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A Note on Surfaces in Space Forms with Pythagorean Fundamental Forms

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Abstract: In the present note we introduce a Pythagorean-like formula for surfaces immersed into 3-dimensional space forms $\mathbb{M}^3(c)$ of constant sectional curvature $c = -1, 0, 1$. More precisely, we consider a surface immersed into $\mathbb{M}^3(c)$ satisfying $\mathrm{I}^2 + \mathrm{II}^2 = \mathrm{III}^2$, where I, II and III are the matrices corresponding to the first, second and third fundamental forms of the surface, respectively. We prove that such a surface is a totally umbilical round sphere with Gauss curvature $\varphi + c$, where φ is the Golden ratio.

Keywords: Pythagorean formula; Golden ratio; Gauss curvature; space form

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1. Introduction and Statements of Results

Let $\mathbb{N}^*$ denote set of all positive integers. For $a, b, c \in \mathbb{N}^*$, let $\{a, b, c\}$ be a triple with $a^2 + b^2 = c^2$, called a *Pythagorean triple*. The *Pythagorean theorem* states that the lengths of the sides of a right triangle turns to a Pythagorean triple. Moreover, if $\{a, b, c\}$ is a Pythagorean triple, so is $\{ka, kb, kc\}$, for any $k \in \mathbb{N}^*$. If $\gcd(a, b, c) = 1$, the triple $\{a, b, c\}$ is called a *primitive Pythagorean triple*. Of course, the most famous one among them is $\{3, 4, 5\}$. The Indian mathematician Brahmagupta (598–665 AD) gave a practical way generating all primitive Pythagorean triples: a triple $\{m^2 - n^2, 2mn, m^2 + n^2\}$ is a primitive Pythagorean triple for every $m, n \in \mathbb{N}^*$ satisfying the following conditions

1. $m > n$,
2. $\gcd(m, n) = 1$,
3. $m + n \equiv 1 \pmod 2$ (see [1]).

Recently, in [2], the authors extended this notion to the triple of integer-valued $n \times n$ matrices. Namely, a triple of such matrices $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ is said to be *Pythagorean* if it satisfies

$$\mathrm{A}^2 + \mathrm{B}^2 = \mathrm{C}^2. \tag{1}$$

As a trivial example, Equation (1) holds for any triple

$$\{\mathrm{A} = \mathtt{diag}[a_1, ..., a_n], \mathrm{B} = \mathtt{diag}[b_1, ..., b_n], \mathrm{C} = \mathtt{diag}[c_1, ..., c_n]\}$$

in which $\{a_i, b_i, c_i\}$ $(i = 1, ..., n)$ are Pythagorean triples. We refer to [2] for non-trivial examples and more details. We notice that this is not the first connection between Pythagorean triples and square matrices, see [3,4].

This interesting extension of Pythagorean triples motivates us to search a counterpart, in Differential Geometry, of this topic of Number Theory. For this purpose, a surface M^2 immersed into a 3-dimensional Riemannian space form $\mathbb{M}^3(c)$, $c = -1, 0, 1$, satisfying

$$\mathrm{I}^2 + \mathrm{II}^2 = \mathrm{III}^2, \tag{2}$$

where I^2, II^2 and III^2 are the squares of the matrices corresponding to the first, second and third fundamental forms of M^2, respectively, is considered. We call Equation (2) the *Pythagorean-like formula* for a surface immersed into $\mathbb{M}^3(c)$.

As an example, let $\mathbb{M}^3(c)$ be the 3-dimensional Euclidean space $\mathbb{E}^3$, i.e., $c = 0$. As usual we denote by $\mathbb{S}^2(r)$ a sphere of radius r in $\mathbb{E}^3$ centered at the origin. As is known, the metric of $\mathbb{S}^2(r)$ is given by $\langle , \rangle_I = du^2 + \cos^2\left(\frac{u}{r}\right) dv^2$; for $r \to \infty$ one naturally obtains the Euclidean metric $du^2 + dv^2$. The second and the third fundamental forms of $\mathbb{S}^2(r)$ are $h = -\frac{1}{r}\langle , \rangle_I$ and $\chi = \frac{1}{r^2}\langle , \rangle_I$. Therefore, $\mathbb{S}^2(r)$ satisfies the Pythagorean-like formula if and only if the following algebraic equation of degree 2 holds

$$x^2 + x - 1 = 0, \tag{3}$$

where $x = r^2$. Equation (3) has only one positive root, i.e., $x = \frac{\sqrt{5}-1}{2}$, which is the conjugate of φ, the *Golden Ratio*. This immediately implies that the Gauss curvature $K = 1/r^2$ of $\mathbb{S}^2(r)$ becomes the Golden Ratio.

Besides the Pythagorean Theorem, since the early ages, Golden ratio φ $\left(\varphi = \frac{1+\sqrt{5}}{2} = 1.61803398874989...\right)$ have had great interest not only for mathematicians but also for other scientists, philosophers, architects, and artists, for example see [5]. Indeed, we can see its importance due to Johannes Kepler (1571–1630), reference ([6]).

> "*Geometry has two great treasures; one is the Theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold; the second we may name a precious jewel*".

The main result is the following.

Theorem 1. *Let M^2 be a compact surface immersed into $\mathbb{M}^3(c)$, $c = -1, 0, 1$, with nonzero extrinsic curvature everywhere. If M^2 satisfies a Pythagorean-like formula given by Equation (2), then it is a totally umbilical round sphere with Gauss curvature $\varphi + c$, where φ is the Golden ratio.*

Remark 1. *For $c = 1$, we take $\mathbb{M}^3(c)$ an open hemisphere $\mathbb{S}^3_+$.*

We also denote by A the matrix corresponding to the shape operator A. The Pythagorean-like formula also can be interpreted in terms of shape operator A as

$$\mathcal{I}^2 + (\mathcal{I}\mathrm{A})^2 = (\mathcal{I}\mathrm{A}^2)^2,$$

which is similar to the equation

$$\mathcal{I} + \mathrm{A} = \mathrm{A}^2, \tag{4}$$

where $\mathcal{I}$ is identity on the tangent bundle of M^2. In [7], Equation (4) was completely solved for the so-called *golden-shaped hypersurfaces* in real space forms.

We notice that the starting point for the main idea of this study is the Pythagorean Theorem in spite of the fact that the Pythagorean-like formula given by Equation (2) is not directly related to the distance between points as in the usual case.

2. Preliminaries

In this section we provide some basics from [8,9].

Let $\mathbb{M}^3(c)$ denote a 3-dimensional Riemannian space form of constant sectional curvature $c = -1, 0, 1$ and $\langle , \rangle$ a Riemannian metric on $\mathbb{M}^3(c)$. Therefore, $\mathbb{M}^3(c)$ turns to the *Euclidean space* $\mathbb{E}^3$, the 3-*sphere* $\mathbb{S}^3$ and the *hyperbolic space* $\mathbb{H}^3$ when $c = 0$, $c = 1$ and $c = -1$, respectively. Here, $\mathbb{S}^3$ is the usual unit sphere of $\mathbb{E}^4$ given by

$$\mathbb{S}^3 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4 : \langle x, x \rangle = 1 \right\},$$

and $\mathbb{H}^3$ the hyperquadric of the Lorentz–Minkowski space $\mathbb{E}^4_1$ given by

$$\mathbb{H}^3 = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4_1 : \langle x, x \rangle_L = -1 \right\},$$

where $\langle , \rangle_L$ is the standard Lorentzian metric. We denote by $\mathbb{S}^3_{i,+}$ the open hemisphere consisting of all points x on $\mathbb{S}^3$ with $x_i > 0$.

Next, let M^2 be an orientable surface immersed into $\mathbb{M}^3(c)$ with metric $\langle , \rangle_I$ induced from Riemannian metric $\langle , \rangle$ on $\mathbb{M}^3(c)$. Denote by ν the unit normal vector field over M^2 and T_pM^2 the tangent space of M^2 at the point p. For $x, y \in T_pM^2$, the *second fundamental form* is the symmetric bilinear form given by

$$h_p(x, y) = \langle d\nu(x), y \rangle_I = \langle \mathrm{A}_p(x), y \rangle_I,$$

where A is the *shape operator.* M^2 is called *totally geodesic* when $h = 0$ and *totally umbilical* when $h = \lambda \langle , \rangle_I$, where λ is a nonzero constant. The eigenvalues of A at p, denoted by κ_1 and κ_2, are called the *principal curvatures* of M^2 at p. Denoting the trace of A by $\mathrm{tr}(\mathrm{A})$, $H(p) = \mathrm{tr}(\mathrm{A}_p)/2 = (\kappa_1 + \kappa_2)/2$ is called the *mean curvature* of M^2 at p. M^2 is said to be *minimal* if H vanishes identically.

The Gauss equation for M^2 gives the *Gauss curvature* K by

$$K = K_{ext} + c,$$

where K_{ext} is the *extrinsic curvature* of M^2, i.e., $K_{ext} = \det \mathrm{A} = \kappa_1\kappa_2$. In the Euclidean setting, obviously we have $K = K_{ext}$.

Noting that A is a self-adjoint linear operator at each point of M^2, we introduce the *third fundamental form* of M^2 at p by

$$\chi_p(x, y) = \langle \mathrm{A}_p(x), \mathrm{A}_p(y) \rangle_I.$$

Therefore, the Cayley–Hamilton Theorem for the matrix A has the form:

$$\mathrm{III} - 2H \cdot \mathrm{II} + K_{ext} \cdot \mathrm{I} = 0. \tag{5}$$

3. Proof of Theorem 1

Let M^2 be an immersed surface into $\mathbb{E}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3_+$, respectively, satisfying the Pythagorean-like formula given by Equation (2). If M^2 is totally geodesic, i.e., $\mathrm{II} = 0$, then it follows $\mathrm{III} = 0$ and hence the Pythagorean-like formula leads to the contradiction $\mathrm{I} = 0$. Furthermore, if II is degenerate, or equivalently $\det \mathrm{II} = 0$, then the Equations (2) and (5) imply

$$\mathrm{I}^2 = \left(4H^2 - 1\right) \mathrm{II}^2,$$

which contradicts the fact that I is positive definite. Therefore, we necessarily assume $\det \mathrm{II} \neq 0$ everywhere. In the Euclidean setting, it is equivalent to assume $K \neq 0$ everywhere. If M^2 is minimal, from Equations (2) and (5) we derive

$$\left(K^2_{ext} - 1\right) \mathrm{I}^2 = \mathrm{II}^2. \tag{6}$$

Taking the determinant, we obtain

$$K_{ext}^4 - 3K_{ext}^2 + 1 = 0, \tag{7}$$

at each point of M^2. Then K_{ext} is a nonzero constant, or equivalently, K is constant. If the ambient space is $\mathbb{E}^3$ or $\mathbb{H}^3$ then M^2 must be totally geodesic (see [10] (Corollary 1)), which gives a contradiction. Otherwise, i.e., the ambient space is $\mathbb{S}^3_+$, there exist two cases (for details, see [11] (Corollary 3)):

Case a. $K = 1$ and M^2 is totally geodesic. This case is not possible, already we discussed it above.

Case b. $K = 0$ and M^2 is an open piece of the Clifford torus. Thus, $K_{ext} = -1$, which does not fulfill Equation (7).

Consequently, an immersed surface into $\mathbb{E}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3_+$ satisfying the Pythagorean-like formula can be neither totally geodesic, nor minimal, nor have degenerate second fundamental form.

Next we present the proof of the main result.

Proof of Theorem. Let M^2 be a compact surface immersed into $\mathbb{E}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3_+$, respectively, with non-degenerate second fundamental form. Assume that M^2 satisfies the Pythagorean-like formula. By substituting (5) into (2), we get

$$\left(1 - K_{ext}^2\right) \mathrm{I}^2 + 2K_{ext}H\mathrm{I} \cdot \mathrm{II} + 2K_{ext}H\mathrm{II} \cdot \mathrm{I} + \left(1 - 4H^2\right) \mathrm{II}^2 = 0. \tag{8}$$

Notice that matrices do not commute by matrix multiplication " $\cdot$ ". Since I is positive definite and everywhere $\det \mathrm{II} \neq 0$, I and II have inverse matrices and thus Equation (8) can be rewritten as

$$\mathrm{I}\left[\left(1 - 4H^2\right) \mathrm{I}^{-1} \cdot \mathrm{II} + 2HK_{ext}\mathcal{I}_2\right] \mathrm{II} = \mathrm{II}\left[\left(K_{ext}^2 - 1\right) \mathrm{II}^{-1} \cdot \mathrm{I} - 2HK_{ext}\mathcal{I}_2\right] \mathrm{I}, \tag{9}$$

where I^{-1} denotes the inverse matrix of I and $\mathcal{I}_2$ is the 2×2 unit matrix. Taking the determinant of the Equation (9), we obtain

$$\det\left[\left(1 - 4H^2\right) \mathrm{I}^{-1}\mathrm{II} + 2HK_{ext}\mathcal{I}_2\right] = \det\left[\left(K_{ext}^2 - 1\right) \mathrm{II}^{-1} \cdot \mathrm{I} - 2HK_{ext}\mathcal{I}_2\right]. \tag{10}$$

Because $\mathrm{I}^{-1} \cdot \mathrm{II} = \mathrm{A}$ and $\mathrm{II}^{-1} \cdot \mathrm{I} = \mathrm{A}^{-1}$, Equation (10) reduces to

$$\begin{aligned} &(1 - 4H^2)^2 \det \mathrm{A} + 2HK_{ext} \left(1 - 4H^2\right) \mathrm{tr}(\mathrm{A}) = \\ &= \left(\frac{K_{ext}^2 - 1}{K_{ext}}\right)^2 \det \mathrm{A} - 2HK_{ext} \left(\frac{K_{ext}^2 - 1}{K_{ext}}\right) \mathrm{tr}(\mathrm{A}). \end{aligned} \tag{11}$$

By substituting $K_{ext} = \det \mathrm{A}$ and $\mathrm{tr}(\mathrm{A}) = 2H$ into Equation (11), we obtain

$$4H^2 \left(K_{ext}^2 - K_{ext} - 1\right) = \frac{\left(K_{ext}^2 - K_{ext} - 1\right)\left(K_{ext}^2 + K_{ext} - 1\right)}{K_{ext}}. \tag{12}$$

Now assume that $K_{ext}^2 - K_{ext} - 1 \neq 0$ in Equation (12). Thereby Equation (11) reduces to

$$4H^2 = \frac{K_{ext}^2 + K_{ext} - 1}{K_{ext}}. \tag{13}$$

Because of compactness of M^2, there exist a point $p \in M^2$ at which K_{ext} is strictly positive, i.e., $K_{ext}(p) > 0$ (see [12] (Theorem 13.36)). Furthermore, because $4H^2(p) \geq 4K_{ext}(p)$, Equation (13) yields

$$3K_{ext}^2(p) - K_{ext}(p) + 1 \leq 0, \tag{14}$$

which is not possible because the left-hand side of formula (14) is strictly positive: contradiction. This implies from Equation (12) that

$$K_{ext}^2 - K_{ext} - 1 = 0, \tag{15}$$

for each point of M^2. Solving Equation (15) yields that K_{ext} is a constant $\pm\varphi$, where φ is the Golden Ratio. Since K_{ext} is strictly positive at least at a point on M^2, one leads to $K_{ext} = \varphi$. Therefore, we obtain $K = \varphi + c$, for $c = -1, 0, 1$. This completes the proof by the fact that every compact surface with $K =$constant is a totally umbilical round sphere (see [13] (Theorem 1)).

□

4. Conclusions

Surfaces immersed into space forms satisfying the Pythagorean-like formula given by Equation (2) were investigated. Of course, the roles of I and II in Equation (2) are symmetric. Moreover, the study of those surfaces satisfying the following equations could be challenging problems:

$$\mathrm{I}^2 + \mathrm{III}^2 = \mathrm{II}^2 \text{ and } \mathrm{II}^2 + \mathrm{III}^2 = \mathrm{I}^2.$$

Furthermore, the above Pythagorean-like formula given for surfaces can be extended to hypersurfaces (or submanifolds of codimension >1) in space forms.

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