t_0), v(t_0)) = 0,\ \Pi(u,v) > 0, \forall t \in [0, t_0)\}. \tag{34}$$

Then, by (31), (32), we have, for all $(u, v) \in Y$,

$$\begin{aligned}
\Lambda(u,v) &= \frac{p-1}{2(p+1)} \left[\left(1 - \int_0^t \varpi_1(\tau)\,d\tau\right) \|u\|_{\mathcal{H}}^2 + \left(1 - \int_0^t \varpi_2(\tau)\,d\tau\right) \|v\|_{\mathcal{H}}^2 \right] \\
&+ \frac{p-1}{2(p+1)} \left[(\varpi_1 \circ u) + (\varpi_2 \circ v) \right] + \frac{1}{p+1} \Pi(u,v) \\
&\geq \frac{p-1}{2(p+1)} \left[l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 + (\varpi_1 \circ u) + (\varpi_2 \circ v) \right].
\end{aligned}$$

Owing to (24), it follows, for $(u, v) \in Y$,

$$l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \Lambda(u,v) \leq \frac{2(p+1)}{p-1} \mathcal{E}(t) \leq \frac{2(p+1)}{p-1} \mathcal{E}(0). \tag{35}$$

By (17) and (21), we have

$$\begin{aligned}
(p+1) \int_{\mathbb{R}^n} \mathcal{F}(u(t_0), v(t_0)) &\leq \eta \left(l \|u(t_0)\|_{\mathcal{H}}^2 + m \|v(t_0)\|_{\mathcal{H}}^2 \right)^{(p+1)/2} \\
&\leq \eta \left(\frac{2(p+1)}{p-1} E(0) \right)^{(p-1)/2} (l \|u(t_0)\|_{\mathcal{H}}^2 + m \|v(t_0)\|_{\mathcal{H}}^2) \\
&\leq \gamma (l \|u(t_0)\|_{\mathcal{H}}^2 + m \|v(t_0)\|_{\mathcal{H}}^2) \\
&< \left(1 - \int_0^{t_0} \varpi_1(\tau) d\tau\right) \|u(t_0)\|_{\mathcal{H}}^2 + \left(1 - \int_0^{t_0} \varpi_2(\tau) d\tau\right) \|v(t_0)\|_{\mathcal{H}}^2 \\
&< \left(1 - \int_0^{t_0} \varpi_1(\tau) d\tau\right) \|u(t_0)\|_{\mathcal{H}}^2 + \left(1 - \int_0^{t_0} \varpi_2(\tau) d\tau\right) \|v(t_0)\|_{\mathcal{H}}^2 \\
&+ (\varpi_1 \circ u) + (\varpi_2 \circ v).
\end{aligned} \tag{36}$$

Hence, $\Pi(u(t_0), v(t_0)) > 0$ on Y. This contradicts the definition of Y since $\Pi(u(t_0), v(t_0)) = 0$. Thus, $\Pi(u,v) > 0,\ \forall t > 0.$ □

We are now ready to prove the decay rate.

Proof. (*Of Theorem 3.*) By (17) and (35), we have, for $t \geq 0$,

$$0 < l \|u\|_{\mathcal{H}}^2 + m \|v\|_{\mathcal{H}}^2 \leq \frac{2(p+1)}{p-1} \mathcal{E}(t). \tag{37}$$

Let

$$I(t) = \frac{\mu(t)}{1-\mu_0(t)},$$

where μ and μ_0 are defined in (14) and (15), respectively.

Note that $\lim_{t\to+\infty} \mu(t) = 0$. Then, we have

$$\lim_{t\to+\infty} I(t) = 0, \ \ I(t) > 0, \ \forall t \geq 0.$$

Take $t_0 > 0$ such that

$$0 < \frac{2(\kappa-1)}{\kappa} I(t) < \min\left\{2\left(\omega\lambda_1 + a\right), \chi'(0)\right\},$$

with (16), for all $t > t_0$. Due to (24), we have

$$\begin{aligned}
\mathcal{E}(t) &\leq \frac{(\kappa-1)}{\kappa}\left(\|u_t\|^{\kappa}_{L^{\kappa}_{\theta}} + \|v_t\|^{\kappa}_{L^{\kappa}_{\theta}} \right) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v)] \\
&+ \frac{1}{2}\left(1 - \int_0^t \varpi_1(\tau)\,d\tau\right) \|u\|^2_{\mathcal{H}} + \frac{1}{2}\left(1 - \int_0^t \varpi_2(\tau)\,d\tau\right) \|v\|^2_{\mathcal{H}}, \\
&\leq \frac{(\kappa-1)}{\kappa}\left(\|u_t\|^{\kappa}_{L^{\kappa}_{\theta}} + \|v_t\|^{\kappa}_{L^{\kappa}_{\theta}} \right) + \frac{1}{2}[(\varpi_1 \circ u) + (\varpi_2 \circ v)] \\
&+ \frac{1}{2}(1-\mu_0(t))[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}].
\end{aligned}$$

Then, by the definition of $I(t)$, we have

$$\begin{aligned}
I(t)\mathcal{E}(t) &\leq \frac{(\kappa-1)}{\kappa} I(t)\left(\|u_t\|^{\kappa}_{L^{\kappa}_{\theta}} + \|v_t\|^{\kappa}_{L^{\kappa}_{\theta}} \right) + \frac{1}{2}\mu(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}] \\
&+ \frac{1}{2} I(t)[(\varpi_1 \circ u) + (\varpi_2 \circ v)].
\end{aligned} \tag{38}$$

By Lemma 4, we have, for all $t_1, t_2 \geq 0$,

$$\begin{aligned}
&\mathcal{E}(t_2) - \mathcal{E}(t_1) \\
&\leq -\int_{t_1}^{t_2} \left(a\,\|u_t\|^2_{L^2_{\theta}} + \omega\,\|u_t\|^2_{\mathcal{H}} + \frac{1}{2}\mu(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}] \right) dt \\
&- \int_{t_1}^{t_2} \left(a\,\|v_t\|^2_{L^2_{\theta}} + \omega\,\|v_t\|^2_{\mathcal{H}} - \frac{1}{2}(\varpi_1' \circ u) - \frac{1}{2}(\varpi_2' \circ v) \right) dt.
\end{aligned}$$

Then, by generalized Poincaré's inequalities, we get

$$\begin{aligned}
\mathcal{E}'(t) &\leq -\left(\omega\lambda_1 + a\right)[\|u_t\|^2_{L^2_{\theta}} + \|v_t\|^2_{L^2_{\theta}}] \\
&- \frac{1}{2}\mu(t)[\|u\|^2_{\mathcal{H}} + \|v\|^2_{\mathcal{H}}] + \frac{1}{2}[(\varpi_1' \circ u) + (\varpi_2' \circ v)].
\end{aligned}$$

Finally, $\forall t \geq t_0$, we have

$$\begin{aligned}
&\mathcal{E}'(t) + I(t)\mathcal{E}(t) \\
&\leq \left\{ \frac{(\kappa-1)}{\kappa} I(t) - (\omega\lambda_1 + a) \right\} \left(\|u_t\|^2_{L^2_{\theta}} + \|v_t\|^2_{L^2_{\theta}} \right) \\
&+ \frac{1}{2}(\varpi_1' \circ u) + \frac{1}{2}(\varpi_2' \circ v) + \frac{1}{2} I(t)((\varpi_1 \circ u) + (\varpi_2 \circ v))
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_0^t \left\{\varpi_1'(t-\tau)+I(t)\varpi_1(t-\tau)\right\}\|u(t)-u(\tau)\|_{\mathcal{H}}^2\,d\tau \\
&+ \frac{1}{2}\int_0^t \left\{\varpi_2'(t-\tau)+I(t)\varpi_2(t-\tau)\right\}\|v(t)-v(\tau)\|_{\mathcal{H}}^2\,d\tau \\
&\leq \frac{1}{2}\int_0^t \left\{\varpi_1'(\tau)+I(t)\varpi_1(\tau)\right\}\|u(t)-u(t-\tau)\|_{\mathcal{H}}^2\,d\tau \\
&+ \frac{1}{2}\int_0^t \left\{\varpi_2'(\tau)+I(t)\varpi_2(\tau)\right\}\|v(t)-v(t-\tau)\|_{\mathcal{H}}^2\,d\tau \\
&\leq \frac{1}{2}\int_0^t \left\{-\chi\Big(\varpi_1(\tau)\Big)+\chi'(0)\varpi_1(\tau)\right\}\|u(t)-u(t-\tau)\|_{\mathcal{H}}^2\,d\tau \\
&+ \frac{1}{2}\int_0^t \left\{-\chi\Big(\varpi_2(\tau)\Big)+\chi'(0)\varpi_2(\tau)\right\}\|v(t)-v(t-\tau)\|_{\mathcal{H}}^2\,d\tau. \\
&\leq 0,
\end{aligned}
$$

By the convexity of χ and (16), we get

$$\chi(\xi) \geq \chi(0) + \chi'(0)\xi = \chi'(0)\xi.$$

Then

$$\mathcal{E}(t) \leq \mathcal{E}(t_0)\exp\left(-\int_{t_0}^t I(s)ds\right).$$

This completes the proof. □

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