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Existence of Positive Solutions to Singular φ-Laplacian Nonlocal Boundary Value Problems when φ is a Sup-multiplicative-like Function

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Abstract: In this paper, using a fixed point index theorem on a cone, we present some existence results for one or multiple positive solutions to φ-Laplacian nonlocal boundary value problems when φ is a sup-multiplicative-like function and the nonlinearity may not satisfy the L^1-Carathéodory condition.

Keywords: multiplicity of positive solutions; sup-multiplicative-like function; singular problem; nonlocal boundary conditions

1. Introduction

The nonlocal boundary value problems play an important role in physics and applied mathematics such as heat conduction, chemical engineering, thermo-elasticity and plasma physics (see, e.g., [1–7]). For this reason, the existence of positive solutions for nonlocal boundary value problems have been extensively studied (see, e.g., [8–30] and references therein). For an introduction to nonlocal boundary value problems, we refer to the survey papers [31–35].

In this paper, we study the existence and multiplicity of positive solutions to the following boundary value problem (BVP)

$$(q(t)\varphi(u'(t)))' + h(t)f(t,u(t)) = 0, \ \ t \in (0,1), \tag{1}$$

$$u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r), \tag{2}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $q \in C([0,1],(0,\infty))$, $f \in C([0,1] \times (0,\infty), \mathbb{R})$, $h \in C((0,1), \mathbb{R}_+)$, $\mathbb{R}_+ := [0,\infty)$, and the integrator functions α_i $(i = 1,2)$ are nondecreasing on $[0,1]$.

All integrals in (2) are meant in the sense of Riemann–Stieljes. Throughout this paper, we assume the following hypotheses, unless otherwise stated.

(F_1) There exists an increasing homeomorphism $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \le \varphi(xy) \text{ for all } \ x,y \in \mathbb{R}_+. \tag{3}$$

(F_2) For $i = 1,2$, $\hat{\alpha}_i := \alpha_i(1) - \alpha_i(0) \in [0,1)$.

(F_3) $h \in \mathcal{H}_\varphi \setminus \{0\}$, where

$$\mathcal{H}_\varphi = \left\{ g \in C((0,1),\mathbb{R}_+) : \int_0^1 \left| \varphi^{-1}\left(\int_s^{\frac{1}{2}} g(\tau)d\tau \right) \right| ds < \infty \right\}.$$

In [36], Wang introduced a condition $(H1)$ on an odd increasing homeomorphism φ:

$(H1)$ there exist increasing homeomorphisms $\psi_1, \psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(x)\psi_1(y) \leq \varphi(xy) \leq \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+$$

and investigated the existence, multiplicity and nonexistence of positive solutions to quasilinear boundary value problems. The proofs are based upon a result on the fixed point index for compact operators on a Banach space. The odd increasing homeomorphism φ satisfying the condition (F_1) is called a sup-multiplicative-like function which was introduced by Karakostas ([13,14]). When φ is super-multiplicative-like, in [13,14], the author provided sufficient conditions for the existence of positive solutions of the one dimensional differential equation with deviating arguments.

Any function of the form

$$\varphi(s) = \sum_{k=1}^{n} c_k |s|^{p_k - 2} s$$

is sup-multiplicative-like, where $c_k \geq 0$ and $p_k \in (1, \infty)$ for $1 \leq k \leq n$ and $c_1 c_n > 0$ for some $n \in \mathbb{N}$. In this case, it is easy to see that ψ_1 in the assumption (F_1) can be defined by $\psi_1(s) = \min\{s^{p_n - 1}, s^{p_1 - 1}\}$ for $s \in \mathbb{R}_+$ (see, e.g., [13] or Remark 5 (3) below). If $n = 1$, it follows that $\varphi(s) = |s|^{p-2}s$ for some $p \in (1, \infty)$, so that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in \mathbb{R}$ and an increasing homeomorphism $\psi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (3) can be chosen as $\psi_1 \equiv \varphi$ on $\mathbb{R}_+$.

As pointed out in [37], the assumption (F_1) is equivalent to the one $(H1)$. Indeed, let us define $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi_2(0) := 0 \text{ and } \psi_2(y) := \Big(\psi_1(y^{-1})\Big)^{-1} \text{ for } y > 0. \tag{4}$$

Then ψ_2 is an increasing homeomorphism on $\mathbb{R}_+$. From (3), it follows that $0 < \varphi(xy)\psi_1(y^{-1}) \leq \varphi(x)$ for $x, y > 0$. Consequently, one has the following inequality:

$$\varphi(xy) \leq \varphi(x)\psi_2(y) \text{ for all } x, y \in \mathbb{R}_+, \tag{5}$$

and the assumption (F_1) is equivalent to the one $(H1)$. Moreover, it is well known that

$$\varphi^{-1}(x)\psi_2^{-1}(y) \leq \varphi^{-1}(xy) \leq \varphi^{-1}(x)\psi_1^{-1}(y) \text{ for all } x, y \in \mathbb{R}_+. \tag{6}$$

(see, e.g., ([37], Remark 1)). Clearly, $L^1(0,1) \cap C(0,1) \subseteq \mathcal{H}_\varphi$ and there may be a function $h \in \mathcal{H}_\varphi \setminus L^1(0,1)$ (see, e.g., Section 4 below). Consequently, the nonlinearity $h(t)f(t,u)$ in the Equation (1) may not satisfy the L^1-Carathéodory condition.

When $\varphi(s) = s$ and $q \equiv h \equiv 1$, Henderson and Thompson [38] proved the existence of at least three symmetric positive solutions to problem (1) subject to Dirichlet boundary conditions $u(0) = u(1) = 0$ (i.e., $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$) under the assumptions on the nonlinear term $f = f(u)$ that $f(0) > 0$, $f(u) < 8a$ for $0 \leq u \leq a$, $f(u) \geq 16b$ for $b \leq u \leq 2b$ and $0 \leq f(u) \leq 8c$ for $0 \leq u \leq c$, where $0 < a < b < \frac{c}{2}$. Liu [15] studied the following four-point boundary value problem

$$\begin{cases} u'' + h(t)f(u) = 0, \ \ t \in (0,1), \\ u(0) = \mu_0 u(\xi_0), \ u(1) = \mu_1 u(\xi_1), \end{cases} \tag{7}$$

which is a special case of BVP (1) and (2). Here, $0 < \xi_0 \leq \xi_1 < 1$, $0 < \mu_0 < \frac{1}{1-\xi_0}$, $0 < \mu_1 < \frac{1}{\xi_1}$, $\mu_0\xi_0(1-\mu_1) + (1-\mu_0)(1-\mu_1\xi_1) > 0$, $h \in C[0,1]$ with $h(t_0) > 0$ for some $t_0 \in [\frac{1}{4}, \frac{3}{4}]$ and $f \in C(\mathbb{R}_+, \mathbb{R}_+)$. Under several assumptions on $f = f(u)$, the existence of one or two positive solutions to problem (7) were shown. Later on, Kwong and Wong [16] improved a result in [15] on the existence of a positive solution to problem (7) with an alternative proof. When $\varphi(s) = s$ and $q \equiv 1$, Webb and Infante [19] studied problem (1) subject to several nonlocal boundary conditions involving a Stieltjes integral with a signed measure. The authors defined a suitable cone in $C[0,1]$ instead of the standard positive cone of nonnegative functions to use fixed point index theory and gave some sufficient conditions on the nonlinear term $f = f(t,u)$ for the existence and multiplicity of positive solutions. When $\varphi(s) = |s|^{p-2}s$ for some $p \in (1,\infty)$, $q \equiv 1$ and $h \in L^1(0,1)$ with $h(t) \not\equiv 0$ on any subinterval of $(0,1)$, Feng, Ge and Jiang [23] presented sufficient conditions on the nonlinear term $f = f(t,u)$ for the existence of multiple positive solutions to problem (1) subject to multi-point boundary conditions. Kim [24] improved on the results in [23] by assuming weaker hypotheses to the weight function h and the nonlinear term $f(t,u)$ than those in [23]. When φ is sub-multiplicative, i.e.,

$$\varphi(xy) \le \varphi(x)\varphi(y)) \text{ for all } x,y \in \mathbb{R}_+.$$

Bachouche, Djebali and Moussaoui [39] studied parameter-dependent φ-Laplacian boundary value problems

$$\begin{cases} (\varphi(u'))' + \lambda f(t,u,u') = 0, \ \ t \in (0,1), \\ u(0) = L_0(u), \ u(1) = L_1(u), \end{cases}$$

where $\lambda > 0$, L_i is a bounded linear operator for $i = 1,2$, and the nonlinearity $f = f(t,u,v)$ satisfies L^1-Carathéodory condition. The existence of a positive solution or a nonnegative solution was shown. For general φ satisfying (F_1) and $q \equiv h \equiv 1$, under some suitable assumptions on more general nonlinear term $f = f(t,u,v)$ satisfying $f(t,0,0) \not\equiv 0$ and $f(t,u,v) = f(1-t,u,-v)$ for $(t,u,v) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}$, Ding [27] showed the existence of at least three symmetric positive solutions to problem (1) subject to boundary conditions (2) with $\alpha_1 \equiv \alpha_2$ satisfying $\alpha_1' \in L^1(0,1)$. For more general φ which does not satisfy (F_1), Kaufmann and Milne [40] studied the following problem

$$\begin{cases} (\varphi(u'))' + \lambda h(t)f(u) = 0, \ \ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{8}$$

where $\lambda > 0$, $0 \le h \in L^1(0,1)$ with $h \not\equiv 0$ and $f \in C(\mathbb{R}_+,\mathbb{R}_+)$. The existence of positive solution to problem (8) was shown for all $\lambda > 0$ under the assumptions on f which induces the sublinear nonlinearity provided $\varphi(s) = |s|^{p-1}s$ with $p > 1$. When φ satisfies (F_1), h satisfies

$$\int_0^1 \left| \psi_1^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau \right) \right| ds < \infty$$

and $\displaystyle\lim_{s\to 0^+} \frac{f(s)}{\varphi(s)} = \lim_{s\to\infty} \frac{f(s)}{\varphi(s)} = \infty$, Lee and Xu [41] showed that there exist $\lambda^* \ge \lambda_* > 0$ such that (8) has at least two positive solutions for $\lambda \in (0,\lambda_*)$, one positive solution for $\lambda \in [\lambda_*,\lambda^*]$ and no positive solution for $\lambda > \lambda_*$. Recently, under the assumption that φ is an increasing homeomorphism such that $\varphi(0) = 0$, Feltrin, Sovrano and Zanolin [42] studied the periodic boundary value problem associated with φ-Laplacian equation of the form $(\varphi(u'))' + g(u)u' + k(t,u) = s$, where s is a real parameter, $g \in C(\mathbb{R},\mathbb{R})$ g and k are continuous functions and k is T-periodic in the variable t. They showed the Ambrosetti–Prodi type alternatives which provide the existence of zero, one or two solutions depending on the choice of the parameter s. For other interesting results, we refer the reader to [43–46] and the references therein.

Motivated by the papers mentioned above, we present some existence results for one or multiple positive solutions to BVP (1) and (2) by means of a fixed point index theorem. To this end, we define a suitable positive cone in $C[0,1]$ on which a solution operator related to BVP (1) and (2) is well defined. We remark that if q is not a positive constant function on $[0,1]$, the solutions to BVP (1) and (2) may not be concave down on $[0,1]$, even though the nonlinearity $h(t)f(t,u)$ is nonnegative for all $(t,u) \in (0,1) \times (0,\infty)$ (see, e.g., ([37], Remark 2 (1))).

The rest of this paper is organized as follows. In Section 2, we give preliminary results which are essential for proving the main results in this paper. In Section 3, the main results are stated and proved. Finally, some examples to illustrate the main results are provided in Section 4.

2. Preliminaries

In this section, a solution operator related to BVP (1) and (2) with $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ is introduced and a well-known theorem of fixed point index theory is recalled.

The usual maximum norm in a Banach space $C[0,1]$ is denoted by

$$\|u\|_\infty := \max_{t\in[0,1]} |u(t)| \text{ for } u \in C[0,1],$$

and let $\alpha_h := \inf\{x \in (0,1) : h(x) > 0\}$, $\beta_h := \sup\{x \in (0,1) : h(x) > 0\}$, $\bar{\alpha}_h := \sup\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (\alpha_h, x)\}$, $\bar{\beta}_h := \inf\{x \in (0,1) : h(y) > 0 \text{ for all } y \in (x, \beta_h)\}$, $\gamma_h^1 := 4^{-1}(3\alpha_h + \bar{\alpha}_h)$, $\gamma_h^2 := 4^{-1}(\bar{\beta}_h + 3\beta_h)$ and $\gamma_h := 2^{-1}(\gamma_h^1 + \gamma_h^2)$. Then, since $h \not\equiv 0$, it follows that

$$0 \le \alpha_h < \gamma_h^1 < \gamma_h < \gamma_h^2 < \beta_h \le 1 \text{ and } h(t) > 0 \text{ for } t \in (\alpha_h, \bar{\alpha}) \cup (\bar{\beta}_h, \beta_h). \tag{9}$$

Let $\rho_h := \rho_1 \min\{\gamma_h^1, 1 - \gamma_h^2\} \in (0,1)$, where

$$q_0 := \min_{t\in[0,1]} q(t) > 0 \text{ and } \rho_1 := \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1].$$

Then $\mathcal{K} := \{u \in C([0,1], \mathbb{R}_+) : u(t) \ge \rho_h \|u\|_\infty \text{ for } t \in [\gamma_h^1, \gamma_h^2]\}$ is a cone in $C[0,1]$. For $r > 0$, let

$$\mathcal{K}_r := \{u \in \mathcal{K} \,:\, \|u\|_\infty < r\}, \partial\mathcal{K}_r := \{u \in \mathcal{K} \,:\, \|u\|_\infty = r\} \text{ and } \overline{\mathcal{K}}_r := \mathcal{K}_r \cup \partial\mathcal{K}_r.$$

Now, we introduce a solution operator related to BVP (1) and (2). Let $g \in \mathcal{H}_\varphi$ be given. Define functions $\nu_g^1, \nu_g^2 : (0,1) \to \mathbb{R}$ by, for $x \in (0,1)$,

$$\nu_g^1(x) = A_1 \int_0^1 \int_0^r I_g(s,x) ds d\alpha_1(r) + \int_0^x I_g(s,x) ds$$

and

$$\nu_g^2(x) = -A_2 \int_0^1 \int_r^1 I_g(s,x) ds d\alpha_2(r) - \int_x^1 I_g(s,x) ds.$$

Here,

$$A_i := (1 - \hat{\alpha}_i)^{-1} \in [1, \infty) \text{ for } i = 1,2 \text{ and } I_g(s,x) = \varphi^{-1}\left(\frac{1}{q(s)} \int_s^x g(\tau) d\tau\right).$$

Remark 1. *We give the properties of $I_g^1(s,x)$ for any given $g \in \mathcal{H}_\varphi$ as follows.*

(1) $I_g(s,x) \ge 0$ *for* $0 < s \le x < 1$ *and* $I_g^1(s,x) \le 0$ *for* $0 < x \le s < 1$.
(2) $I_g(s,x_1) \le I_g^1(s,x_2)$ *for any* $s \in (0,1)$ *and* $0 < x_1 \le x_2 < 1$.
(3) *Let* $x \in (0,1)$ *be given. For any* $\epsilon \in [0, \min\{x, 1-x\})$, *there exists* $C = C(x,\epsilon) > 0$ *satisfying*

$$\int_0^1 |I_g(s,x)| \, ds \le C.$$

Indeed, by (6),

$$\begin{aligned}\int_0^1 |I_g(s,x)| \, ds &= \int_0^x \varphi^{-1}\left(\frac{1}{q(s)} \int_s^x g(\tau) d\tau\right) ds + \int_x^1 \varphi^{-1}\left(\frac{1}{q(s)} \int_x^s g(\tau) d\tau\right) ds \\ &\le \psi_1^{-1}\left(\frac{1}{q_0}\right)\left[\int_0^{x+\epsilon} \varphi^{-1}\left(\int_s^{x+\epsilon} g(\tau) d\tau\right) ds + \int_{x-\epsilon}^1 \varphi^{-1}\left(\int_{x-\epsilon}^s g(\tau) d\tau\right) ds\right] < \infty.\end{aligned}$$

From Remark 1, it follows that ν_g^1 is well defined and it is a monotonically increasing continuous function on $(0,1)$ (see, e.g., [37]). Similarly, ν_g^2 is a monotonically decreasing continuous function on $(0,1)$.

For $g \equiv 0$, clearly $\nu_0^1 \equiv \nu_0^2 \equiv 0$ on $[0,1]$.

Lemma 1. *Assume that $(F_1), (F_2)$ and $g \in \mathcal{H}_\varphi \setminus \{0\}$ hold. Then there exist an interval $[\sigma_g^1, \sigma_g^2] \subseteq (0,1)$ and a constant C_g satisfying $\nu_g^1 \equiv \nu_g^2 \equiv C_g$ on $[\sigma_g^1, \sigma_g^2]$.*

Proof. Let $g \in \mathcal{H}_\varphi \setminus \{0\}$ be given. First we prove $\lim_{x \to 0^+} \nu_g^1(x) \in [-\infty, 0]$. In order to show it, we rewrite $\nu_g^1(x)$ by, for $x \in (0,1)$,

$$\begin{aligned}\nu_g^1(x) &= A_1 \left(\int_0^1 \int_0^r I_g(s,x) ds d\alpha_1(r) + \left(1 - \int_0^1 d\alpha_1(r)\right) \int_0^x I_g(s,x) ds \right) \\ &= A_1 \left(\int_0^1 \int_x^r I_g(s,x) ds d\alpha_1(r) + \int_0^x I_g(s,x) ds \right).\end{aligned}$$

For any $x \in (0,1)$, by Remark 1 (1),

$$\int_0^1 \int_x^r I_g(s,x) ds d\alpha_1(r) = -\int_0^x \int_r^x I_g(s,x) ds d\alpha_1(r) + \int_x^1 \int_x^r I_g(s,x) ds d\alpha_1(r) \le 0.$$

Consequently,

$$\nu_g^1(x) \le A_1 \int_0^x I_g(s,x) ds \text{ for any } x \in (0,1). \tag{10}$$

Since $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$\lim_{x \to 0^+} \int_0^x I_g(s,x) ds = 0$$

and thus, by (10),

$$\lim_{x \to 0^+} \nu_g^1(x) \in [-\infty, 0].$$

Next we show $\lim_{x \to 1^-} \nu_g^1(x) \in (0, \infty]$. For any $x \in (0,1)$,

$$\begin{aligned}\nu_g^1(x) = A_1 \Big[\int_0^x \int_0^r I_g(s,x) ds d\alpha_1(r) \quad &+ \int_x^1 \int_0^x I_g(s,x) ds d\alpha_1(r) \\ &+ \int_x^1 \int_x^r I_g(s,x) ds d\alpha_1(r) \Big] + \int_0^x I_g(s,x) ds.\end{aligned}$$

Then, by $g \in \mathcal{H}_\varphi \setminus \{0\}$,

$$\lim_{x \to 1^-} \int_0^x I_g(s,x) ds \in (0, \infty]$$

and, for any $x \in (0,1)$,

$$\int_0^x \int_0^r I_g(s,x) ds d\alpha_1(r) + \int_x^1 \int_0^x I_g(s,x) ds d\alpha_1(r) \ge 0.$$

For $x > 1/2$, by (6),

$$\begin{aligned}\left| \int_x^1 \int_x^r I_g(s,x) ds d\alpha_1(r) \right| &= \int_x^1 \int_x^r \varphi^{-1}\left(\frac{1}{q(s)} \int_x^s g(\tau) d\tau \right) ds d\alpha_1(r) \\ &\le \psi_1^{-1}\left(\frac{1}{q_0}\right) \int_x^1 d\alpha_1(r) \int_x^1 \varphi^{-1}\left(\int_{\frac{1}{2}}^s g(\tau) d\tau \right) ds,\end{aligned}$$

which implies

$$\lim_{x\to 1^-}\int_x^1\int_x^r I_g(s,x)dsd\alpha_1(r)=0.$$

Consequently, $\lim\limits_{x\to 1^-}\nu_g^1(x)\in(0,\infty]$.

Similarly, it can be shown that

$$\lim_{x\to 0^+}\nu_g^2(x)\in(0,\infty] \text{ and } \lim_{x\to 1^-}\nu_g^2(x)\in[-\infty,0].$$

Thus, there exist an interval $[\sigma_g^1,\sigma_g^2]\subseteq(0,1)$ and a constant C_g satisfying

$$\nu_g^1\equiv\nu_g^2\equiv C_g \text{ on } [\sigma_g^1,\sigma_g^2],$$

by continuity and monotonicity of ν_g^1 and ν_g^2. Thus the proof is complete. □

Define an operator $T:\mathcal{H}_\varphi\to C[0,1]$ by, for $g\in\mathcal{H}_\varphi$,

$$T(g)(t)=\begin{cases} A_1\int_0^1\int_0^r I_g(s,\sigma)dsd\alpha_1(r)+\int_0^t I_g(s,\sigma)ds, & \text{if } 0\le t\le\sigma,\\ -A_2\int_0^1\int_r^1 I_g(s,\sigma)dsd\alpha_2(r)-\int_t^1 I_g(s,\sigma)ds, & \text{if } \sigma\le t\le 1,\end{cases} \tag{11}$$

where $\sigma=\sigma(g)$ is a constant satisfying $\nu_g^1(\sigma)=\nu_g^2(\sigma)$, i.e.,

$$A_1\int_0^1\int_0^r I_g(s,\sigma)dsd\alpha_1(r)+\int_0^\sigma I_g(s,\sigma)ds=-A_2\int_0^1\int_r^1 I_g(s,\sigma)dsd\alpha_2(r)-\int_\sigma^1 I_g(s,\sigma)ds. \tag{12}$$

Clearly, $T(0)\equiv 0$, and for any $g\in\mathcal{H}_\varphi$ and any $\sigma\in[\sigma_g^1,\sigma_g^2]$, $T(g)$ is monotone increasing on $[0,\sigma)$ and monotone decreasing on $(\sigma,1]$.

Remark 2. *We notice that, although $\sigma=\sigma(g)$ is not necessarily unique, by Lemma 1, $T(g)$ is independent of the choice of $\sigma\in[\sigma_g^1,\sigma_g^2]$. Indeed, let $\sigma\in(\sigma_g^1,\sigma_g^2]$ be fixed and T_1 be the operator defined as (11) with $\sigma=\sigma_g^1$. By Lemma 1 and Remark 1* (1) *and* (2)*,*

$$\begin{aligned} & A_1\int_0^1\int_0^r I_g(s,\sigma_g^1)dsd\alpha_1(r)+\int_0^{\sigma_g^1} I_g(s,\sigma_g^1)ds\\ = & T_1(\sigma_g^1)=\nu_g^1(\sigma_g^1)=\nu_g^1(\sigma)=T(g)(\sigma)\\ = & A_1\int_0^1\int_0^r I_g(s,\sigma)dsd\alpha_1(r)+\int_0^{\sigma} I_g(s,\sigma)ds\\ \ge & A_1\int_0^1\int_0^r I_g(s,\sigma_g^1)dsd\alpha_1(r)+\int_0^{\sigma_g^1} I_g(s,\sigma)ds,\end{aligned}$$

which implies $\int_0^{\sigma_g^1}\left(I_g(s,\sigma)ds-I_g(s,\sigma_g^1)\right)ds=0$. Consequently, $g(\tau)=0$ for all $\tau\in[\sigma_g^1,\sigma]$, so that $I_g(s,\sigma_g^1)=I_g(s,\sigma)$ for all $s\in(0,1)$, which implies that $T_1\equiv T(g)$. Thus $T(g)$ is independent of the choice of $\sigma\in[\sigma_g^1,\sigma_g^2]$.

Consider the following problem

$$\begin{cases}(q(t)\varphi(u'(t)))'+g(t)=0, \ \ t\in(0,1),\\ u(0)=\int_0^1 u(r)d\alpha_1(r), u(1)=\int_0^1 u(r)d\alpha_2(r).\end{cases} \tag{13}$$

Lemma 2. *Assume that* (F_1), (F_2) *and* $g\in\mathcal{H}_\varphi$ *hold. Then* $T(g)$ *is a unique solution to problem* (13) *satisfying the following properties:*

(*i*) $T(g)(t)\ge\min\{T(g)(0),T(g)(1)\}\ge 0$ *for* $t\in[0,1]$;

(*ii*) *for any* $g \not\equiv 0$, $\max\{T(g)(0), T(g)(1)\} < \|T(g)\|_\infty$*;*
(*iii*) σ *is a constant satisfying* (12) *if and only if* $T(g)(\sigma) = \|T(g)\|_\infty$*;*
(*iv*) $T(g)(t) \geq \rho_1 \min\{t, 1-t\}\|T(g)\|_\infty$ *for* $t \in [0,1]$ *and* $T(g) \in \mathcal{K}$.

Proof. First, we show that $T(g)$ is a solution to problem (13). Clearly, $T(0) \equiv 0$ is a solution to problem (13) with $g \equiv 0$. Let $g \in \mathcal{H}_\varphi \setminus \{0\}$. Then, by Lemma 1 and (11), $\sigma \in [\sigma_g^1, \sigma_g^2] \subseteq (0,1)$ and

$$(T(g))'(s) = I_g(s,\sigma) \text{ for } s \in (0,1),$$

which implies $T(g)$ satisfies the equation in (13) and

$$T(g)(r) = T(g)(0) + \int_0^r I_g(s,\sigma)ds \text{ for } r \in [0,1].$$

Integrating this from 0 to 1 and using the fact

$$\int_0^1 \int_0^r I_g(s,\sigma)dsd\alpha_1(r) = A_1^{-1}T(g)(0) = (1-\hat{\alpha}_1)T(g)(0),$$

it follows that

$$\int_0^1 T(g)(r)d\alpha_1(r) = \hat{\alpha}_1 T(g)(0) + \int_0^1 \int_0^r I_g(s,\sigma)dsd\alpha_1(r) = T(g)(0).$$

Similarly, it can be shown that

$$T(g)(1) = \int_0^1 T(g)(r)d\alpha_2(r).$$

Thus $T(g)$ is a solution to problem (13).

Since $T(g)$ is monotone increasing on $[0,\sigma)$ and is monotone decreasing on $(\sigma, 1]$,

$$T(g)(t) \geq \min\{T(g)(0), T(g)(1)\} \text{ for } t \in [0,1].$$

We only consider the case

$$\min\{T(g)(t) : 0 \leq t \leq 1\} = T(g)(0),$$

since the case $\min\{T(g)(t) : 0 \leq t \leq 1\} = T(g)(1)$ is similar. Then

$$T(g)(0) = \int_0^1 T(g)(r)d\alpha_1(r) \geq \hat{\alpha}_1 T(g)(0),$$

which implies $T(g)(0) \geq 0$, since $\hat{\alpha}_1 = \int_0^1 d\alpha_1(r) \in [0,1)$. Consequently,

$$T(g) \geq \min\{T(g)(0), T(g)(1)\} \geq 0.$$

Moreover, it is easy to see that σ is a constant satisfying (12) if and only if $T(g)(\sigma) = \|T(g)\|_\infty$.

Let u_g be a solution to problem (13) with $g \in \mathcal{H}_\varphi$. Since

$$u_g(0) = \int_0^1 u_g(r)d\alpha_1(r) \leq \hat{\alpha}_1\|u_g\|_\infty,$$

$u_g(0) < \|u_g\|_\infty$, provided $u_g \not\equiv 0$. Similarly, $u_g(1) < \|u\|_\infty$, provided $u_g \not\equiv 0$.

For $g \equiv 0$, 0 is a unique solution to problem (13). Indeed, assume on the contrary that $u_0 \not\equiv 0$, which implies $\max\{u_0(0), u_0(1)\} < \|u_0\|_\infty$. From the equation in (13), it follows that

$$u_0(t) = u_0(0) + \int_0^t \varphi^{-1}\left(\frac{C}{q(s)}\right) ds \text{ for some constant } C \text{ and for } t \in [0,1].$$

Then u_0 is a monotonic function on $[0,1]$, so that

$$\min\{u_0(t) : t \in [0,1]\} = \min\{u_0(0), u_0(1)\} \text{ and } \|u_0\|_\infty = \max\{|u_0(0)|, |u_0(1)|\}.$$

From boundary conditions in (13), it follows that $u_0(0) \geq 0$ and $u_0(1) \geq 0$. Consequently

$$\|u_0\|_\infty = \max\{u_0(0), u_0(1)\},$$

which contradicts $u_0 \not\equiv 0$. Thus, $u_0 \equiv 0 \equiv T(0)$.

Let $g \not\equiv 0$. Then $u_g \not\equiv 0$, so that $\max\{u_g(0), u_g(1)\} < \|u_g\|_\infty$ and there exists $\sigma \in (0,1)$ satisfying $u_g'(\sigma) = 0$. Direct calculation yields

$$u_g(r) = u_g(0) + \int_0^r I_g(s,\sigma)ds = u_g(1) - \int_r^1 I_g(s,\sigma)ds \text{ for } r \in [0,1].$$

By boundary conditions in (13),

$$u_g(0) = A_1 \int_0^1 \int_0^r I_g(s,\sigma)dsd\alpha_1(r) \text{ and } u_g(1) = -A_2 \int_0^1 \int_r^1 I_g(s,\sigma)dsd\alpha_2(r).$$

Consequently, $u_g \equiv T(g)$.

So far we have shown that $T(g)$ is a unique solution to problem (13) satisfying (i), (ii) and (iii). Finally, we show that $T(g)$ satisfies (iv). For $g \equiv 0$, the conclusion is clear. Let $g \in \mathcal{H}_\varphi \setminus \{0\}$ and let $\sigma \in (0,1)$ be a constant satisfying (12), i.e., $T(g)(\sigma) = \|T(g)\|_\infty > 0$. For $t \in [0,\sigma]$, by (6),

$$T(g)(t) = T(g)(0) + \int_0^t \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma g(\tau)d\tau\right)ds \geq T(g)(0) + \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)w_1(t). \tag{14}$$

Here $w_1(t) := \int_0^t \varphi^{-1}\left(\int_s^\sigma g(\tau)d\tau\right)ds$ for $t \in [0,\sigma]$. Similarly,

$$\|T(g)\|_\infty \leq T(g)(0) + \psi_1^{-1}\left(\frac{1}{q_0}\right)w_1(\sigma). \tag{15}$$

We claim that $\sigma \in (\alpha_g, \beta_g)$, where $\alpha_g := \inf\{x \in (0,1) : g(x) > 0\}$ and $\beta_g := \sup\{x \in (0,1) : g(x) > 0\}$. Indeed, if $\sigma \in [0,\alpha_g] \cup [\beta_g, 1]$, then $0 < T(g)(\sigma) = \max\{T(g)(0), T(g)(1)\}$, which contradicts Theorem 2(ii). Thus w_1 is a nondecreasing concave function on $[0,\sigma]$ with $w_1(0) = 0$ and $w_1(t) > 0$ for $t \in (0,\sigma]$, so that $w_1(t) \geq tw_1(\sigma)$ for $t \in [0,\sigma]$. Consequently, by (14) and (15),

$$T(g)(t) - T(g)(0) \geq \rho_1 t(\|T(g)\|_\infty - T(g)(0)),$$

which implies

$$T(g)(t) \geq \rho_1 t\|T(g)\|_\infty + (1-\rho_1 t)T(g)(0) \geq \rho_1 t\|T(g)\|_\infty \text{ for } t \in [0,\sigma].$$

Recall that $\rho_1 = \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\left[\psi_1^{-1}\left(\frac{1}{q_0}\right)\right]^{-1} \in (0,1]$. Similarly, it can be shown that

$$T(g)(t) \geq \rho_1(1-t)\|T(g)\|_\infty \text{ for } t \in [\sigma,1].$$

Consequently, $T(g)(t) \geq \rho_1 \min\{t, 1-t\}\|T(g)\|_\infty$ for $t \in [0,1]$. Clearly, $T(g) \in \mathcal{K}$ for any $g \in \mathcal{H}_\varphi$, and thus the proof is complete. □

For the rest of this section, we assume $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. Define a function $F : \mathcal{K} \to C(0,1)$ by

$$F(u)(t) = h(t)f(t,u(t)) \text{ for } u \in \mathcal{K} \text{ and } t \in (0,1).$$

Clearly, $F(u) \in \mathcal{H}_\varphi$ for any $u \in \mathcal{K}$, since $h \in \mathcal{H}_\varphi$. Let us define an operator $H : \mathcal{K} \to \mathcal{K}$ by

$$H(u) = T(F(u)) \text{ for } u \in \mathcal{K}.$$

By Lemma 2 (iv), $H(\mathcal{K}) \subseteq \mathcal{K}$ and consequently, H is well defined.

Lemma 3. *Assume that (F_1), (F_2), (F_3) and $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ hold. Let $M > 0$ be given and let (u_n) be a bounded sequence in $\mathcal{K}$ with $\|u_n\|_\infty \leq M$. If $\lim_{n\to\infty} \sigma_n \in \{0,1\}$, then*

$$\|H(u_n)\|_\infty \to 0 \text{ and } F(u_n)(t) \to 0 \text{ for any } t \in (0,1) \text{ as } n \to \infty.$$

Here, σ_n is a constant satisfying (12) with $g = F(u_n)$ for each $n \in \mathbb{N}$.

Proof. We only prove the case $\lim_{n\to\infty}\sigma_n = 0$, since the other case can be dealt in a similar manner. Since there exists $N > 0$ such that

$$F(u)(t) = h(t)f(t,u(t)) \le Nh(t) \text{ for all } (t,u) \in [0,1]\times[0,M],$$

by (6) and (10),

$$\begin{aligned} \|H(u_n)\|_\infty &= A_1\int_0^1\int_0^r I_{F(u_n)}(s,\sigma_n)dsd\alpha_1(r) + \int_0^{\sigma_n} I_{F(u_n)}(s,\sigma_n)ds \\ &\le A_1\int_0^{\sigma_n} I_{F(u_n)}(s,\sigma_n)ds \le A_1\psi_1^{-1}\left(\frac{N}{q_0}\right)\int_0^{\sigma_n}\varphi^{-1}\left(\int_s^{\sigma_n} h(\tau)d\tau\right)ds. \end{aligned}$$

Thus, from $h \in \mathcal{H}_\varphi$, it follows that $\|H(u_n)\|_\infty \to 0$ as $n\to\infty$.

Since $H(u_n)(1) \ge 0$ for all $n \in \mathbb{N}$, by (6),

$$0 \le \psi_2^{-1}\left(\frac{1}{\|q\|_\infty}\right)\int_{\sigma_n}^1 \varphi^{-1}\left(\int_{\sigma_n}^s F(u_n)(\tau)d\tau\right)ds \le -\int_{\sigma_n}^1 I_{F(u_n)}(s,\sigma_n)ds \le \|H(u_n)\|_\infty,$$

which implies that

$$\int_{\sigma_n}^1 \varphi^{-1}\left(\int_{\sigma_n}^s F(u_n)(\tau)d\tau\right)ds \to 0 \text{ as } n\to\infty.$$

Consequently, $\lim_{n\to\infty} F(u_n)(t) = 0$ for any $t\in(0,1)$. □

Using Lemma 3 and (6), by the similar arguments in the proof of ([24], Lemma 2.4), the following lemma can be proved, and so we omit the proof of it.

Lemma 4. *Assume that* $(F_1), (F_2), (F_3)$ *and* $f \in C([0,1]\times\mathbb{R}_+,\mathbb{R}_+)$ *hold. Then the operator* $H : \mathbb{R}_+\times\mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Remark 3. *Assume that all the assumptions in Lemma 4 are satisfied. By Lemma 2, it is easy to see that BVP* (1) *and* (2) *has a positive solution if and only if H has a fixed point in* $\mathcal{K}\setminus\{0\}$. *Moreover,* σ *is a constant satisfying* (12) *with* $g = F(u) \not\equiv 0$ *if and only if* $H(u)(\sigma) = \|H(u)\|_\infty > 0$.

Finally, we recall a well-known theorem of the fixed point index theory.

Theorem 1. (see, e.g., [47,48]) *Assume that, for some* $r > 0$, $H : \overline{\mathcal{K}}_r \to \mathcal{K}$ *is completely continuous. Then the following assertions are true.*

$$(i)\ i(H,\mathcal{K}_r,\mathcal{K}) = 1 \text{ if } \|H(u)\|_\infty < \|u\|_\infty \text{ for } u\in\partial\mathcal{K}_r;$$

$$(ii)\ i(H,\mathcal{K}_r,\mathcal{K}) = 0 \text{ if } \|H(u)\|_\infty > \|u\|_\infty \text{ for } u\in\partial\mathcal{K}_r.$$

3. Main Results

Let

$$C_1 := \frac{1}{\psi_1^{-1}(q_0^{-1})}\min\left\{\left(A_1\int_0^{\gamma_h}\varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\right)^{-1}, \left(A_2\int_{\gamma_h}^1\varphi^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right)^{-1}\right\}$$

and

$$C_2 := \frac{1}{\psi_2^{-1}(\|q\|_\infty^{-1})}\max\left\{\left(\int_{\gamma_h^1}^{\gamma_h}\varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right)ds\right)^{-1}, \left(\int_{\gamma_h}^{\gamma_h^2}\varphi^{-1}\left(\int_{\gamma_h}^s h(\tau)d\tau\right)ds\right)^{-1}\right\}.$$

Then, from (6) and (9), it follows that $0 < C_1 < C_2$.

Definition 1. *For* $g \in C([0,1]\times\mathbb{R}_+,\mathbb{R}_+)$, *we say that*

(i) *g satisfies (H_1^r) for some $r > 0$ if*

$$g^*(r) := \max\{g(t,s) : (t,s) \in [0,1] \times [0,r]\} < \psi_1(C_1)\psi_1(r);$$

(ii) *g satisfies (H_2^R) for some $R > 0$ if*

$$g_*(R) := \min\{g(t,s) : (t,s) \in \left[\gamma_h^1, \gamma_h^2\right] \times [\rho_h R, R]\} > \psi_2(C_2)\psi_1(R).$$

Lemma 5. *Assume that (F_1), (F_2), (F_3) and $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ hold, and that f satisfies (H_1^r) for some $r > 0$. Then $\|H(u)\|_\infty < r$ for any $u \in \partial\mathcal{K}_r$ and $i(H, \mathcal{K}_r, \mathcal{K}) = 1$.*

Proof. Let $u \in \partial\mathcal{K}_r$ be fixed. Since f satisfies (H_1^r) and $0 \le u(t) \le r$ for $t \in [0,1]$,

$$f(t, u(t)) < \psi_1(C_1)\psi_1(r) \text{ for } t \in [0,1]. \tag{16}$$

Let σ be a constant satisfying $H(u)(\sigma) = \|H(u)\|_\infty$. We only consider $\sigma \in [0, \gamma_h)$, since the case $\sigma \in [\gamma_h, 1]$ can be dealt in a similar manner. By (6), (10), (16) and the choice of C_1,

$$\begin{aligned} \|H(u)\|_\infty &= H(u)(0) + \int_0^\sigma I_{F(u)}(s,\sigma)ds \le A_1 \int_0^\sigma I_{F(u)}(s,\sigma)ds \\ &< A_1\psi_1^{-1}(q_0^{-1}) \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\psi_1(C_1)\psi_1(r)\right) ds \\ &\le A_1\psi_1^{-1}(q_0^{-1}) \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\psi_1(r)\right) ds\, C_1 \\ &\le A_1\psi_1^{-1}(q_0^{-1}) \int_0^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) dsC_1 r \le r = \|u\|_\infty. \end{aligned}$$

Thus, by Theorem 1 (i), $i(H, \mathcal{K}_r, \mathcal{K}) = 1$. □

Lemma 6. *Assume that (F_1), (F_2), (F_3) and $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ hold, and that f satisfies (H_2^R) for some $R > 0$. Then $\|H(u)\|_\infty > R$ for any $u \in \partial\mathcal{K}_R$ and $i(H, \mathcal{K}_R, \mathcal{K}) = 0$.*

Proof. Let $u \in \partial\mathcal{K}_R$ be fixed. Then $\rho_h R \le u(t) \le R$ for $t \in [\gamma_h^1, \gamma_h^2]$. Since f satisfies (H_2^R),

$$f(t, u(t)) > \psi_2(C_2)\psi_2(R) \text{ for } t \in [\gamma_h^1, \gamma_h^2]. \tag{17}$$

Let σ be a constant satisfying $H(u)(\sigma) = \|H(u)\|_\infty$. We only consider the case $\sigma \in [\gamma_h, 1)$, since the case $\sigma \in (0, \gamma_h)$ can be dealt in a similar manner. Since $H(u)(0) \ge 0$, it follows from (6), (17) and the choice of C_2 that

$$\begin{aligned} \|H(u)\|_\infty &\ge \int_0^\sigma \varphi^{-1}\left(\frac{1}{q(s)}\int_s^\sigma h(\tau)f(\tau, u(\tau))d\tau\right) ds \\ &> \psi_2^{-1}(\|q\|_\infty^{-1}) \int_{\gamma_h^1}^{\gamma_h} \varphi^{-1}\left(\int_s^{\gamma_h} h(\tau)d\tau\right) dsC_2 R \ge R = \|u\|_\infty. \end{aligned}$$

Then, by Theorem 1 (ii), $i(H, \mathcal{K}_R, \mathcal{K}) = 0$. □

Now, we give a result on the existence of positive solutions to BVP (1) and (2) with $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$.

Theorem 2. *Assume that (F_1), (F_2), (F_3) and $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ satisfies (H_1^r) and (H_2^R) for some r and R. Then the following assertions are true:*

(1) *If* $0 < r < R$, *then BVP* (1) *and* (2) *has a positive solution* u *satisfying* $r < \|u\|_\infty < R$. *Moreover, if* $f(t,0) \not\equiv 0$, *then BVP* (1) *and* (2) *has another positive solution* v *satisfying* $\|v\|_\infty < r$.

(2) *If* $0 < R < r$, *then BVP* (1) *and* (2) *has a positive solution* u *satisfying* $R < \|u\|_\infty < r$.

Proof. We only give the proof of (1), since the proof of (2) is similar. Since f satisfies (H_1^r) and (H_2^R), by Lemma 5 and 6,

$$i(H, \mathcal{K}_r, \mathcal{K}) = 1 \text{ and } i(H, \mathcal{K}_R, \mathcal{K}) = 0.$$

By the additivity property,

$$i(H, \mathcal{K}_R \setminus \overline{\mathcal{K}}_r, \mathcal{K}) = -1.$$

Then there exists $u \in \mathcal{K}_R \setminus \overline{\mathcal{K}}_r$ such that $H(u) = u$ by the solution property. Thus, by Remark 3, BVP (1) and (2) has a positive solution u satisfying $r < \|u\|_\infty < R$. Moreover, since $i(H, \mathcal{K}_r, \mathcal{K}) = 1$, BVP (1) and (2) has a nonnegative solution v satisfying $\|v\|_\infty < r$. If $f(t,0) \not\equiv 0$, then $v \not\equiv 0$, so that v is a positive solution to BVP (1) and (2) by Remark 3. □

The following corollary directly follows from Theorem 2.

Corollary 1. *Assume that* (F_1), (F_2), (F_3) *and* $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ *hold. Then the following assertions are true.*

(1) *Assume that* f *satisfies* $(H_1^{r_1})$, $(H_2^{R_1})$ *and* $(H_1^{r_2})$ *for some* r_1, R_1 *and* r_2 *satisfying* $0 < r_1 < R_1 < r_2$. *Then BVP* (1) *and* (2) *has two positive solutions* u_1, u_2 *satisfying* $r_1 < \|u_1\|_\infty < R_1 < \|u_2\|_\infty < r_2$. *Moreover, if* $f(t,0) \not\equiv 0$, *then BVP* (1) *and* (2) *has another positive solution* v *satisfying* $\|v\|_\infty < r_1$.

(2) *Assume that* f *satisfies* $(H_2^{R_1})$, $(H_1^{r_1})$ *and* $(H_2^{R_2})$ *for some* R_1, r_1 *and* R_2 *satisfying* $0 < R_1 < r_1 < R_2$. *Then BVP* (1) *and* (2) *has two positive solutions* u_1, u_2 *satisfying* $R_1 < \|u_1\|_\infty < r_1 < \|u_2\|_\infty < R_2$.

Definition 2. *For* $g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$, *we say that*

(i) *g satisfies* (H_1^0) *if* $g_0^* := \limsup_{v \to 0^+} \max_{t \in [0,1]} \dfrac{g(t,v)}{\psi_1(v)} < \psi_1(C_1)$;

(ii) *g satisfies* (H_1^∞) *if* $g_\infty^* := \limsup_{v \to \infty} \max_{t \in [0,1]} \dfrac{g(t,v)}{\psi_1(v)} < \psi_1(C_1)$;

(iii) *g satisfies* (H_2^0) *if* $g_*^0 := \liminf_{v \to 0^+} \min_{t \in [\gamma_h^1, \gamma_h^2]} \dfrac{g(t,v)}{\psi_2(\rho_h^{-1} v)} > \psi_2(C_2)$;

(iv) *g satisfies* (H_2^∞) *if* $g_*^\infty := \liminf_{v \to \infty} \min_{t \in [\gamma_h^1, \gamma_h^2]} \dfrac{g(t,v)}{\psi_2(\rho_h^{-1} v)} > \psi_2(C_2)$.

Lemma 7. *Assume that* (F_1) *and* $g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ *hold. Then the following assertions are true.*

(1) *If g satisfies* (H_1^0), *then there exists* $r_* > 0$ *such that g satisfies* (H_1^r) *for all* $r \in (0, r_*)$.
(2) *If g satisfies* (H_1^∞), *then there exists* $r^* > 0$ *such that g satisfies* (H_1^r) *for all* $r \in [r^*, \infty)$.
(3) *If g satisfies* (H_2^0), *then there exists* $R_* > 0$ *such that g satisfies* (H_2^R) *for all* $R \in (0, R_*)$.
(4) *If g satisfies* (H_2^∞), *then there exists* $R^* > 0$ *such that g satisfies* (H_2^R) *for all* $R \in [R^*, \infty)$.

Proof. (1) Since $g_0^* < \psi_1(C_1)$, there exists $r_* > 0$ such that

$$\frac{g(t,v)}{\psi_1(v)} < \psi_1(C_1) \text{ for all } (t,v) \in [0,1] \times (0, r_*),$$

which implies that for any $r \in (0, r_*)$,

$$g(t,v) < \psi_1(C_1)\psi_1(v) \leq \psi_1(C_1)\psi_1(r) \text{ for all } (t,v) \in [0,1] \times [0,r].$$

Thus g satisfies (H_1^r) for all $r \in (0, r_*)$.

(2) Let $\epsilon \in (0, \psi_1(C_1) - g_\infty^*) > 0$ be given. Then $g_\infty^* < \psi_1(C_1) - \epsilon$, so that there exists $r_1 > 0$ satisfying

$$g(t, v) \le (\psi_1(C_1) - \epsilon)\psi_1(v) \text{ for all } (t, v) \in [0, 1] \times [r_1, \infty). \tag{18}$$

For $r \ge r_1$, by (18),

$$g^*(r) = \max\{g(t, v) : (t, v) \in [0, 1] \times [0, r]\} \le g^*(r_1) + g(t^*, v^*) \le g^*(r_1) + (\psi_1(C_1) - \epsilon)\psi_1(v^*),$$

where $(t^*, v^*) \in [0, 1] \times [r_1, r]$ is a point satisfying

$$g(t^*, v^*) = \max\{g(t, v) : (t, v) \in [0, 1] \times [r_1, r]\}.$$

Then

$$\frac{g^*(r)}{\psi_1(r)} \le \frac{g^*(r_1)}{\psi_1(r)} + \psi_1(C_1) - \epsilon,$$

and consequently there exists $r^* \ge r_1$ satisfying

$$\frac{g^*(r)}{\psi_1(r)} < \psi_1(C_1) \text{ for all } r \ge r^*.$$

Thus g satisfies (H_1^r) for all $r \ge r^*$.

(3) Since $g_*^0 > \psi_2(C_2)$, there exists $R_* > 0$ such that

$$g(t, v) > \psi_2(C_2)\psi_2(\rho_h^{-1}v) \text{ for } (t, v) \in [\gamma_h^1, \gamma_h^2] \times [0, R_*),$$

which implies that, for any $R \in (0, R_*)$,

$$g(t, v) > \psi_2(C_2)\psi_2(\rho_h^{-1}v) \ge \psi_2(C_2)\psi_2(R) \text{ for } (t, v) \in [\gamma_h^1, \gamma_h^2] \times [\rho_h R, R].$$

Thus g satisfies (H_2^R) for all $R \in (0, R_*)$.

(4) Since $g_*^\infty > \psi_2(C_2)$, there exists $R^* > 0$ such that

$$g(t, v) > \psi_2(C_2)\psi_2(\rho_h^{-1}v) \text{ for } t \in [\gamma_h^1, \gamma_h^2] \text{ and } v \ge \rho_h R^*,$$

which implies that, for $R \ge R^*$,

$$g(t, v) > \psi_2(C_2)\psi_2(R) \text{ for } (t, v) \in [\gamma_h^1, \gamma_h^2] \times [\rho_h R, R].$$

Thus g satisfies (H_2^R) for all $R \ge R^*$. □

The following corollary directly follows from Theorem 2 and Lemma 7.

Corollary 2. *Assume that (F_1), (F_2), (F_3) and $f \in C([0, 1] \times \mathbb{R}_+, \mathbb{R}_+)$ hold. Then the following assertions are true.*

(1) Assume that f satisfies (H_1^0) and (H_2^∞). Then BVP (1) and (2) has a positive solution u_1.

(2) Assume that f satisfies (H_1^0), (H_1^∞) and (H_2^R) for some $R > 0$. Then BVP (1) and (2) has two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < R < \|u_2\|$.

(3) In (1) and (2), if $f(t, 0) \not\equiv 0$, then BVP (1) and (2) has another positive solution v satisfying $\|v\|_\infty < \|u_1\|_\infty$.

(4) Assume that f satisfies (H_2^0) and (H_1^∞). Then BVP (1) and (2) has a positive solution.

(5) Assume that f satisfies (H_2^0), (H_2^∞) and (H_1^r) for some $r > 0$. Then BVP (1) and (2) has two positive solutions u_1, u_2 satisfying $0 < \|u_1\| < r < \|u_2\|$.

Remark 4. *Assume that $f \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ and $\hat{\alpha}_i \in (0,1)$ for $i = 1,2$. For any positive solutions u to BVP (1) and (2) satisfying $r_1 < \|u\|_\infty < r_2$,*

$$\hat{\rho} r_1 < u(t) < r_2 \text{ for all } t \in [0,1]. \tag{19}$$

Here

$$\hat{\rho} := \rho_1 \min\left\{ \int_0^1 \min\{r, 1-r\} d\alpha_i(r) : i = 1,2 \right\} \in (0,1).$$

In fact, by boundary conditions in (1) and Lemma 2 (i),

$$u(t) \geq \min\{u(0), u(1)\} = \min\left\{ \int_0^1 u(r) d\alpha_i(r) : i = 1,2 \right\} \text{ for } t \in [0,1].$$

Consequently, by Lemma 2 (iv), (19) is satisfied.

Now, we give a result on the existence of positive solutions to BVP (1) and (2) with $f \in C([0,1] \times (0,\infty), \mathbb{R})$ and $\hat{\alpha}_i \in (0,1)$ for $i = 1,2$.

Theorem 3. *Assume that $(F_1), (F_3)$ and $f \in C([0,1] \times (0,\infty), \mathbb{R})$ hold, and that $\hat{\alpha}_i \in (0,1)$ for $i = 1,2$. Then the following assertions are true.*

(1) *Assume that there exist positive constants r, R such that $0 < r < R$,*

$$f(t,s) \geq 0 \text{ for } (t,s) \in [0,1] \times [\hat{\rho} r, R], \tag{20}$$

$$f(t,s) < \psi_1(C_1)\psi_1(r) \text{ for } (t,s) \in [0,1] \times [\hat{\rho} r, r], \tag{21}$$

and

$$f(t,s) > \psi_2(C_2)\psi_2(R) \text{ for } (t,s) \in [\gamma_h^1, \gamma_h^2] \times [\rho_h R, R] \tag{22}$$

Then BVP (1) and (2) has a positive solution u satisfying $r < \|u\|_\infty < R$.

(2) *Assume that there exist constants $r, R > 0$ such that $0 < R < r$,*

$$f(t,s) > \psi_2(C_2)\psi_2(R) \text{ for } (t,s) \in [\gamma_h^1, \gamma_h^2] \times [\max\{\rho_h, \hat{\rho}\} R, R] \tag{23}$$

and

$$0 \leq f(t,s) < \psi_1(C_1)\psi_1(r) \text{ for } (t,s) \in [0,1] \times [\hat{\rho} R, r]. \tag{24}$$

Then BVP (1) and (2) has a positive solution u satisfying $R < \|u\|_\infty < r$.

(3) Assume that there exist positive constants r_1, r_2, R_1 such that $0 < r_1 < R_1 < r_2$,

$$f(t,s) \geq 0 \text{ for } (t,s) \in [0,1] \times [\hat{\rho} r_1, r_2], \tag{25}$$

$$f(t,s) < \psi_1(C_1)\psi_1(r_1) \text{ for } (t,s) \in [0,1] \times [\hat{\rho} r_1, r_1], \tag{26}$$

$$f(t,s) > \psi_2(C_2)\psi_2(R_1) \text{ for } (t,s) \in [\gamma_h^1, \gamma_h^2] \times [\rho_h R_1, R_1] \tag{27}$$

and

$$f(t,s) < \psi_1(C_1)\psi_1(r_2) \text{ for } (t,s) \in [0,1] \times [\hat{\rho} R_1, r_2]. \tag{28}$$

Then BVP (1) and (2) has two positive solutions u_1, u_2 satisfying $r_1 < \|u_1\|_\infty < R_1 < \|u_2\|_\infty < r_2$.

(4) Assume that there exist positive constants R_1, R_2, r_1 such that $0 < R_1 < r_1 < R_2$,

$$f(t,s) \geq 0 \text{ for } (t,s) \in [0,1] \times [\hat{\rho} R_1, R_2],$$

$$f(t,s) > \psi_2(C_2)\psi_2(R_1) \text{ for } (t,s) \in [\gamma_h^1, \gamma_h^2] \times [\max\{\hat{\rho}, \rho_h\} R_1, R_1]$$

$$f(t,s) < \psi_1(C_1)\psi_1(r_1) \text{ for } (t,s) \in [0,1] \times [\hat{\rho}R_1, r_1],$$

and

$$f(t,s) > \psi_2(C_2)\psi_2(R_2) \text{ for } (t,s) \in [\gamma_h^1, \gamma_h^2] \times [\rho_h R_2, R_2].$$

Then BVP (1) *and* (2) *has two positive solutions* u_1, u_2 *satisfying* $R_1 < \|u_1\|_\infty < r_1 < \|u_2\|_\infty < R_2$.

Proof. Consider the following modified problem

$$\begin{cases} (q(t)\varphi(u'(t)))' + h(t)g(t,u(t)) = 0, \ \ t \in (0,1), \\ u(0) = \int_0^1 u(r)d\alpha_1(r), u(1) = \int_0^1 u(r)d\alpha_2(r), \end{cases} \tag{29}$$

where $g(t,s) = f(t, m(s))$ for $(t,s) \in [0,1] \times \mathbb{R}_+$ and $m \in C(\mathbb{R}_+, (0,\infty))$ will be defined appropriately so that $g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$.

(1) Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$m(s) = \begin{cases} \hat{\rho}r, & \text{for } s \in [0, \hat{\rho}r); \\ s, & \text{for } s \in [\hat{\rho}r, R); \\ R, & \text{for } s \in [R, \infty). \end{cases}$$

Then, by (20) and the definition of m, $g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$. By (3) and (5), $\psi_1(y) \le \psi_2(y)$ for all $y \in \mathbb{R}_+$. Since $0 < C_1 < C_2$,

$$0 < \psi_1(C_1)\psi_1(s) < \psi_2(C_2)\psi_2(s) \text{ for } s > 0.$$

Then, by (21), (22) and the definition of m, it is easy to see that g satisfies (H_1^r) and (H_2^R). By Theorem 2, problem (29) has a positive solution u satisfying $r < \|u\|_\infty < R$. By Remark 4, $\hat{\rho}r < u(t) < R$ for all $t \in [0,1]$, so that $g(t,u(t)) = f(t,u(t))$ for $t \in [0,1]$. Consequently, BVP (1) and (2) has a positive solution u satisfying $r < \|u\|_\infty < R$.

(2) Let $m : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by

$$m(s) = \begin{cases} \hat{\rho}R, & \text{for } s \in [0, \hat{\rho}R); \\ s, & \text{for } s \in [\hat{\rho}R, r); \\ r, & \text{for } s \in [r, \infty). \end{cases}$$

Then, by (23),(24) and the definition of m, $g \in C([0,1] \times \mathbb{R}_+, \mathbb{R}_+)$ satisfies (H_1^r) and (H_2^R). By the same argument in the proof of (1) above, BVP (1) and (2) has a positive solution u satisfying $R < \|u\|_\infty < r$.

(3) By (25), (26) and (27), it is easy to see that (20), (21) and (22) are satisfied with $r = r_1$ and $R = R_1$. By Theorem 3 (1), BVP (1) and (2) has a positive solution u_1 satisfying $r_1 < \|u_1\|_\infty < R_1$. On the other hand, from (25), (27) and (28), it follows that (23) and (24) are satisfied with $R = R_1$ and $r = r_2$. Consequently, BVP (1) and (2) has another positive solution u_2 satisfying $R_1 < \|u_2\|_\infty < r_2$.

(4) By the similar argument as in the proof of the case (3), one can prove the case (4), so that we omit the proof. □

Remark 5. (1) *In Theorem* 2 (1) *(resp., Theorem* 3 (1)*),* $0 < r < \rho_h R$ *should be satisfied, since* f *satisfies* (H_1^r) *and* (H_2^R) *(resp.,* (21) *and* (22)*). Similarly, in Theorem* 2 (2) *and Theorem* 3 (2)*,* $\psi_2(C_2)\psi_2(R) < \psi_1(C_1)\psi_1(r)$ *should be satisfied.*

(2) *In Theorem* 3*, it is not needed that* f *is defined on* $[0,1] \times (0,\infty)$*. For example, for Theorem* 3 (1)*, it is sufficient to assume that* $f \in C([0,1] \times (r - \epsilon_1, R + \epsilon_2), \mathbb{R})$ *for any* $\epsilon_1 \in ((1-\hat{\rho})r, r]$ *and* $\epsilon_2 > 0$.

(3) *Let* $\varphi_i : \mathbb{R} \to \mathbb{R}$ *be an odd increasing homeomorphism satisfying* (F_1) *with* $\psi_1 = \psi_1^i$ *for* $i \in \{1,2\}$*. Then* $\varphi = \varphi_1 + \varphi_2 : \mathbb{R} \to \mathbb{R}$ *is an odd increasing homeomorphism satisfying* (F_1) *with* $\psi_1 = \min\{\psi_1^1, \psi_2^2\}$.

4. Examples

In this section, we give some examples to illustrate the results obtained in Section 3.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism defined by

$$\varphi(s) = s + |s|s \text{ for } s \in \mathbb{R}.$$

By Remark 5 (3), it is easy to see that (F_1) is satisfied with $\psi_1(y) = \min\{y, y^2\}$ for $y \in \mathbb{R}_+$ and, by (4), an increasing homeomorphism $\psi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (5) can be defined by $\psi_2(y) = \max\{y, y^2\}$ for $y \in \mathbb{R}_+$. Then

$$\psi_1^{-1}(y) = \max\{y, y^{\frac{1}{2}}\} \text{ and } \psi_2^{-1}(y) = \min\{y, y^{\frac{1}{2}}\} \text{ for } y \in \mathbb{R}_+.$$

Define $h : (0, 1] \to \mathbb{R}_+$ by

$$h(t) = \frac{9}{16}t^{-2} - 1 \text{ for } t \in (0, \tfrac{3}{4}] \text{ and } h(t) = 0 \text{ for } t \in (\tfrac{3}{4}, 1].$$

Then

$$\alpha_h = \bar{\beta}_h = 0,\ \beta_h = \bar{\alpha}_h = \frac{3}{4},\ \gamma_h^1 = \frac{3}{16},\ \gamma_h^2 = \frac{9}{16} \text{ and } \gamma_h = \frac{3}{8}.$$

Since $\varphi^{-1}(x) = 2^{-1}(-1 + \sqrt{1 + 4x})$ for $x \in \mathbb{R}_+$,

$$\varphi^{-1}\left(\int_s^{\frac{1}{2}} h(\tau)d\tau\right) = \varphi^{-1}\left(-\frac{13}{8} + \frac{9}{16}s^{-1} + s\right) \in L^1\left(0, \frac{1}{2}\right).$$

Consequently, since $h \in C(0, 1]$, (F_3) holds. Note that $h \notin L^1(0, 1)$.

Let $q(t) = (t + 1)^{-1}$ for $t \in [0, 1]$. Then

$$q_0 = \tfrac{1}{2},\ \|q\|_\infty = 1,\ \rho_1 = \frac{1}{2} \text{ and } \rho_h = \frac{3}{32}.$$

Case I. Let $\alpha_1 \equiv \alpha_2 \equiv C \in \mathbb{R}$. Then (F_2) holds, and it follows that

$$\hat{\alpha}_1 = \hat{\alpha}_2 = 0 \text{ and } A_1 = A_2 = 1.$$

Then C_1 and C_2 are well defined. Using MATLAB, approximate values of C_1 and C_2 can be calculated, i.e., $C_1 \approx 0.8956$ and $C_2 \approx 33.4276$.

(1) Let $R_1 > \rho_h^{-1}$ and $\epsilon \in [0, \psi_1(1))$ be fixed, and let $f(t, s) = f_1(s)$ for $(t, s) \in [0, 1] \times \mathbb{R}_+$, where $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$f_1(s) = \begin{cases} 2^{-1}\psi_1(C_1)[\psi_1(s) + \epsilon], & \text{for } t \in [0, 1], \\ \frac{f_1(\rho_h R) - f_1(1)}{\rho_h R - 1}(s - 1) + f_1(1), & \text{for } t \in (1, \rho_h R_1), \\ \psi_2(C_2)\psi_2(\rho_h^{-1}s) + 1, & \text{for } t \in [\rho_h R_1, \infty). \end{cases}$$

Since f_1 is strictly increasing on $\mathbb{R}_+$,

$$f^*(s) = f_1(s) \text{ and } f_*(s) = f_1(\rho_h s) \text{ for any } s > 0.$$

Consequently, f satisfies (H_1^1) and $(H_2^{R_1})$. Thus, by Theorem 2 (1), BVP (1) and (2) has a positive solution u satisfying $1 < \|u\|_\infty < R_1$ for $\epsilon = 0$ and it has two positive solutions u, v satisfying $0 < \|v\|_\infty < 1 < \|u\|_\infty < R_1$ for $\epsilon \in (0, \psi_1(1))$.

(2) Let $f(t, s) = \begin{cases} f_1(s), & \text{for } (t, s) \in [0, 1] \times [0, R_1], \\ f_2(s), & \text{for } (t, s) \in [0, 1] \times [R_1, \infty). \end{cases}$

Here f_1 is the function defined above and f_2 is defined by

$$f_2(s) = f_1(R_1)[\psi_1(R_1)]^{-\frac{1}{2}}[\psi_1(s)]^{\frac{1}{2}} \text{ for } s \in \mathbb{R}_+.$$

Then $\lim_{s\to\infty}\frac{f_2(s)}{\psi_1(s)}=0$, so that f satisfies (H_1^∞). Consequently, by Lemma 7 (2), f satisfies $(H_1^{r_2})$ for sufficiently large $r_2(>R_1)$. Note that f satisfies (H_1^1) and $(H_2^{R_1})$. Thus, Corollary 1 (1), BVP (1) and (2) has two positive solutions u_1,u_2 satisfying $1<\|u_1\|_\infty<R_1<\|u_2\|_\infty<r_2$ for $\epsilon=0$ and it has three positive solutions u_1,u_2,v satisfying $0<\|v\|_\infty<1<\|u_1\|_\infty<R_1<\|u_2\|_\infty<r_2$ for $\epsilon\in(0,\psi_1(1))$.

Case II. Let $\alpha_1(r)=\frac{1}{2}r^2$ and $\alpha_2(r)=\frac{1}{3}r^3$ for $r\in[0,1]$. Then (F_2) holds, and it follows that $\hat{\alpha}_1=\frac{1}{2}$, $\hat{\alpha}_2=\frac{1}{3}$, $A_1=2$, $A_2=\frac{3}{2}$, $\hat{\rho}=\frac{7}{192}$, $\max\{\rho_h,\hat{\rho}\}=\rho_h=\frac{3}{32}$, $C_1\approx 0.4478$ and $C_2\approx 33.4276$.

(1) Let $f(t,s)=s^{-1}f_3(t,s)$ for $(t,s)\in[0,1]\times(0,\infty)$, where $f_3\in C([0,1]\times\mathbb{R}_+,(0,\infty))$ is a given bounded function. Then $f\in C([0,1]\times(0,\infty),(0,\infty))$ satisfies that

$$\lim_{s\to 0^+}\min_{t\in[0,1]}f(t,s)=\infty \text{ and } \lim_{s\to\infty}f^*(s)=0.$$

It is easy to show the existence of r_1 and R_1 such that $0<R_1<r_1$ and f satisfies (23) and (24) with $R=R_1$ and $r=r_1$. Consequently, by Theorem 3 (2), BVP (1) and (2) has a positive solution u satisfying $R_1<\|u\|_\infty<r_1$.

(2) Let $f(t,s)=\begin{cases} s^{-1}f_3(t,s), & \text{for } (t,s)\in[0,1]\times(0,r_1],\\ f_4(t,s), & \text{for } (t,s)\in[0,1]\times[r_1,\rho_hR_2),\\ \psi_2(C_2)\psi_2(\rho_h^{-1}s)+\sin t, & \text{for } (t,s)\in[0,1]\times[\rho_hR_2,\infty).\end{cases}$

Here, f_3 is the function defined above, R_2 is a fixed constant satisfying $r_1<\rho_hR_2$ and f_4 is any nonnegative continuous function satisfying

$$f_4(t,r_1)=r_1^{-1}f_3(t,r_1) \text{ and } f_4(t,\rho_hR_2)=\psi_2(C_2)\psi_2(R_2)+\sin t \text{ for } t\in[0,1].$$

Then $f\in C([0,1]\times(0,\infty),\mathbb{R}_+)$ satisfies all the assumptions in Theorem 3 (4). Consequently, BVP (1) and (2) has two positive solutions u_1,u_2 satisfying $R_1<\|u_1\|_\infty<r_1<\|u_2\|_\infty<R_2$.

(3) Let $f(t,s)=f_5(s)$ for $(t,s)\in[0,1]\times(0,\infty)$, where $f_5:(0,\infty)\to\mathbb{R}$ is defined by

$$f_5(s)=\begin{cases}\frac{\psi_1(C_1)\psi_1(1)}{2\ln(\hat{\rho}^{-1})}\ln(\hat{\rho}^{-1}s), & \text{for } s\in(0,1],\\ \frac{\psi_1(C_1)\psi_1(1)}{2\psi_2(\rho_h^{-1})}s\psi_2(\rho_h^{-1}s), & \text{for } s\in(1,\infty).\end{cases}$$

Then f_5 is a strictly increasing continuous function on $(0,\infty)$ satisfying

$$\lim_{s\to 0^+}f_5(s)=-\infty,\ f_5(\hat{\rho})=0,\ f_5(1)=\frac{1}{2}\psi_1(C_1)\psi_1(1) \text{ and } \lim_{s\to\infty}\frac{f_5(s)}{\psi_2(\rho_h^{-1}s)}=\infty.$$

It is easy to see that, for $r=1$ and sufficiently large $R=R_1$, all the assumptions in Theorem 3 (1) are satisfied. Consequently, BVP (1) and (2) has a positive solution u_1 satisfying $1<\|u_1\|_\infty<R_1$.

(4) Let R_1 and f_5 is the constant and the function defined in Case II (3), respectively. Define $f:[0,1]\times(0,\infty)\to\mathbb{R}$ by

$$f(t,s)=f_5(s) \text{ for } (t,s)\in[0,1]\times[0,R_1] \text{ and } f(t,s)=f_6(t,s) \text{ for } (t,s)\in[0,1]\times(R_1,\infty).$$

Here $f_6:[0,1]\times[R_1,\infty)\to\mathbb{R}_+$ is any bounded continuous function satisfying $f_6(t,R_1)=f_5(R_1)$ for all $t\in[0,1]$. Then (28) is satisfied for sufficiently large $r_2\in(R_1,\infty)$, so that all the assumptions in Theorem 3 (3) are satisfied. Consequently, BVP (1) and (2) has two positive solutions u_1,u_2 satisfying $1<\|u_1\|_\infty<R_1<\|u_2\|_\infty<r_2$.

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