mathematics

Article

New Comparison Theorems for the Nth Order Neutral Differential Equations with Delay Inequalities

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Received: 20 February 2020; Accepted: 18 March 2020; Published: 22 March 2020

Abstract: In this work, we present a new technique for the oscillatory properties of solutions of higher-order differential equations. We set new sufficient criteria for oscillation via comparison with higher-order differential inequalities. Moreover, we use the comparison with first-order differential equations. Finally, we provide an example to illustrate the importance of the results.

Keywords: higher-order differential equations; differential inequalities; oscillatory properties

1. Introduction

This work is concerned with studying the oscillatory behavior of the higher-order neutral differential equation

$$((z^{(n-1)}(\varsigma))^{\alpha})' + f(\varsigma, x(\sigma(\varsigma))) = 0,\ \varsigma \geq \varsigma_0 > 0, \tag{1}$$

where $n \geq 2$ is an even natural number and $z(\varsigma) = x(\varsigma) + p(\varsigma)\, x(\tau(\varsigma))$. Through the paper, we assume that $0 < \alpha < 1$ is a ratio of odd positive integers, $p \in C^1([\varsigma_0, \infty), \mathbb{R}^+)$, with $\lim_{\varsigma\to\infty} p(\varsigma) = 0$, $\sigma, \tau \in C^1([\varsigma_0, \infty), \mathbb{R})$ such that $\sigma(\varsigma) \leq \varsigma$, $\lim_{\varsigma\to\infty} \sigma(\varsigma) = \infty$. $\tau(\varsigma) \leq \varsigma$, $\lim_{\varsigma\to\infty} \tau(\varsigma) = \infty$ and there exists a nonnegative function h such that $f(\varsigma, x) \geq h(\varsigma)\, x^{\beta}$, β is a ratio of odd positive integers.

If there exists a $\varsigma_x \geq \varsigma_0$ such that the real valued function x is continuous, $\left(z^{(n-1)}\right)^{\alpha} \in C^1([\varsigma_x, \infty), \mathbb{R})$ and satisfies (1), for all $\varsigma \in [\varsigma_x, \infty)$, then x is said to be a solution of (1). We restrict our discussion to those solutions x of (1) which satisfy $\sup\{|x(\varsigma)| : \varsigma_1 \leq \varsigma_0\} > 0$ for every $\varsigma_1 \in [\varsigma_x, \infty)$ and tacitly assume that (1) possesses such solutions.

Definition 1. *A solution x of (1) is said to be non-oscillatory if it is positive or negative, ultimately; otherwise, it is said to be oscillatory. The equation itself is termed oscillatory if all its solutions oscillate.*

Differential equations are of great importance in various applied sciences, for example, kinetic chemistry [1,2], eigenproblems [3,4] and wave functions [5]. A neutral differential equation is an equation in which the delayed argument occurs in the highest derivative of the state variable. These equations has become an significant field of research due to the fact that such equations appear in many applications, for example population dynamics, mixing liquids, etc., see Hale [6].

The study of the problem of oscillation of differential equations with higher order has attracted the attention of many researchers in recent times. As a result of this interest, many different techniques and methods have been introduced to study the oscillation problem. Below we show some works closely related to this paper.

Moaaz et al. [7] and El-Nabulsi et al. [8] studied the equations

$$\left[r(\varsigma)\left(x'''(\varsigma)\right)^{\alpha}\right]' + \int_a^b q(\varsigma,\xi)\, f\left(x\left(g(\varsigma,\xi)\right)\right) d(\xi) = 0,$$

under the conditions

$$\int_{\varsigma_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s < \infty$$

and

$$\int_{\varsigma_0}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = \infty,$$

respectively. Baculikova et al. [9] and Moaaz et al. [10,11] use the Riccati-technique to study the oscillation of quasi-linear neutral equation

$$\left(\left(\left(x(\varsigma) + p(\varsigma)\, x(\tau(\varsigma))\right)^{(n-1)}\right)^{\gamma}\right)' + q(\varsigma)\, x^{\gamma}(\sigma(\varsigma)) = 0,\ \varsigma \geq \varsigma_0,$$

Moaaz et al. [12] established the oscillation criteria for solutions of the even-order equation

$$\left[r(\varsigma)\left(z^{(n-1)}(\varsigma)\right)\right]' + \int_a^b q(\varsigma,s)\, f\left(x\left(g(\varsigma,s)\right)\right) d(s) = 0,$$

where $z(\varsigma) = x^{\alpha}(\varsigma) + p(\varsigma)\, x(\sigma(\varsigma))$. Xing et al. [13] investigated the oscillation behavior of neutral equation

$$\left(r(\varsigma)\left(\left(x(\varsigma) + p(\varsigma)\, x(\tau(\varsigma))\right)^{(n-1)}\right)^{\gamma}\right)' + q(\varsigma)\, x^{\gamma}(\sigma(\varsigma)) = 0,\ n \geq 2,$$

by using comparison technique with first-order delay equations. Baculikova and Dzurina [14] investigated the asymptotic properties and oscillation of the nth order neutral equations

$$\left(r(\varsigma)\left(\left(x(\varsigma) + p(\varsigma)\, x(\tau(\varsigma))\right)^{(n-1)}\right)\right)' + q(\varsigma)\, x(\sigma(\varsigma)) = 0,\ n \geq 3.$$

Elabbasy et al. [15] studied the asymptotic properties of solutions of even order neutral delay differential equations of the form

$$\left(r(\varsigma)\left|z^{(n-1)}(\varsigma)\right|^{\alpha_1 - 1} z^{(n-1)}(\varsigma)\right)' + \sum_{i=1}^{m} q_i(\varsigma)\left|x(\sigma_i(\varsigma))\right|^{\alpha_i - 1} x(\sigma_i(\varsigma)) = 0.$$

We have greatly fewer results for oscillation of the neutral equation than for oscillation of the delay equation. Therefore, the importance of this study lies in linking between delay and neutral equations by comparison. This comparison enables us to benefit from the previous results—in the literature—of oscillation of delay equations. In this paper, we establish some new oscillation criteria via comparison with higher-order differential inequalities and with first-order differential equations. An example illustrating the results is also given.

Lemma 1 ([16])**.** *Let $f \in C^n([\varsigma_0,\infty),(0,\infty))$. If the derivative $f^{(n)}(\varsigma)$ is eventually of one sign for all large ς, then there exist a ς_x such that $\varsigma_x \geq \varsigma_0$ and an integer l, $0 \leq l \leq n$, with $n+l$ even for $f^{(n)}(\varsigma) \geq 0$, or $n+l$ odd for $f^{(n)}(\varsigma) \leq 0$ such that*

$$l > 0 \text{ implies } f^{(k)}(\varsigma) > 0 \text{ for } \varsigma \geq \varsigma_x,\ k = 0, 1, ..., l - 1,$$

and

$$l \leq n - 1 \text{ implies } (-1)^{l+k} f^{(k)}(\varsigma) > 0 \text{ for } \varsigma \geq \varsigma_x,\ k = l, l + 1, ..., n - 1.$$

Lemma 2 ([17]). *If a and b are nonnegative and $0 < \delta < 1$, then*

$$a^{\delta} - \delta a b^{\delta-1} - (1 - \delta) b^{\delta} \leq 0,$$

where equality holds if and only if $a = b$.

Lemma 3 ([18]). *Let $x \in C^n([\varsigma_0, \infty), (0, \infty))$, $x^{(n-1)}(\varsigma) x^{(n)}(\varsigma) \leq 0$ for $\varsigma \geq \varsigma_x$ and assume that $\lim_{\varsigma \to \infty} x(\varsigma) \neq 0$, then for every $\theta \in (0, 1)$, there exists a $\varsigma_\theta \in [\varsigma_x, \infty)$ such that*

$$x(\varsigma) \geq \frac{\theta}{(n-1)!} \varsigma^{n-1} \left| x^{(n-1)}(\varsigma) \right| \text{ for all } \varsigma \in [\varsigma_\theta, \infty).$$

Remark 1. *We will prove the results when $x(\varsigma)$ eventually positive while at $x(\varsigma)$ is eventually negative the proof is similar, so we omit the details of that case in this paper.*

2. Main Results

Theorem 1. *Assume that $\beta > 1$. For every constant $N > 0$ if the even-order differential inequality*

$$((z^{(n-1)}(\varsigma))^{\alpha})' + N h(\varsigma) z(\sigma(\varsigma)) \leq 0 \tag{2}$$

has no positive solution, then Equation (1) is oscillatory.

Proof. Assume to the contrary that x is a nonoscillatory solution of (1), say $x(\varsigma) > 0$, $x(\tau(\varsigma)) > 0$ and $x(\sigma(\varsigma)) > 0$ for $\varsigma \geq \varsigma_1$ for some $\varsigma_1 \geq \varsigma_0$. From (1) we have

$$((z^{(n-1)}(\varsigma))^{\alpha})' \leq -h(\varsigma) x^{\beta}(\sigma(\varsigma)) < 0, \tag{3}$$

from Lemma 1, there exists a $\varsigma_2 \geq \varsigma_1$ such that

$$z'(\varsigma) > 0 \text{ and } z^{(n-1)}(\varsigma) > 0, \tag{4}$$

for $\varsigma \geq \varsigma_2$. From definition $z(\varsigma)$, we have

$$x(\varsigma) = z(\varsigma) - (p(\varsigma) x^{\alpha}(\tau(\varsigma)) - p(\varsigma) x(\tau(\varsigma))) + p(\varsigma) x^{\alpha}(\tau(\varsigma)) - 2p(\varsigma) x(\tau(\varsigma)). \tag{5}$$

By using Lemma 2 with $a = p^{1/\alpha}(\varsigma) x(\tau(\varsigma))$, $b = \left(\alpha^{-1} p^{(\alpha-1)/\alpha}(\varsigma)\right)^{1/(\alpha-1)}$ and $\delta = \alpha$, we get

$$p(\varsigma) x^{\alpha}(\tau(\varsigma)) - p(\varsigma) x(\tau(\varsigma)) \leq (1 - \alpha) \alpha^{\alpha/(1-\alpha)} p(\varsigma). \tag{6}$$

From (5) and (6), we have

$$x(\varsigma) \geq z(\varsigma) + p(\varsigma) x^{\alpha}(\tau(\varsigma)) - 2p(\varsigma) x(\tau(\varsigma)) - (1 - \alpha) \alpha^{\alpha/(1-\alpha)} p(\varsigma), \tag{7}$$

that is

$$x(\varsigma) \geq z(\varsigma) - 2p(\varsigma) x(\tau(\varsigma)) - (1 - \alpha) \alpha^{\alpha/(1-\alpha)} p(\varsigma). \tag{8}$$

Since $x(\varsigma) \leq z(\varsigma)$, we have

$$x(\varsigma) \geq z(\varsigma) - 2p(\varsigma) z(\tau(\varsigma)) - (1 - \alpha) \alpha^{\alpha/(1-\alpha)} p(\varsigma). \tag{9}$$

Since $z(\varsigma) > 0$ and $z'(\varsigma) > 0$ on $[\varsigma_2, \infty)$, there exist a $\varsigma_3 \geq \varsigma_2$ and a constant $\mu > 0$ such that

$$z(\varsigma) \geq \mu, \tag{10}$$

for $\varsigma \geq \varsigma_3$. From (9) and (10), we obtain

$$\begin{aligned} x(\varsigma) &\geq z(\varsigma)(1 - 2p(\varsigma)) - (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} p(\varsigma) \\ &\geq z(\varsigma)\left(1 - 2p(\varsigma) - (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} \frac{p(\varsigma)}{z(\varsigma)}\right) \\ &\geq z(\varsigma)\left(1 - 2p(\varsigma) - (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} \frac{p(\varsigma)}{\mu}\right) \\ &\geq z(\varsigma)\left(1 - p(\varsigma)\left(2 + (1-\alpha)\,\alpha^{\alpha/(1-\alpha)} \frac{1}{\mu}\right)\right), \end{aligned} \tag{11}$$

for $\varsigma \geq \varsigma_3$. From (11) and the fact that $\lim_{\varsigma\to\infty} p(\varsigma) = 0$, for any $\lambda \in (0,1)$ there exists $\varsigma_\lambda \geq \varsigma_3$ such that

$$x(\varsigma) \geq \lambda z(\varsigma) \;\text{ for } \varsigma \geq \varsigma_\lambda. \tag{12}$$

Fix $\lambda \in (0,1)$ and choose ς_λ by (12). Since $\lim_{\varsigma\to\infty} \sigma(\varsigma) = \infty$, we can choose $\varsigma_5 \geq \varsigma_\lambda$ such that $\sigma(\varsigma) \geq \varsigma_\lambda$ for all $\varsigma \geq \varsigma_5$. Thus, from (12) we have

$$x(\sigma(\varsigma)) \geq \lambda z(\sigma(\varsigma)), \tag{13}$$

for $\varsigma \geq \varsigma_5$. From (13) and (1), we have

$$((z^{(n-1)}(\varsigma))^\alpha)' + \lambda^\beta h(\varsigma) z^\beta(\sigma(\varsigma)) \leq 0. \tag{14}$$

It can be reformulated as follow

$$((z^{(n-1)}(\varsigma))^\alpha)' + \lambda^\beta h(\varsigma) z^{\beta-1}(\sigma(\varsigma)) z(\sigma(\varsigma)) \leq 0, \text{ for } \varsigma \geq \varsigma_5. \tag{15}$$

By using (10), $z'(\varsigma) > 0$ and $\sigma(\varsigma) \geq \varsigma_\lambda$, (15) become to

$$((z^{(n-1)}(\varsigma))^\alpha)' + \lambda^\beta \mu^{\beta-1} h(\varsigma) z(\sigma(\varsigma)) \leq 0,$$

that is

$$((z^{(n-1)}(\varsigma))^\alpha)' + N h(\varsigma) z(\sigma(\varsigma)) \leq 0, \text{ for } \varsigma \geq \varsigma_5, \tag{16}$$

where $N = \lambda^\beta \mu^{\beta-1}$. We conclude that (2) has a positive solution, which is a contradiction. The proof is complete. □

Theorem 2. *Assume that* $0 < \beta < 1$. *For every constant* $J > 0$ *if the even-order differential inequality*

$$((z^{(n-1)}(\varsigma))^\alpha)' + J\left(\sigma^{n-1}(\varsigma)\right)^{\beta-1} h(\varsigma) z(\sigma(\varsigma)) \leq 0 \tag{17}$$

has no positive solution, then Equation (1) is oscillatory.

Proof. Assume to the contrary that x is a nonoscillatory solution of (1), say $x(\varsigma) > 0$, $x(\tau(\varsigma)) > 0$ and $x(\sigma(\varsigma)) > 0$ for $\varsigma \geq \varsigma_1$ for some $\varsigma_1 \geq \varsigma_0$. We follow the same method as in proof Theorem 1, we get (14) which can be reformulated as follow

$$((z^{(n-1)}(\varsigma))^\alpha)' + \frac{\lambda^\beta h(\varsigma)}{z^{1-\beta}(\sigma(\varsigma))} z(\sigma(\varsigma)) \leq 0, \text{ for } \varsigma \geq \varsigma_5. \tag{18}$$

Since $z^{(n-1)}(\varsigma) > 0$ and $z^{(n-1)}(\varsigma)$ is decreasing on $[\varsigma_5, \infty) \subset [\varsigma_2, \infty)$, there exist a ς_6 such that $\varsigma_6 \geq \varsigma_5$ and a constant $\eta > 0$ such that

$$z^{(n-1)}(\varsigma) \leq \eta, \tag{19}$$

for $\varsigma \geq \varsigma_6$. Integrating (19) from ς_6 to ς $n-1$ times, we get

$$z(\varsigma) \leq U\varsigma^{n-1},\ \varsigma \geq \varsigma_6, \tag{20}$$

for some constant $U > 0$, and so,

$$z(\sigma(\varsigma)) \leq U\sigma^{n-1}(\varsigma),\ \varsigma \geq \varsigma_7 \geq \varsigma_6, \tag{21}$$

where we assume $\sigma(\varsigma) \geq \varsigma_6$ for $\varsigma \geq \varsigma_7$. By using (21), and (18), we have

$$((z^{(n-1)}(\varsigma))^{\alpha})' + \lambda^{\beta}U^{\beta-1}\left(\sigma^{n-1}(\varsigma)\right)^{\beta-1} h(\varsigma)\, z(\sigma(\varsigma)) \leq 0$$

or

$$((z^{(n-1)}(\varsigma))^{\alpha})' + J\left(\sigma^{n-1}(\varsigma)\right)^{\beta-1} h(\varsigma)\, z(\sigma(\varsigma)) \leq 0,\ \text{for } \varsigma \geq \varsigma_7, \tag{22}$$

where $J = \lambda^{\beta}U^{\beta-1}$. We conclude that (17) has a positive solution, which is a contradiction. The proof is complete. □

Theorem 3. *Assume that* $\beta = 1$. *For any* $\lambda \in (0,1)$ *if the even-order differential inequality*

$$((z^{(n-1)}(\varsigma))^{\alpha})' + \lambda h(\varsigma)\, z(\sigma(\varsigma)) \leq 0 \tag{23}$$

has no positive solution, then Equation (1) is oscillatory.

The Proof of the previous theorem is produced directly from (14) with $\beta = 1$ and Theorem 1; so we omit the details.

Theorem 4. *Assume that* $\beta > 1$. *For every constant* $N > 0$, *if the first-order differential equation*

$$y'(\varsigma) + \frac{\theta_0}{(n-1)!} N\sigma^{n-1}(\varsigma)\, h(\varsigma)\, y^{1/\alpha}(\sigma(\varsigma)) = 0 \tag{24}$$

is oscillatory for some $\theta_0 \in (0,1)$, *then Equation (1) is oscillatory.*

Proof. Assume to the contrary that x is a nonoscillatory solution of (1), say $x(\varsigma) > 0$, $x(\tau(\varsigma)) > 0$ and $x(\sigma(\varsigma)) > 0$ for $\varsigma \geq \varsigma_1$ for some $\varsigma_1 \geq \varsigma_0$. We follow the same method as in proof Theorem 1, we get (16) for $\varsigma \geq \varsigma_5$. Since $\lim_{\varsigma\to\infty} z(\varsigma) \neq 0$, where $z(\varsigma) > 0$ and $z'(\varsigma) > 0$ on $[\varsigma_5, \infty) \subset [\varsigma_2, \infty)$. By using Lemma 3 we have

$$z(\varsigma) \geq \frac{\theta}{(n-1)!}\varsigma^{n-1} z^{(n-1)}(\varsigma), \tag{25}$$

for $\varsigma \geq \varsigma_6$. From (16) and (25), we have

$$((z^{(n-1)}(\varsigma))^{\alpha})' + \frac{\theta}{(n-1)!} N\sigma^{n-1}(\varsigma)\, h(\varsigma)\, z^{(n-1)}(\sigma(\varsigma)) \leq 0,\ \varsigma \geq \varsigma_7 \geq \varsigma_6. \tag{26}$$

Set $y(\varsigma) = (z^{(n-1)}(\varsigma))^{\alpha}$, we have

$$y'(\varsigma) + \frac{\theta}{(n-1)!} N\sigma^{n-1}(\varsigma)\, h(\varsigma)\, y^{1/\alpha}(\sigma(\varsigma)) \leq 0, \tag{27}$$

for $\varsigma \geq \varsigma_7$. Thus $y(\varsigma)$ is a positive solution of the above inequality. Integrating inequality (27) from ς to ∞ where $\varsigma \geq \varsigma_7$, we have

$$y(\varsigma) \geq \int_{\varsigma}^{\infty} \frac{\theta}{(n-1)!} N\sigma^{n-1}(s)\, h(s)\, y^{1/\alpha}(\sigma(s))\, \mathrm{d}s.$$

From [19] (Theorem 1) we see that there exists a positive solution $y(\varsigma)$ of Equation (24) with $\lim_{\varsigma\to\infty} y(\varsigma) = 0$, which contradicts (24), is oscillatory. The proof is complete. □

Theorem 5. *Assume that $0 < \beta < 1$. For every constant $J > 0$ if the first-order differential equation*

$$y'(\varsigma) + \frac{\theta_0}{(n-1)!} J\left(\sigma^{n-1}(\varsigma)\right)^{\beta} h(\varsigma)\, y^{1/\alpha}(\sigma(\varsigma)) = 0 \tag{28}$$

is oscillatory for some $\theta_0 \in (0,1)$, then Equation (1) is oscillatory.

The Proof of the previous theorem is produced directly from (22) and (25) and Theorem 4, so we omit the details.

Theorem 6. *Assume that $\beta = 1$. For any $\lambda \in (0,1)$ if the first-order differential equation*

$$y'(\varsigma) + \frac{\theta_0}{(n-1)!} \lambda\sigma^{n-1}(\varsigma)\, h(\varsigma)\, y^{1/\alpha}(\sigma(\varsigma)) = 0 \tag{29}$$

is oscillatory for some $\theta_0 \in (0,1)$, then Equation (1) is oscillatory.

The Proof of the previous theorem is produced directly from (14) with $\beta = 1$, (25) and Theorem 4, so we omit the details.

Corollary 1. *Assume that $\beta \geq 1$. If*

$$\lim_{\varsigma\to\infty} \int_{\sigma(\varsigma)}^{\varsigma} \sigma^{n-1}(s)\, h(s)\, \mathrm{d}s = \infty,$$

then Equation (1) is oscillatory.

Corollary 2. *Assume that $0 < \beta < 1$. If*

$$\lim_{\varsigma\to\infty} \int_{\sigma(\varsigma)}^{\varsigma} \left(\sigma^{n-1}(s)\right)^{\beta} h(s)\, \mathrm{d}s = \infty,$$

then Equation (1) is oscillatory.

Example 1. *Consider the differential equation*

$$\left(\left(\left(x(\varsigma) + \frac{1}{\varsigma} x\left(\frac{\varsigma}{4}\right)\right)^{(n-1)}\right)^{\alpha}\right)' + \frac{3}{\varsigma^2} x^2\left(\frac{\varsigma}{3}\right) = 0,\ \varsigma \geq \varsigma_0 > 0 \text{ and } n > 2. \tag{30}$$

From (30) we have $0 < \alpha < 1$, $\beta = 3$, $p(\varsigma) = 1/\varsigma$, $\tau(\varsigma) = \varsigma/4$, $\sigma(\varsigma) = \varsigma/3$ and $h(\varsigma) = 3/\varsigma^2$. By using Corollary 1 we find

$$\begin{aligned}\int_{\sigma(\varsigma)}^{\varsigma} \sigma^{n-1}(s)\, h(s)\, \mathrm{d}s &= \int_{\varsigma/3}^{\varsigma} \left(\frac{s}{3}\right)^{n-1} \frac{3}{s^2} \mathrm{d}s \\ &= \frac{1}{3^{n-2}(n-2)} \left(\frac{3^{n-2}-1}{3^{n-2}}\right) \varsigma^{n-2},\end{aligned}$$

therefore

$$\lim_{\varsigma\to\infty}\int_{\sigma(\varsigma)}^{\varsigma}\sigma^{n-1}(s)\,h(s)\,\mathrm{d}s=\infty,$$

then (30) is oscillatory.

3. Conclusions

Several techniques have been used in studying the oscillation of solutions of higher-order differential equations. Unusually, our method here is based on presenting new comparison theorems that compare the higher-order neutral equation with higher or first-order delay inequality or equation. Through our results, Theorems 1–6 guaranteed the oscillation of the solutions of Equation (1) by ensuring the oscillation of one of the inequalities (2), (17) and (23) or Equations (24), (28) and (29).

Author Contributions: Formal analysis, S.F.; Investigation, O.M.; Supervision, O.M.; Writing–original draft, A.M.; Writing–review & editing, O.M., S.F. and A.M. These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The author (S.F.) was partially supported by JSPS KAKENHI Grant Number 16K05257.

Conflicts of Interest: The authors declare no conflict of interest.

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