mathematics

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A Note on the Topological Transversality Theorem for Weakly Upper Semicontinuous, Weakly Compact Maps on Locally Convex Topological Vector Spaces

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Received: 8 January 2020; Accepted: 18 February 2020; Published: 25 February 2020

Abstract: A simple theorem is presented that automatically generates the topological transversality theorem and Leray–Schauder alternatives for weakly upper semicontinuous, weakly compact maps. An application is given to illustrate our results.

Keywords: weakly upper semicontinuous; essential maps; homotopy

1. Introduction

Many problems arising in natural phenomena give rise to problems of the form $x \in F\,x$, for some map F. In applications for a complicated F, the intent is to attempt to relate it to a simpler (and solvable) problem $x \in G\,x$, where the map G is homotopic (in an appropriate way) to F, and then to hopefully deduce that $x \in F\,x$ is solvable. This approach was initiated by Leray and Schauder and extended to a very general formulation in, for example, [1,2]. The goal, to begin with, is to consider a class of maps that arise in applications and then to present the notion of homotopy for the class of maps that are fixed point free on the boundary of the considered set.

In this paper we consider weakly upper semicontinuous, weakly compact maps F and G, with $F \cong G$. We present the topological transversality theorem, which states that F is essential if, and only if, G is essential. The proof is based on a new result (Theorem 1) for weakly upper semicontinuous, weakly compact maps. Our topological transversality theorem will then immediately generate Leray–Schauder type alternatives (see Theorem 4 and Corollary 1). In addition, we note that these results are useful from an application viewpoint (see Theorem 5).

2. Topological Transversality Theorem

Let X be a Hausdorff locally convex topological vector space and U be a weakly open subset of C, where C is a closed convex subset of X. First we present the class of maps, M, that we will consider in this paper.

Definition 1. *We say* $F \in M(\overline{U^w}, C)$ *if* $F : \overline{U^w} \to K(C)$ *is a weakly upper semicontinuous, weakly compact map; here* $\overline{U^w}$ *denotes the weak closure of* U *in* C *and* $K(C)$ *denotes the family of nonempty, convex, weakly compact subsets of* C*.*

Definition 2. *We say* $F \in M_{\partial U}(\overline{U^w}, C)$ *if* $F \in M(\overline{U^w}, C)$ *and* $x \notin F(x)$ *for* $x \in \partial U$*; here* ∂U *denotes the weak boundary of* U *in* C*.*

Now we present the notion of homotopy for the class of maps, M, with the fixed point free on the boundary.

Definition 3. *Let $F, G \in M_{\partial U}(\overline{U^w}, C)$. We write $F \cong G$ in $M_{\partial U}(\overline{U^w}, C)$ if there exists a weakly upper semicontinuous, weakly compact map $\Psi : \overline{U^w} \times [0,1] \to K(C)$ with $x \notin \Psi_t(x)$ for $x \in \partial U$ and $t \in (0,1)$ (here $\Psi_t(x) = \Psi(x,t)$), $\Psi_0 = F$ and $\Psi_1 = G$.*

Definition 4. *Let $F \in M_{\partial U}(\overline{U^w}, C)$. We say that F is essential in $M_{\partial U}(\overline{U^w}, C)$ if, for every map $J \in M_{\partial U}(\overline{U^w}, C)$ with $J|_{\partial U} = F|_{\partial U}$, there exists a $x \in U$ with $x \in J(x)$.*

We present a simple theorem that will immediately yield the so called topological transversality theorem (motivated from [1]) for weakly upper semicontinuous, weakly compact maps (see Theorem 2). The topological transversality theorem essentially states that if a map F is essential and $F \cong G$ then the map G is essential (and so in particular has a fixed point).

Theorem 1. *Let X be a Hausdorff locally convex topological vector space, U be a weakly open subset of C, C be a closed convex subset of X, $F \in M_{\partial U}(\overline{U^w}, C)$ and $G \in M_{\partial U}(\overline{U^w}, C)$ is essential in $M_{\partial U}(\overline{U^w}, C)$. Also suppose*

$$\begin{cases} \text{for any map } J \in M_{\partial U}(\overline{U^w}, C) \text{ with } J|_{\partial U} = F|_{\partial U} \\ \text{we have } G \cong J \text{ in } M_{\partial U}(\overline{U^w}, C). \end{cases} \tag{1}$$

Then F is essential in $M_{\partial U}(\overline{U^w}, C)$.

Proof. Let $J \in M_{\partial U}(\overline{U^w}, C)$ with $J|_{\partial U} = F|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Let $H^J : \overline{U^w} \times [0,1] \to K(C)$ be a weakly upper semicontinuous, weakly compact map with $x \notin H^J_t(x)$ for any $x \in \partial U$ and $t \in (0,1)$ (here $H^J_t(x) = H^J(x,t)$), $H^J_0 = G$ and $H^J_1 = J$ (this is guaranteed from (2.1)). Let

$$\Omega = \left\{ x \in \overline{U^w} : \ x \in H^J(x,t) \ \text{ for some } \ t \in [0,1] \right\}$$

and

$$D = \left\{ (x,t) \in \overline{U^w} \times [0,1] : \ x \in H^J(x,t) \right\}.$$

Now recall that $X = (X, w)$, the space X endowed with the weak topology, is completely regular. First, $D \neq \emptyset$ (note G is essential in $M_{\partial U}(\overline{U^w}, C)$) and D is weakly closed (note H^J is weakly upper semicontinuous) and so D is weakly compact (note H^J is a weakly compact map). Let $\pi : \overline{U^w} \times [0,1] \to \overline{U^w}$ be the projection. Now $\Omega = \pi(D)$ is weakly closed (see Kuratowski's theorem ([3] p. 126)) and so in fact weakly compact. Also note that $\Omega \cap \partial U = \emptyset$ (since $x \notin H^J_t(x)$ for any $x \in \partial U$ and $t \in [0,1]$). Thus there exists a weakly continuous map $\mu : \overline{U^w} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(\Omega) = 1$. We define the map R by $R(x) = H^J(x, \mu(x)) = H^J \circ g(x)$, where $g : \overline{U^w} \to \overline{U^w} \times [0,1]$ is given by $g(x) = (x, \mu(x))$. Note that $R \in M_{\partial U}(\overline{U^w}, C)$ with $R|_{\partial U} = G|_{\partial U}$ (note, if $x \in \partial U$, then $R(x) = H^J(x,0) = G(x)$) so the essentiality of G guarantees a $x \in U$ with $x \in R(x)$ i.e., $x \in H^J_{\mu(x)}(x)$). Thus $x \in \Omega$ so $\mu(x) = 1$ and as a result $x \in H^J_1(x) = J(x)$. $\square$

Before we state the topological transversality theorem we note two things:

(a). If Λ, $\Theta \in M_{\partial U}(\overline{U^w}, C)$ with $\Lambda|_{\partial U} = \Theta|_{\partial U}$ then $\Lambda \cong \Theta$ in $M_{\partial U}(\overline{U^w}, C)$. To see this let $\Psi(x,t) = (1-t)\Lambda(x) + t\Theta(x)$ and note that $\Psi : \overline{U^w} \times [0,1] \to K(C)$ is a weakly upper semicontinuous, weakly compact map [some authors prefer to assume (but it is not necessary) the following property:

$$\begin{cases} \text{if } \ W \ \text{ is a weakly compact subset of} \\ C \ \text{ then } \ \overline{co}\,(W) \ \text{ is weakly compact} \end{cases}$$

to guarantee that Ψ is weakly compact. Note, this property is a Krein–Šmulian type property [4,5], which we know is true if X is a quasicomplete locally convex linear topological space]. Note, $x \notin \Psi_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ (note, $\Lambda|_{\partial U} = \Theta|_{\partial U}$).
(b). A standard argument guarantees that $\cong$ in $M_{\partial U}(\overline{U^w}, C)$ is an equivalence relation.

Theorem 2. *Let X be a Hausdorff locally convex topological vector space, U be a weakly open subset of C, and C be a closed convex subset of X. Suppose F and G are two maps in $M_{\partial U}(\overline{U^w}, C)$ with $F \cong G$ in $M_{\partial U}(\overline{U^w}, C)$. Then F is essential in $M_{\partial U}(\overline{U^w}, C)$ if, and only if, G is essential in $M_{\partial U}(\overline{U^w}, C)$.*

Proof. Assume G is essential in $M_{\partial U}(\overline{U^w}, C)$. To show that F is essential in $M_{\partial U}(\overline{U^w}, C)$ let $J \in M_{\partial U}(\overline{U^w}, C)$ with $J|_{\partial U} = F|_{\partial U}$. Now since $F \cong G$ in $M_{\partial U}(\overline{U^w}, C)$, then (a) and (b) above guarantees that $G \cong J$ in $M_{\partial U}(\overline{U^w}, C)$ i.e., (2.1) holds. Then Theorem 1 guarantees that F is essential in $M_{\partial U}(\overline{U^w}, C)$. A similar argument shows that if F is essential in $M_{\partial U}(\overline{U^w}, C)$, then G is essential in $M_{\partial U}(\overline{U^w}, C)$. □

Next, we present an example of an essential map in $M_{\partial U}(\overline{U^w}, C)$, which will be useful from an application viewpoint (see Corollary 1 and Theorem 5).

Theorem 3. *Let X be a Hausdorff locally convex topological vector space, U be a weakly open subset of C, $0 \in U$, and C be a closed convex subset of X. Then the zero map is essential in $M_{\partial U}(\overline{U^w}, C)$.*

Proof. Let $J \in M_{\partial U}(\overline{U^w}, C)$ with $J|_{\partial U} = \{0\}|_{\partial U}$. We must show there exists a $x \in U$ with $x \in J(x)$. Consider the map R given by

$$R(x) = \begin{cases} J(x), & x \in \overline{U^w} \\ \{0\}, & x \in C \setminus \overline{U^w}. \end{cases}$$

Note, $R : C \to K(C)$ is a weakly upper semicontinuous, weakly compact map, thus [6] guarantees that there exists a $x \in C$ with $x \in R(x)$. If $x \in C \setminus \overline{U^w}$ then since $R(x) = \{0\}$ and $0 \in U$ we have a contradiction. Thus $x \in U$ so $x \in R(x) = J(x)$. □

We combine Theorem 2 and Theorem 3 and we obtain:

Theorem 4. *Let X be a Hausdorff locally convex topological vector space, U be a weakly open subset of C, $0 \in U$, and C be a closed convex subset of X. Suppose $F \in M_{\partial U}(\overline{U^w}, C)$ with*

$$x \notin t\,F(x) \ \textit{ for } \ x \in \partial U \ \textit{ and } \ t \in (0,1). \tag{2}$$

Then F is essential in $M_{\partial U}(\overline{U^w}, C)$ (in particular there exists a $x \in U$ with $x \in F(x)$).

Proof. Note, Theorem 3 guarantees that the zero map is essential in $M_{\partial U}(\overline{U^w}, C)$. The result will follow from Theorem 2 if we note the usual homotopy between the zero map and F, namely, $\Psi(x,t) = t\,F(x)$ (note $x \notin \Psi_t(x)$ for $x \in \partial U$ and $t \in [0,1]$; see (2.2)). □

Corollary 1. *Let X be a Hausdorff locally convex topological vector space, U be a weakly open subset of C, $0 \in U$, C be a closed convex subset of X, and $\overline{U^w}$ be a Šmulian space (i.e., for any $\Omega \subseteq \overline{U^w}$ if $x \in \overline{\Omega^w}$ then there exists a sequence $\{x_n\}$ in Ω with $x_n \rightharpoonup x$). Suppose $F : \overline{U^w} \to K(C)$ is a weakly sequentially upper semicontinuous i.e., for any weakly closed set A of C we have that $F^{-1}(A) = \{x \in \overline{U^w} : \; F(x) \cap A \neq \emptyset\}$ is a weakly sequentially closed), weakly compact map with*

$$x \notin t\,F(x) \;\; for \;\; x \in \partial U \;\; and \;\; t \in (0,1]. \tag{3}$$

Then F is essential in $M_{\partial U}(\overline{U^w}, C)$ (in particular there exists a $x \in U$ with $x \in F(x)$).

Proof. The result follows from Theorem 4, as $F \in M_{\partial U}(\overline{U^w}, C)$. To see this we simply need to show that $F : \overline{U^w} \to K(C)$ is weakly upper semicontinuous. The argument is similar to that in [2,7]. Let A be a weakly closed subset of C and let $x \in \overline{F^{-1}(A)}^w$. As $\overline{U^w}$ is Šmulian then there exists a sequence $\{x_n\}$ in $F^{-1}(A)$ with $x_n \rightharpoonup x$. Now, $x \in F^{-1}(A)$ since $F^{-1}(A)$ is weakly sequentially closed. Thus, $\overline{F^{-1}(A)}^w = F^{-1}(A)$ so $F^{-1}(A)$ is weakly closed. □

We consider the second order differential inclusion

$$\begin{cases} y'' \in f(t, y, y') \;\; \text{a.e. on} \;\; [0,1] \\ y(0) = y(1) = 0 \end{cases} \tag{4}$$

where $f : [0,1] \times \mathbf{R}^2 \to CK(\mathbf{R})$ is a L^p–Carathéodory function (here $p > 1$ and $CK(\mathbf{R})$ denotes the family of nonempty, convex, compact subsets of $\mathbf{R}$); by this we mean
(a). $t \mapsto f(t, x, y)$ is measurable for every $(x, y) \in \mathbf{R}^2$,
(b). $(x, y) \mapsto f(t, x, y)$ is upper semicontinuous for a.e. $t \in [0,1]$,
and
(c). for each $r > 0$, $\exists\, h_r \in L^p[0,1]$ with $|f(t, x, y)| \le h_r(t)$ for a.e. $t \in [0,1]$ and every $(x, y) \in \mathbf{R}^2$ with $|x| \le r$ and $|y| \le r$.

We present an existence principle for (2.4) using Corollary 1. For notational purposes for appropriate functions u, let

$$\|u\|_0 = \sup_{[0,1]} |u(t)|, \;\; \|u\|_1 = \max\{\|u\|_0, \|u'\|_0\} \text{ and } \|u\|_{L^p} = \left(\int_0^1 |u(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Recall that $W^{k,p}[0,1]$, $1 \le p < \infty$ denotes the space of functions $u : [0,1] \to \mathbf{R}^n$, with $u^{(k-1)} \in AC[0,1]$ and $u^{(k)} \in L^p[0,1]$. Note, $W^{k,p}[0,1]$ is reflexive if $1 < p < \infty$.

Theorem 5. *Let $f : [0,1] \times \mathbf{R}^2 \to CK(\mathbf{R})$ be a L^p–Carathéodory function ($1 < p < \infty$) and assume there exists a constant M_0 (independent of λ) with $\|y\|_1 \neq M_0$ for any solution $y \in W^{2,p}[0,1]$ to*

$$\begin{cases} y'' \in \lambda\, f(t, y, y') \;\; \textit{a.e. on} \;\; [0,1] \\ y(0) = y(1) = 0 \end{cases}$$

for $0 < \lambda \le 1$. Then (2.4) has a solution in $W^{2,p}[0,1]$.

Proof. Since f is L^p–Carathéodory, there exists $h_{M_0} \in L^p[0,1]$ with

$$\begin{cases} |f(t, u, v)| \le h_{M_0}(t) \;\; \text{for a.e.} \;\; t \in [0,1] \;\; \text{and} \\ \text{every} \;\; (u, v) \in \mathbf{R}^2 \;\; \text{with} \;\; |u| \le M_0 \;\; \text{and} \;\; |v| \le M_0. \end{cases}$$

Let

$$G(t,s) = \begin{cases} (t-1)\,s, & 0 \le s \le t \le 1 \\ (s-1)\,t, & 0 \le t \le s \le 1 \end{cases}$$

and $N = \max\{N_0, N_1, M_0\}$ where (here $\frac{1}{p} + \frac{1}{q} = 1$),

$$N_0 = \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q \, ds \right)^{\frac{1}{q}}$$

and

$$N_1 = \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G_t(t,s)|^q \, ds \right)^{\frac{1}{q}}.$$

We also let

$$N_2 = \|h_{M_0}\|_{L^p}.$$

We will apply Corollary 1 with $X = W^{2,p}[0,1]$,

$$C = \left\{ u \in W^{2,p}[0,1] : \ \|u\|_1 \le N \ \text{ and } \ \|u''\|_{L^p} \le N_2 \right\}$$

and

$$U = \left\{ u \in W^{2,p}[0,1] : \ \|u\|_1 < M_0 \ \text{ and } \ \|u''\|_{L^p} \le N_2 \right\}.$$

Now, let

$$F = L \circ N_f : C \to 2^X$$

where $L : L^p[0,1] \to W^{2,p}[0,1]$ and $N_f : W^{2,p}[0,1] \to 2^{L^p[0,1]}$ are given by

$$L\,y(t) = \int_0^1 G(t,s)\,y(s)\,ds$$

and

$$N_f\,u = \left\{ y \in L^p[0,1] : \ y(t) \in f(t,u(t),u'(t)) \ \text{ a.e. } \ t \in [0,1] \right\}.$$

Note, N_f is well defined, since if $x \in C$ then ([8] p. 26 or [9], p. 56) guarantees that $N_f\,x \neq \emptyset$.

Notice that C is a convex, closed, bounded subset of X. We first show that U is weakly open in C. To do this, we will show that $C \setminus U$ is weakly closed. Let $x \in \overline{C \setminus U}^w$. Then there exists $x_n \in C \setminus U$ (see [10] p. 81) with $x_n \rightharpoonup x$ (here $W^{2,p}[0,1]$ is endowed with the weak topology and $\rightharpoonup$ denotes weak convergence). We must show $x \in C \setminus U$. Now since the embedding $j : W^{2,p}[0,1] \to C^1[0,1]$ is completely continuous ([11], p. 144 or [12], p. 213), there is a subsequence S of integers with

$$x_n \to x \ \text{ in } \ C^1[0,1] \ \text{ and } \ x_n'' \rightharpoonup x'' \ \text{ in } \ L^p[0,1]$$

as $n \to \infty$ in S. Also

$$\|x\|_1 = \lim_{n\to\infty} \|x_n\|_1 \ \text{ and } \ \|x''\|_{L^p} \le \liminf \|x_n''\|_{L^p} \le N_2.$$

Note, $M_0 \le \|x\|_1 \le N$ since $M_0 \le \|x_n\|_1 \le N$ for all n. As a result, $x \in C \setminus U$, so $\overline{C \setminus U}^w = C \setminus U$. Thus, U is weakly open in C. Also,

$$\partial U = \{u \in C : \ \|u\|_1 = M_0\} \quad \text{and} \quad \overline{U^w} = \{u \in C : \ \|u\|_1 \leq M_0\};$$

note, $\overline{U^w} = \overline{U}$ ([5] p. 66) since U is convex (alternatively take $x \in \overline{U^w}$ and follow a similar argument as above). Also note that $\overline{U^w}$ is weakly compact (note $W^{2,p}[0,1]$ is reflexive) so $\overline{U^w}$ is Šmulian. Notice also that $F : \overline{U^w} \to 2^C$ since if $y \in \overline{U^w}$ then from above we have

$$\|F\,y\|_0 \leq \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G(t,s)|^q \, ds \right)^{\frac{1}{q}} = N_0,$$

$$\|(F\,y)'\|_0 \leq \|h_{M_0}\|_{L^p} \sup_{t\in[0,1]} \left(\int_0^1 |G_t(t,s)|^q \, ds \right)^{\frac{1}{q}} = N_1,$$

and

$$\|(F\,y)''\|_0 \leq \|h_{M_0}\|_{L^p} = N_2.$$

A standard argument (see for example ([13] p. 283)) guarantees that $F : \overline{U^w} \to K(C)$ is weakly sequentially upper semicontinuous.

Now we apply Corollary 1 to deduce our result: Note that (2.3) holds since, if there exists $x \in \partial U$ and $\lambda \in (0,1]$ with $x \in \lambda\, F\, x$, then $\|x\|_1 = M_0$ (since $x \in \partial U$) and $\|x\|_1 \neq M_0$ by assumption. Thus, F is essential in $M_{\partial U}(\overline{U^w}, C)$, so in particular, F has a fixed point in U. □

Conflicts of Interest: The author declares no conflict of interest.

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