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On Istrăţescu Type Contractions in *b*-Metric Spaces

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Abstract: In this paper, we introduce the notions of α-almost Istrăţescu contraction of type E and of type E^* in the setting of b-metric space. The existence of fixed points for such mappings is investigated and some examples to illustrate the validity of the main results are considered. In the last part of the paper, we list some immediate consequences.

Keywords: Istrăţescu contraction; fixed point; b-metric space

1. Introduction and Preliminaries

Fixed point theory is an important tool in the investigation of the solutions of integral and differential equations via the successive approximations approach. The idea was abstracted and then solely formulated in 1922 by Banach, under the name of Contraction Mapping Principle. After 1922, the result was extended and generalized by many researchers. One of the most significant fixed point result was given by Istrăţescu [1]. Roughly speaking, the idea of Istrăţescu [1] can be considered as a Second-Order Contraction Principle. In what follows, we recall this interesting fixed point theorem of Istrăţescu (see [1,2]).

Theorem 1. *Given a complete metric space $(\mathcal{M}, d)$, every map $T : \mathcal{M} \to \mathcal{M}$ is a Picard operator provided that there exist $a_1, a_2 \in (0,1)$ such that $a_1 + a_2 < 1$ and*

$$d(T^2x, T^2y) \leq a_1 \cdot d(Tx, Ty) + a_1 \cdot d(x, y),$$

for all $x, y \in \mathcal{M}$.

Another interesting extension of the contraction mapping was given by Berinde [3] under the name of almost contraction. A self-mapping T on a metric space (M, d) is called almost contraction if there exist a constant $\kappa \in (0,1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \kappa d(x, y) + Ld(y, Tx), \text{ for all } x, y \in M.$$

On the other hand, the notion of metric space has been generalized in several directions and the above-mentioned Contraction Principle has been extended in these new settings. Among this new generalizations, we mention here the case of b-metric space (see, e.g., Bakhtin [4] and Czerwik [5]). The notion was also proposed as quasi-metric spaces (see, e.g., Berinde [6]).

Assume that d is a distance function on a non-empty set $\mathcal{M}$, that is, $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$. If the following conditions are satisfied, then d is called a b-metric:

$(b1)$ $d(x, y) = 0$ if and only if $x = y$.
$(b2)$ $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{M}$.
$(b3)$ $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in M$, where $s \geq 1$

Further, the triple $(\mathcal{M}, d, s)$ is called a b-metric space. It is evident that, for $s = 1$, the b-metric turns into a standard metric. We first underline the fact that unlike the standard metric, b-metric is not necessarily continuous due to modified triangle inequality (see, e.g., [7]).

The following lemma demonstrates one of the basic observations in the setting of b-metric spaces (see, e.g., [8–13] and the references therein).

Lemma 1. *Every sequence $\{x_n\}$ with elements from a b-metric space $(\mathcal{M}, d, s)$ satisfies for every $n \in \mathbb{N}$ the inequality*

$$d(x_0, x_l) \leq s^n \sum_{j=0}^{l-1} d(x_j, x_{j+1}), \tag{1}$$

where $l \in \{1, 2, 3, ..., 2^n - 1, 2^n\}$.

The following is one of the characterizations of Cauchy criteria in the setting of b-metric spaces (see, e.g., [13]).

Lemma 2. *A sequence $\{x_n\}$ with elements from a b-metric space $(\mathcal{M}, d, s)$ is a Cauchy if there exists $c \in [0, 1)$ such that*

$$d(x_n, x_{n+1}) \leq c \cdot d(x_n, x_{n-1}) \tag{2}$$

for every $n \in \mathbb{N}$.

Let $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ and $T : \mathcal{M} \to \mathcal{M}$ be mapping such that

(O) $\alpha(x, Tx) \geq 1 \Rightarrow \alpha(Tx, T^2x) \geq 1, \quad$ for all $x \in \mathcal{M}$.

Then, f is called an $\alpha-$*orbital admissible* mapping [14].

In this paper, inspired from the results of Istrăţescu and Berinde, we consider two new types of generalized contractions in the framework of b-metric space. We examine the existence of a fixed point for these new mappings. We then provide examples to support our main theorems and list some useful consequences.

2. Main Results

We first introduce the notion of α-almost Istrăţescu contraction of type E.

Definition 1. *Let $(\mathcal{M}, d, s)$ be a b-metric space and $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ be a function. A mapping $T : \mathcal{M} \to \mathcal{M}$ is called α-almost Istrăţescu contraction of type E if there exist $k \in [0, 1)$, $\lambda \geq 0$ such that for any $x, y \in \mathcal{M}$*

$$\alpha(x, y) d(T^2x, T^2y) \quad \leq k \cdot E(x, y) + \lambda \cdot N(x, y) \tag{3}$$

where

$$E(x, y) = d(Tx, Ty) + \left| d(Tx, T^2x) - d(Ty, T^2y) \right| \tag{4}$$

and

$$N(x, y) = \min\left\{ d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(Tx, T^2y), d(Ty, T^2x) \right\}. \tag{5}$$

Theorem 2. *Let $(\mathcal{M}, d, s)$ be a complete b-metric space and $T : \mathcal{M} \to \mathcal{M}$ an α-almost Istrăţescu contraction of type E such that either:*

(i) T is continuous; or
(ii) T^2 is continuous and $\alpha(Tu, u) \geq 1$ for any $u \in Fix_{T^2}(\mathcal{M})$.

If T is $\alpha-$orbital admissible and there exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof. Let $x_0 \in \mathcal{M}$ be the given point with the property that $\alpha(x_0, Tx_0) \geq 1$. Because of the $\alpha-$orbital admissible property of the mapping T, we have that $\alpha(Tx_0, T^2x_0) \geq 1$, and continuing this process we get

$$\alpha(T^n x_0, T^{n+1} x_0) \geq 1, \text{ for } n \in \mathbb{N}. \tag{6}$$

Replacing x by x_0 and y by Tx_0 in (3), we have

$$\begin{aligned}
d(T^2x_0, T^3x_0) &\leq \alpha(x_0, Tx_0) d(T^2x_0, T^2(Tx_0)) \leq k \cdot E(x_0, Tx_0) + \lambda \cdot N(x_0, Tx_0)\\
&= k \cdot \big(d(Tx_0, T(Tx_0)) + \big|d(Tx_0, T^2x_0) - d(T(Tx_0), T^2(Tx_0))\big|\big) +\\
&\quad +\lambda \cdot \min\{d(x_0, Tx_0), d(Tx_0, T(Tx_0)), d(x_0, T(Tx_0)), d(Tx_0, Tx_0),\\
&\qquad d(Tx_0, T^2(Tx_0)), d(T(Tx_0), T^2x_0)\}\\
&\leq k \cdot \big(d(Tx_0, T^2(x_0)) + \big|d(Tx_0, T^2x_0) - d(T^2x_0, T^3x_0)\big|\big) +\\
&\quad +\lambda \cdot \min\{d(x_0, Tx_0), d(Tx_0, T^2x_0), d(x_0, T^2x_0), d(Tx_0, Tx_0),\\
&\qquad d(Tx_0, T^3x_0), d(T^2x_0), T^2x_0)\}\\
&= k \cdot \big(d(Tx_0, T^2(x_0)) + \big|d(Tx_0, T^2x_0) - d(T^2x_0, T^3x_0)\big|\big)
\end{aligned} \tag{7}$$

If $d(Tx_0, T^2x_0) \leq d(T^2x_0, T^3x_0)$, then we have

$$\begin{aligned}
d(T^2x_0, T^3x_0) &\leq k \cdot \big(d(Tx_0, T^2x_0) + d(T^2x_0, T^3x_0) - d(Tx_0, T^2x_0)\big)\\
&\leq k \cdot \big(d(Tx_0, T^2x_0) + d(T^2x_0, T^3x_0) - d(Tx_0, T^2x_0)\big)\\
&= k \cdot d(T^2x_0, T^3x_0) < d(T^2x_0, T^3x_0),
\end{aligned}$$

which is a contradiction, thus $d(Tx_0, T^2x_0) > d(T^2x_0, T^3x_0)$ and the inequality in Equation (7) becomes

$$\begin{aligned}
d(T^2x_0, T^3x_0) &\leq k \cdot (d(Tx_0, T^2x_0) + d(Tx_0, T^2x_0) - d(T^2x_0, T^3x_0))\\
&= k \cdot \big(2d(Tx_0, T^2x_0) - d(T^2x_0, T^3x_0)\big) \qquad \Leftrightarrow\\
&d(T^2x_0, T^3x_0) \leq \frac{2k}{1+k} d(Tx_0, T^2x_0)
\end{aligned} \tag{8}$$

For $x = Tx_0, y = T^2x_0$, taking Equation (6) into account,

$$\begin{aligned}
d(T^3x_0, T^4x_0) &\leq \alpha(Tx_0, T^2x_0) d(T^2(Tx_0), T^2(T^2x_0)) \leq k \cdot E(Tx_0, T^2x_0) + \lambda \cdot N(Tx_0, T^2x_0) \\
&\leq k \cdot \left(d(T(Tx_0), T(T^2x_0)) + \left| d(T(Tx_0), T^2(Tx_0)) - d(T(T^2x_0), T^2(T^2x_0)) \right| \right) + \\
&\quad + L \cdot \min \left\{ d(Tx_0, T(Tx_0)), d(T(Tx_0), T(T^2x_0)), d(Tx_0, T(T^2x_0)), d(T(T(x_0), Tx_0), \right. \\
&\qquad \left. d(Tx_0, T^2(T^2x_0)), d(T(T^2x_0), T^2(Tx_0)) \right\} \\
&= k \cdot \left(d(T^2x_0, T^3x_0) + \left| d(T^2x_0, T^3x_0) - d(T^3x_0, T^4x_0) \right| \right) + \\
&\quad + \lambda \cdot \min \left\{ d(Tx_0, T^2x_0), d(T^2x_0, T^3x_0), d(Tx_0, T^3x_0), d(Tx_0, Tx_0), \right. \\
&\qquad \left. d(T^2x_0, T^4x_0), d(T^3x_0), T^3x_0) \right\} \\
&= k \cdot \left(d(T^2x_0, T^3x_0) + \left| d(T^2x_0, T^3x_0) - d(T^3x_0, T^4x_0) \right| \right)
\end{aligned}$$

Since for the case $d(T^2x_0, T^3x_0) \leq d(T^3x_0, T^4x_0)$ we get

$$\begin{aligned}
d(T^3x_0, T^4x_0) &\leq k \cdot \left(d(T^2x_0, T^3x_0) + d(T^3x_0, T^4x_0) - d(T^2x_0, T^3x_0) \right) \\
&\leq k \cdot d(T^3x_0, T^4x_0),
\end{aligned}$$

a contradiction, we have $d(T^2x_0, T^3x_0) > d(T^3x_0, T^4x_0)$ and

$$\begin{aligned}
d(T^3x_0, T^4x_0) &\leq k \cdot \left(d(T^2x_0, T^3x_0) + d(T^2x_0, T^3x_0) - d(T^3x_0, T^4x_0) \right) \\
&\leq k \cdot \left(2d(T^2x_0, T^3x_0) - d(T^3x_0, T^4x_0) \right), \qquad \Leftrightarrow \\
& d(T^3x_0, T^4x_0) \leq \frac{2k}{1+k} d(T^2x_0, T^3x_0). \qquad (9)
\end{aligned}$$

By proceeding in the same way,

$$d(T^nx_0, T^{n+1}x_0) \leq \frac{2k}{1+k} d(T^{n-1}x_0, T^nx_0) \leq \left(\frac{2k}{1+k} \right)^{n-1} d(Tx_0, T^2x_0) \to 0 \text{ as } n \to \infty, \qquad (10)$$

because $l = \frac{2k}{1+k} < 1$.

On the other hand, considering the sequence $\{x_n\}_{n \in \mathbb{N}}$ defined as follows

$$x_1 = Tx_0, \quad x_2 = T^2x_0, \quad \ldots \quad x_n = T^nx_0,$$

where $x_0 \in \mathcal{M}$, from Equation (10), we have

$$d(x_n, x_{n+1}) \leq l \cdot d(x_{n-1}, x_n),$$

for $n \in \mathbb{N}$. Therefore, from Lemma 2, we gather that $\{x_n\}_{n \in \mathbb{N}}$ forms a Cauchy sequence on a complete b-metric space. Attendantly, it is convergent. Then, there exists $u \in \mathcal{M}$ such that

$$\lim_{n \to \infty} d(x_n, u) = 0. \qquad (11)$$

When the mapping T is continuous, it follows that $\lim_{n \to \infty} d(x_n, Tu) = \lim_{n \to \infty} d(Tx_{n-1}, Tu) = 0$ and thus we conclude that $Tu = u$, that is u forms a fixed point of T.
Keeping the continuity of T^2 in mind, we derive $\lim_{n \to \infty} d(x_n, T^2u) = \lim_{n \to \infty} d(T^2x_{n-2}, T^2u) = 0$. Since each sequence in b-metric space has a unique limit, we get that $T^2u = u$. That is, u is a fixed

point of T^2. on the purpose of showing that u forms also a fixed point of T, we employ the method of reductio ad absurdum. In an attempt to deduce the result, we presume that $u \neq Tu$. Thereupon, from Equation (3), we have

$$\begin{aligned}
0 < d(Tu,u) &= d(T^2(Tu),T^2u) \leq \alpha(fu,u)d(T^2(fu),T^2u) \leq k \cdot E(Tu,u) + \lambda \cdot N(Tu,u) \\
&= k \cdot \left(d(Tu,T^2u) + \left|d(Tu,T^2u) - d(T^2u,T^3u)\right|\right) + \\
&\quad + \lambda \cdot \min\left\{d(u,Tu), d(Tu,T^2u), d(u,T^2u), d(Tu,Tu), d(Tu,T^3u), d(T^2u,T^2u)\right\} \\
&= k \cdot (d(Tu,u) + |d(Tu,u) - d(u,Tu)|) \\
&= k \cdot (d(Tu,u)) < d(Tu,u).
\end{aligned}$$

Hence, $u = Tu$. □

Example 1. *Let $\mathcal{M} = [0,\infty)$ and the function $d : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ with $d(x,y) = (x-y)^2$, which is a 2-metric. Define a mapping $T : \mathcal{M} \to \mathcal{M}$ by* $Tx = \begin{cases} x^2, & \text{if } x \in [0,1) \\ 1, & \text{if } x \in [1,2) \\ \frac{6x^2+3x+1}{4x^2+4x+6}, & \text{if } x \in [2,\infty). \end{cases}$

We can notice that T is discontinuous at the point $x = 2$, but T^2 is continuous on $\mathcal{M}$ since $T^2x = \begin{cases} x^4, & \text{if } x \in [0,1) \\ 1, & \text{if } x \in [1,\infty). \end{cases}$ *Let the function $\alpha : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ be given by*

$$\alpha(x,y) = \begin{cases} 3, & \text{if } x,y \in [1,\infty) \\ 0, & \text{otherwise}. \end{cases}$$

It is easy to see that T is an α-almost Istrăţescu contraction of type E. Indeed, due to definition of function α, we see the only interesting case is for $x,y \in [1,\infty)$; we have for any $k \in [0,1)$

$$0 = 3 \cdot d(1,1) = \alpha(x,y)d(T^2x,T^2y) \leq k \cdot E(x,y) + \lambda \cdot N(x,y).$$

We can conclude that for any $x,y \in \mathcal{M}$, all the conditions of Theorem 3 are satisfied, and $Fix_T\mathcal{M} = \{0,1\}$.

Theorem 3. *Under the assumptions of Theorem 2, the mapping T has a unique the fixed point, provided that for any $y \in \mathcal{M}$*

$$\alpha(u,y) \geq 1, \text{ where } u \in Fix_T(\mathcal{M}). \tag{12}$$

Proof. By Theorem 2, we already have that $Fix_T(\mathcal{M}) \neq \emptyset$, thus let $u,v \in Fix_T(\mathcal{M})$ such that $v \neq u$. We have

$$
\begin{aligned}
d(u,v) &= d(T^2u,T^2v) \le \alpha(u,y)d(T^2u,T^2v) \le k\cdot E(u,v) + \lambda\cdot N(u,v)\\
&\le k\cdot \left(d(Tu,Tv) + \left|d(Tu,T^2u) - d(Tv,T^2v)\right|\right) +\\
&\quad +\lambda\cdot \min\left\{d(u,Tu), d(v,Tv), d(u,Tv), d(v,Tu), d(Tu,T^2v), d(Tv,T^2u)\right\}\\
&= k\cdot (d(u,v) + |d(u,u) - d(v,v)|) +\\
&\quad +\lambda\cdot \min\{d(u,u), d(v,v), d(u,v), d(v,u), d(u,v), d(v,u)\}\\
&= k\cdot d(u,v) < d(u,v),
\end{aligned}
$$

a contradiction. Thereupon, T possesses exactly one fixed point. □

Example 2. *Let $(\mathcal{M},d,2)$ be a complete b-metric space, where $\mathcal{M}=[0,2]$ and the function $d:\mathcal{M}\times\mathcal{M}\to[0,\infty)$ with $d(x,y)=(x-y)^2$. Let $T:\mathcal{M}\to\mathcal{M}$ be a mapping, defined by $Tx=\begin{cases}1, & \text{if } x\in[0,1]\\ \frac{x^2}{6}, & \text{if } x\in(1,2].\end{cases}$*

In this case, $T^2x=1$, so that the mapping T is discontinuous in $x=1$, but T^2 is continuous on $\mathcal{M}$. On the other hand, considering $\alpha:\mathcal{M}\times\mathcal{M}\to[0,\infty)$, where, for example $\alpha(x,y)=ln(x^2+y^2+4)$, we can easily get that T is α-orbital admissible and α-almost Istrăţescu contraction of type E (since $d(T^2x,T^2y)=0$), so that from Theorem 2 T has a fixed point, which is $x=1$. On the other hand, for any $y\in\mathcal{M}$, we have $\alpha(1,y)=ln(1+y^2+5)\ge1$ so that from Theorem 3 we get that the fixed point is unique.

Definition 2. *Let $(\mathcal{M},d,s)$ be a b-metric space. A mapping $T:\mathcal{M}\to\mathcal{M}$ is called almost Istrăţescu contraction of type E if there exist $k\in[0,1)$, $\lambda\ge0$ such that for any $x,y\in\mathcal{M}$*

$$d(T^2x,T^2y) \quad \le k\cdot E(x,y)+\lambda\cdot N(x,y) \tag{13}$$

where $E(x,y)$ and $N(x,y)$ are defined by Equations (4) and (5) respectively.

Theorem 4. *Let $(\mathcal{M},d,s)$ be a complete b-metric space and $T:\mathcal{M}\to\mathcal{M}$ an almost Istrăţescu contraction of type E such that either T is continuous or T^2 is continuous. Then, T has a unique fixed point.*

Proof. It is sufficient to set $\alpha(x,y)=1$ in Theorem 3. □

Corollary 1. *Suppose that a self-mapping T, on a complete b-metric space $(\mathcal{M},d,s)$ fulfills*

$$d(T^2x,T^2y) \quad \le k\cdot E(x,y), \tag{14}$$

for all $x,y\in\mathcal{M}$. If either T or T^2 is continuous, then T possesses a unique fixed point.

Proof. Put $\lambda=0$ in Theorem 4. □

In what follows we define α-almost Istrăţescu contraction of type E^*.

Definition 3. *Let $(\mathcal{M},d,s)$ be a complete b-metric space and $\alpha:\mathcal{M}\times\mathcal{M}\to[0,\infty)$ be a function. A mapping $T:\mathcal{M}\to\mathcal{M}$ is called α-almost Istrăţescu contraction of type E^* if there exist $k\in[0,1)$, $\lambda\ge0$ such that for any $x,y\in\mathcal{M}$*

$$\alpha(x,y)d(T^2x,T^2y) \quad \le k\cdot E^*(x,y)+\lambda\cdot N(x,y) \tag{15}$$

where

$$E^*(x,y) = \Big|d(x,Tx) - d(Ty,T^2y)\Big| + d(x,y) + \Big|d(y,Ty) - d(Tx,T^2x)\Big| \tag{16}$$

and

$$N(x,y) = \min\Big\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(Tx,T^2y), d(Ty,T^2x)\Big\}. \tag{17}$$

Theorem 5. *Let $(\mathcal{M}, d, s)$ be a complete b-metric space and $T : M \to M$ an α-almost Istrăţescu contraction of type E^* such that either:*

(i) *T is continuous; or*
(ii) *T^2 is continuous and $\alpha(fu,u) \geq 1$ for any $u \in Fix_{T^2}(\mathcal{M})$.*
(iii) *If T is $\alpha-$orbital admissible and there exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, Tx_0) \geq 1$,*

then T has a fixed point.

Proof. Let $x_0 \in \mathcal{M}$ and we consider the sequence $\{x_n\}$, defined as in Theorem 2. Then, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} E^*(x_{n-1},x_n) &= +\left|d(x_{n-1},Tx_{n-1}) - d(Tx_n,T^2x_n)\right| + d(x_{n-1},x_n)\left|d(x_n,Tx_n) - d(Tx_{n-1},T^2x_{n-1})\right| \\ &= |d(x_{n-1},x_n) - d(x_{n+1},x_{n+2})| + d(x_{n-1},x_n) + |d(x_n,x_{n+1}) - d(x_n,x_{n+1})| \\ &= |d(x_{n-1},x_n) - d(x_{n+1},x_{n+2})| + d(x_{n-1},x_n), \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1},x_n) &= \min\{d(x_{n-1},Tx_{n-1}), d(x_n,Tx_n), d(x_{n-1},Tx_n), d(x_n,Tx_{n-1}), d(Tx_{n-1},T^2x_n), d(Tx_n,T^2x_{n-1})\} \\ &= \min\{d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), d(x_n,x_n), d(x_n,x_{n+2}), d(x_{n+1},x_{n+1})\} = 0. \end{aligned}$$

Taking into account Equation (6), by Equation (15) we have

$$\begin{aligned} d(x_{n+1},x_{n+2}) &= d(T^2x_{n-1},T^2x_n) \leq \alpha(x_{n-1},x_n)d(T^2x_{n-1},T^2x_n) \leq k\cdot E^*(x_{n-1},x_n) + \lambda\cdot N(x_{n-1},x_n) \\ &= k\cdot\left(d(x_{n-1},x_n) + |d(x_{n-1},x_n) - d(x_{n+1},x_{n+2})|\right). \end{aligned} \tag{18}$$

If we suppose that $d(x_{n-1},x_n) \leq d(x_{n+1},x_{n+2})$, by Equation (18) we get

$$d(x_{n+1},x_{n+2}) \leq k\cdot(d(x_{n+1},x_{n+2})) < d(x_{n+1},x_{n+2}),$$

a contradiction. If $d(x_{n-1},x_n) > d(x_{n+1},x_{n+2})$, then

$$d(x_{n+1},x_{n+2}) \leq k\cdot(2d(x_{n-1},x_n) - d(x_{n+1},x_{n+2}))$$

which turns into

$$d(x_{n+1},x_{n+2}) \leq \frac{2k}{k+1}d(x_{n-1},x_n), \text{ for any } n \in \mathbb{N}. \tag{19}$$

Denoting by $c := \frac{2k}{k+1} < 1$ and $\gamma = \max\{d(x_0,x_1), d(x_1,x_2)\}$, respectively, and continuing in the same way, we get

$$\begin{aligned} d(x_{n+1},x_{n+2}) &\leq c\cdot d(x_{n-1},x_n) \leq c^2\cdot d(x_{n-3},x_{n-2}) \leq \\ &\ldots \\ &\leq c^{\left[\frac{n}{2}\right]}\cdot\max\{d(x_0,x_1), d(x_1,x_2)\} \\ &= c^{\left[\frac{n}{2}\right]}\cdot\gamma. \end{aligned}$$

Therefore,

$$d(x_{n+1},x_{n+2}) \leq c^{\left[\frac{n}{2}\right]}\cdot\gamma, \text{ for } n \in \mathbb{N} \tag{20}$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{21}$$

By Lemma 2, the sequence $\{x_n\}$ is Cauchy on a complete b-metric space, so that there exists u such that $\lim_{n\to\infty} d(x_n, u) = 0$. If Assumption (i) holds, we obtain $Tu = u$.
On the other hand, if we use Assumption (ii), we get $T^2u = u$ and $\alpha(Tu, u) \geq 1$. On account of reductio ad absurdum, we assume that u is not a fixed point of T, by Equation (15) we have

$$\begin{aligned} d(Tu,u) &= d(T^2(Tu), T^2u) \leq \alpha(Tu,u)d(T^2(Tu), T^2u) \leq k \cdot E^*(Tu,u) + \lambda \cdot N(Tu,u) \\ &= k \cdot \left(d(Tu,u) + \left|d(Tu,T^2u) - d(Tu,T^2u)\right| + \left|d(u,Tu) - d(T^2u,T^3u)\right|\right) \\ &= k \cdot d(Tu,u) < d(Tu,u), \end{aligned}$$

a contradiction. Thereupon, $Tu = u$ and u is a fixed point of the mapping T. □

Example 3. *Let $(\mathcal{M}, d, 2)$ be a complete b-metric space, where $\mathcal{M} = [0,\infty)$ and the function $d : \mathcal{M} \times \mathcal{M} \to [0,\infty)$ is defined as $d(x,y) = (x-y)^2$.*

Let $T : \mathcal{M} \to \mathcal{M}$ be a continuous mapping, defined by $Tx = \begin{cases} -\frac{x}{2}, & \text{if } x \in [-1,0) \\ 2x, & \text{if } x \geq 0. \end{cases}$

Then, $T^2x = \begin{cases} -x, & \text{if } x \in [-1,0) \\ 4x, & \text{if } x \geq 0. \end{cases}$

In addition, let the function $\alpha : \mathcal{M} \times \mathcal{M} \to [0,\infty)$, $\alpha(x,y) = \begin{cases} 1, & \text{for } x,y \in [-1,0] \\ 0, & \text{otherwise .} \end{cases}$

Of course, T is α-orbital admissible and $\alpha(0, T0) = \alpha(T0, 0) = \alpha(0,0) = 1$.

If $x, y \in [-1,0]$, then we have $d(T^2x, T^2y) = (x-y)^2$ and

$$\begin{aligned} E^*(x,y) &= d(x,y) + \left|d(x,Tx) - d(Ty,T^2y)\right| + \left|d(y,Ty) - d(Tx,T^2x)\right| \\ &= (x-y)^2 + \left|(x+\tfrac{x}{2})^2 - (y-\tfrac{y}{2})^2\right| + \left|(y+\tfrac{y}{2})^2 - (x-\tfrac{x}{2})^2\right| \\ &= (x-y)^2 + \left|(\tfrac{3x}{2})^2 - (\tfrac{y}{2})^2\right| + \left|(\tfrac{3y}{2})^2 - (\tfrac{x}{2})^2\right| \\ &= (x-y)^2 + \left|\tfrac{9x^2-y^2}{4}\right| + \left|\tfrac{9y^2-x^2}{4}\right|. \end{aligned}$$

Thus, we can find $k \in [0,1)$ such that

$$\alpha(x,y)d(T^2x,T^2y) = (x-y)^2 \leq k \cdot \left((x-y)^2 + \left|\frac{9x^2-y^2}{4}\right| + \left|\frac{9y^2-x^2}{4}\right|\right) = k \cdot E^*(x,y).$$

Otherwise, we have $\alpha(x,y) = 0$.
Consequently, from Theorem 5 the mapping T has a fixed point.

Theorem 6. *Under the assumption of Theorem 5, if $\alpha(u,v) \geq 1$ for every $u, v \in Fix_T(\mathcal{M})$, then the mapping T has a unique fixed point.*

Proof. If you suppose that there are two points $u, v \in \mathcal{M}$ such that $Tu = u \neq v = Tu$, whose existence is ensured by Theorem 5, then we have

$$\begin{aligned} d(u,v) &= d(T^2u, T^2v) \leq \alpha(u,v)d(T^2u,T^2v) \\ &\leq k \cdot E^*(u,v) + \lambda \cdot N(u,v) \\ &\leq k \cdot d(u,v) < d(u,v). \end{aligned}$$

That is a contradiction, so that $d(u,v) = 0$ and then the fixed point of T is unique. □

Theorem 7. *On a complete b-metric space $(\mathcal{M}, d, s)$, each self-mapping T has a unique fixed point provided that:*

(i) There exist $k \in [0,1)$ and $\lambda > 0$ such that

$$d(T^2x, T^2y) \leq k \cdot E^*(x,y) + \lambda \cdot N(x,y)$$

for any $x, y \in \mathcal{M}$.

(ii) Either T is continuous or T^2 is continuous.

Proof. It is enough to take $\alpha(x,y) = 1$ in Theorem 6. □

3. Consequences for the Case of Metric Spaces

Letting $s = 1$ in our previous theorems, we get the following results in complete metric spaces.

Theorem 8. *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ an α-almost Istrăţescu contraction of type E such that:*

1. *T is continuous; or*
2. *T^2 is continuous and $\alpha(Tu, u) \geq 1$ for any $u \in Fix_{T^2}(\mathcal{M})$.*

Suppose that T is $\alpha-$orbital admissible and there exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, Tx_0) \geq 1$. Then, T has a fixed point.

Theorem 9. *Let $(\mathcal{M}, d)$ be a complete metric space and $T : \mathcal{M} \to \mathcal{M}$ an α-almost Istrăţescu contraction of type E^* such that:*

1. *T is continuous; or*
2. *T^2 is continuous and $\alpha(Tu, u) \geq 1$ for any $u \in Fix_{T^2}(\mathcal{M})$.*

Suppose that T is $\alpha-$orbital admissible and there exists $x_0 \in \mathcal{M}$ such that $\alpha(x_0, Tx_0) \geq 1$. Then, T has a fixed point.

In the following examples, we show that there are mappings that are α-almost Istrăţescu contraction of type E^* but not α-almost Istrăţescu contraction of type E.

Example 4. *For $\mathcal{M} = [0,4]$, consider the standard metric $d : \mathcal{M} \times \mathcal{M} \to [0, \infty)$, that is, $d(x,y) = |x - y|$. Let the mapping $\alpha : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ defined as $\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in (2,4] \\ 2, & \text{if } x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$.*

A self-mapping T on $\mathcal{M}$ is defined by $T(x) = \begin{cases} 1, & \text{if } x, y \in [0,1] \\ 2x, & \text{if } x, y \in (1,2] \\ \frac{x}{2}, & \text{if } x, y \in (2,4] \end{cases}$.

We have $T^2(x,y) = \begin{cases} 1, & \text{if } x, y \in [0,1] \\ x, & \text{if } x, y \in (1,4] \end{cases}$ and we can remark that the mapping T^2 is continuous, but T is not. Withal, T is α-orbital admissible and, for example we have $\alpha(1, T1) = \alpha(T1, 1) = \alpha(1,1) = 2 > 1$. For $x, y \in [0,1]$, we have $d(T^2x, T^2y) = d(1,1) = 0$, thus T is an α-almost Istrăţescu contraction of type E^*. For $x, y \in (2,4]$,

$$\begin{aligned} E^*(x,y) &= d(x,y) + \left| d(x,Tx) - d(Ty, T^2y) \right| + \left| d(y,Ty) - d(Tx, T^2x) \right| \\ &= |x-y| + 2\left| \left| x - \tfrac{x}{2} \right| - \left| y - \tfrac{y}{2} \right| \right| \\ &= 2\,|x-y|, \end{aligned}$$

and for $k = \frac{3}{4}$ and $\lambda = 0$ we have

$$\begin{aligned} \alpha(x,y)d(T^2u,T^2v) &= d(T^2x,T^2y) = |x-y| \\ &\leq \tfrac{6}{4} \cdot |x-y| = k \cdot E^*(x,y) \end{aligned}$$

The other cases are not interesting due to the way the function α is defined. Accordingly all the assumption of Theorem 9 are satisfied, so that T has a fixed point.

On the other hand, for any $x, y \in (2,4]$, $x \neq y$, we have

$$\begin{aligned} E(x,y) &= d(Tx,Ty) + \left|d(Tx,T^2x) - d(Ty,T^2y)\right| \\ &= \left|\tfrac{x}{2} - \tfrac{y}{2}\right| + \left|\left|x - \tfrac{x}{2}\right| - \left|y - \tfrac{y}{2}\right|\right| \\ &= \tfrac{|x-y|}{2} + \tfrac{|x-y|}{2} = |x-y| \end{aligned}$$

and then

$$\alpha(x,y)d(T^2x,T^2y) \quad = d(T^2x,T^2y) = |x-y| > k\,|x-y| = E(x,y)$$

for every $k \in [0,1)$, so T is not an α-almost Istrăţescu contraction of type E.

Theorem 10. *Under the assumptions of Theorems 8 and 9 , respectively, the mapping T has a unique the fixed point, provided that for any $y \in \mathcal{M}$*

$$\alpha(u,y) \geq 1, \text{ where } u \in Fix_T(\mathcal{M}). \tag{22}$$

Moreover, taking $\alpha(x,y) = 1$ and $\lambda = 0$, we have:

Corollary 2. *Suppose that a self-mapping T, on a complete metric space $(\mathcal{M}, d)$, fulfills*

$$d(T^2x,T^2y) \quad \leq k \cdot E(x,y), \tag{23}$$

for all $x, y \in \mathcal{M}$. If, eitherT or T^2 is continuous, then T possesses a unique fixed point.

Corollary 3. *Suppose that a self-mapping T, on a complete metric space, fulfills*

$$d(T^2x,T^2y) \quad \leq k \cdot E^*(x,y), \tag{24}$$

for all $x, y \in \mathcal{M}$. If, eitherT or T^2 is continuous, then T possesses a unique fixed point.

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