mathematics

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Infinitely Many Homoclinic Solutions for Fourth Order p-Laplacian Differential Equations

Stepan Tersian

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences (BAS), 1113 Sofia, Bulgaria; sterzian@uni-ruse.bg

Received: 17 March 2020; Accepted: 1 April 2020; Published: 2 April 2020

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Abstract: The existence of infinitely many homoclinic solutions for the fourth-order differential equation $\left(\varphi_p\left(u''(t)\right)\right)'' + w\left(\varphi_p\left(u'(t)\right)\right)' + V(t)\varphi_p\left(u(t)\right) = a(t)f(t,u(t)), t \in \mathbb{R}$ is studied in the paper. Here $\varphi_p(t) = |t|^{p-2}t, p \geq 2$, w is a constant, V and a are positive functions, f satisfies some extended growth conditions. Homoclinic solutions u are such that $u(t) \to 0, |t| \to \infty$, $u \neq 0$, known in physical models as ground states or pulses. The variational approach is applied based on multiple critical point theorem due to Liu and Wang.

Keywords: homoclinic solutions; fourth-order p-Laplacian differential equations; minimization theorem; Clark's theorem

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1. Introduction

In this paper, we study the existence of infinitely many nonzero solutions homoclinic solutions for the fourth-order p-Laplacian differential equation

$$\left(\varphi_p\left(u''(t)\right)\right)'' + w\left(\varphi_p\left(u'(t)\right)\right)' + V(t)\varphi_p\left(u(t)\right) = a(t)f(t,u(t)), \tag{1}$$

where $t \in \mathbb{R}$, w is a constant, $\varphi_p(t) = |t|^{p-2}t$, for $p \geq 2$, V is a positive bounded function, a is a positive continuous function and $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies some growth conditions with respect to p. As usual, we say that a solution u of (1) is a nontrivial homoclinic solution to zero solution of (1) if

$$u \neq 0, u(t) \to 0, \qquad |t| \to \infty. \tag{2}$$

They are known in phase transitions models as ground states or pulses (see [1]). The existence of homoclinic and heteroclinic solutions of fourth-order equations is studied by various authors (see [2–12] and references therein). Sun and Wu [4] obtained existence of two homoclinic solutions for a class of fourth-order differential equations:

$$u^{(4)} + wu'' + a(t)u = f(t,u) + \lambda h(t)\,|u|^{p-2}\,u,\ t \in \mathbb{R},$$

where w is a constant, $\lambda > 0, 1 \leq p < 2$, $a \in C\left(\mathbb{R},\mathbb{R}^+\right)$ and $h \in L^{\frac{2}{2-p}}(\mathbb{R})$ by using mountain pass theorem.

Yang [8] studies the existence of infinitely many homoclinic solutions for a the fourth-order differential equation:

$$u^{(4)} + wu'' + a(t)u = f(t,u),\ t \in \mathbb{R},$$

where w is a constant, $a \in C(\mathbb{R})$ and $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$. A critical point theorem, formulated in the terms of Krasnoselskii's genus (see [13], Remark 7.3), is applied, which ensures the existence of infinitely many homoclinic solutions.

We suppose the following conditions on the functions a, f and V.

(A) $a \in C(\mathbb{R}, \mathbb{R}^+)$ and $a(t) \to 0$ as $|t| \to +\infty$.

(F_1) There are numbers p and q s.t. $1 < q < 2 \leq p$ and for $f \in C^1(\mathbb{R}, \mathbb{R})$

$$uf(t,u) \leq qF(t,u), \ \forall u \in \mathbb{R}, u \neq 0,$$

where $F(t,u) = \int_0^u f(t,x)dx$.

(F_2) $|f(t,u)| \leq b(t)|u|^{q-1}, \quad \forall (t,u) \in \mathbb{R} \times \mathbb{R}$, where b is a positive function, s.t. $b \in L^r(\mathbb{R}) \bigcap L^{\frac{p}{2-q}}(\mathbb{R})$, where $r = \frac{p}{p-q}$.

(F_3) There exists an interval $J \subset \mathbb{R}$ and a constant $c > 0$ s. t. $F(t,u) \geq c|u|^q, \quad \forall (t,u) \in J \times \mathbb{R}$.

(F_4) $F(t,-u) = F(t,u)$ for all $(t,u) \in \mathbb{R} \times \mathbb{R}$.

(V) There exist positive constants v_1 and v_2 such that $0 < v_1 \leq V(t) \leq v_2, \quad \forall t \in \mathbb{R}$.

Let

$$w^* = \inf_{u \neq 0} \frac{\int_{\mathbb{R}} \left(|u''(t)|^p + |u(t)|^p\right) dt}{\int_{\mathbb{R}} |u'(t)|^p \, dt}.$$

Denote by X the Sobolev's space

$$X := W^{2,p}(\mathbb{R}) = \{u \in L^p(\mathbb{R}) : u' \in L^p(\mathbb{R}), u'' \in L^p(\mathbb{R})\},$$

equipped by the usual norm

$$||u||_X := \left(\int_{\mathbb{R}} \left(|u''(t)|^p + |u'(t)|^p + |u(t)|^p \right) dt \right)^{1/p}.$$

The functional $I : X \to \mathbb{R}$ is defined as follows

$$I(u) = \int_{\mathbb{R}} (\Phi_p(u''(t)) - w\Phi_p(u'(t)) + V(t)\Phi_p(u(t)))dt - \int_{\mathbb{R}} a(t)F(t,u(t)dt, \tag{3}$$

where $\Phi(t) = \frac{|t|^p}{p}$ for $p \geq 2$.

Under conditions $(A), (F_1) - (F_3)$ and V the functional I is differentiable and for all $u, v \in X$ we have

$$\begin{aligned} \langle I'(u), v \rangle \quad &= \quad \int_{\mathbb{R}} \left(\varphi_p\left(u''(t)\right) v''(t) - w\varphi_p\left(u'(t)\right) v'(k)\right) dt + V(t)\varphi_p\left(u(t)\right) v(t)dt \\ &\quad - \int_{\mathbb{R}} a(t) f\left(t, u(t)\right) v(t) dt. \end{aligned}$$

where $\langle ., . \rangle$ means the duality pairing between X and it's dual space X^*. The homoclinic solutions of the Equation (1) are the critical points of the functional I, i.e., u_0 is a homoclinic solution of the problem if $\langle I'(u_0), v \rangle = 0$ for every $v \in X$ (see [6,11,12]).

Let $v_0 = \min\{1, v_1\}$, where v_1 is the positive constant from condition (V). Our main result is:

Theorem 1. *Let $p \geq 2, \quad w < v_0 w^*$ and the functions a, f and V satisfy the assumptions (A), $(F_1) - (F_3)$ and (V). Then the Equation (1) has at least one nonzero homoclinic solution $u_0 \in X$. Additionally if (F_4) holds, the Equation (1) has infinitely many nonzero solutions u_j such that $||u_j||_\infty \to 0$ as $j \to \infty$.*

Remark 1. *An example of a function $f(t,u)$, which satisfies the assumptions $(F_1) - (F_4)$ is as follows. Let $p = 3, q = \frac{3}{2}$ and $f(t,u) = \alpha(t)|u|^{1/2}u$, where*

$$\alpha(t) = \begin{cases} \frac{3-t^2}{2}, & |t| \leq 1, \\ \frac{1}{|t|}, & |t| \geq 1. \end{cases}$$

We have that $r = \frac{p}{p-q} = 2$, $\frac{p}{2-q} = 6$ *and* $b(t) = \alpha(t) \in L^2(\mathbb{R}) \cap L^6(\mathbb{R})$, *because* $\int\limits_1^\infty \frac{1}{t^2} dt = 1$ *and* $\int\limits_1^\infty \frac{1}{t^6} dt = \frac{1}{5}$. *Moreover* $\alpha(t) \geq 1$ *if* $t \in (-1,1) = J$. *Next, we have*

$$\begin{aligned} |f(t,u)| &= \alpha(t)|u|^{3/2}, \\ F(t,u) &= \frac{2}{5}\alpha(t)|u|^{5/2}, \end{aligned}$$

and $F(t,u) \geq \frac{2}{5}|u|^{5/2}$, $t \in J = (-1,1)$.

As an open problem we state the existence of weak solutions of the problem when $1 < q < p < 2$.

This paper is organized as follows. In Section 2 we present the variational formulation of the problem and critical point theorems used in the proof of the main result. In Section 3, we give the proof of Theorem 1.

2. Preliminaries

In this section we give the variational formulation of the problem and present two critical point theorems.

Let X_1 be the Sobolev's space

$$X_1 := \{u \in X : \int_{\mathbb{R}} \left(|u''(t)|^p - w\,|u'(t)|^p + V(t)\,|u(t)|^p \right) dt < \infty\},$$

equipped by the norm

$$||u|| := \left(\int_{\mathbb{R}} \left(|u''(t)|^p - w\,|u'(t)|^p + V(t)\,|u(t)|^p \right) dt \right)^{1/p}.$$

Denote

$$w^* = \inf_{u \neq 0} \frac{\int_{\mathbb{R}} \left(|u''(t)|^p + |u(t)|^p\right) dt}{\int_{\mathbb{R}} |u'(t)|^p\, dt}.$$

and $v_0 = \min\{1, v_1\}$. The next lemma shows that under condition (V) for $w < v_0 w^*$ the norms $||.||$ and $||.||_X$ are equivalent and $X = X_1$.

Lemma 1. *Let* $w < v_0 w^*$. *Then, there exists a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}} \left(|u''(t)|^p - w\,|u'(t)|^p + V(t)\,|u(t)|^p \right) dt \geq C\,\|u\|_X^p\ ,\ \forall u \in X. \tag{4}$$

Proof of Lemma 1. In view of Lemma 4.10 in [14], there exists a positive constant $K = K(p)$ depending only on p such that

$$\int_{\mathbb{R}} |u'(t)|^p\, dt \leq K \int_{\mathbb{R}} \left(|u''(t)|^p + |u(t)|^p \right) dt.$$

Then

$$\frac{1}{K} \leq w^* = \inf_{u \neq 0} \frac{\int_{\mathbb{R}} \left(|u''(t)|^p + |u(t)|^p\right) dt}{\int_{\mathbb{R}} |u'(t)|^p\, dt}.$$

Let

$$C_0 = \frac{v_0 w^* - w}{(K+1) v_0 w^*}$$

and $C = v_0 C_0$. We have

$$
\begin{aligned}
& \int_{\mathbb{R}} \left(\left| u''(t) \right|^p - w \left| u'(t) \right|^p + V(t) \left| u(t) \right|^p \right) dt \\
\geq\ & v_0 \int_{\mathbb{R}} \left(\left| u''(t) \right|^p - \frac{w}{v_0} \left| u'(t) \right|^p + \left| u(t) \right|^p \right) dt \\
=\ & v_0((1 - \frac{w}{v_0 w^*}) \int_{\mathbb{R}} \left(\left| u''(t) \right|^p + \left| u(t) \right|^p \right) dt \\
& + \frac{w}{v_0 w^*} \int_{\mathbb{R}} \left(\left| u''(t) \right|^p - w^* \left| u'(t) \right|^p + \left| u(t) \right|^p \right) dt) \\
\geq\ & v_0(1 - \frac{w}{v_0 w^*}) \int_{\mathbb{R}} \left(\left| u''(t) \right|^p + \left| u(t) \right|^p \right) dt \\
=\ & v_0 C_0 (K+1) \int_{\mathbb{R}} \left(\left| u''(t) \right|^p + \left| u(t) \right|^p \right) dt \\
\geq\ & v_0 C_0 \int_{\mathbb{R}} \left(\left| u''(t) \right|^p + \left| u'(t) \right|^p + \left| u(t) \right|^p \right) dt = C||u||_X^p,
\end{aligned}
$$

which completes the proof. □

By Brezis [15], Theorem 8.8 and Corollary 8.9 for $u \in X$ and $s > p$

$$
\begin{aligned}
||u||_\infty \quad &: \quad = ||u||_{L^\infty(\mathbb{R})} \leq C_1 ||u||_X, \\
\int_{\mathbb{R}} |u(t)|^s dt \quad &\leq \quad ||u||_\infty^{s-p} ||u||_X^p,
\end{aligned}
$$

and $\lim_{|t| \to \infty} u(t) = 0$.

We consider the functional $I : X \to \mathbb{R}$

$$
I(u) = \int_{\mathbb{R}} (\Phi_p(u''(t)) - w\Phi_p(u'(t)) + V(t)\Phi_p(u(t)))dt - \int_{\mathbb{R}} a(t)F(t, u(t)dt, \tag{5}
$$

where $\Phi(t) = \frac{|t|^p}{p}$ for $p \geq 2$.

One can show that under conditions $(A), (F_1) - (F_3)$ and V the functional I is differentiable and for all $u, v \in X$ we have

$$
\begin{aligned}
\langle I'(u), v \rangle = \int_{\mathbb{R}} \left(\varphi_p \left(u''(t) \right) v''(t) - w\varphi_p \left(u'(t) \right) v'(k) \right) dt + V(t)\varphi_p \left(u(t) \right) v(t) dt \\
- \int_{\mathbb{R}} a(t) f \left(t, u(t) \right) v(t) dt.
\end{aligned} \tag{6}
$$

Let $L_a^p(\mathbb{R}), p \geq 1$ be the weighted Lebesque space of functions $u : \mathbb{R} \to \mathbb{R}$ with norm $||u||_{p,a} := \left(\int_{\mathbb{R}} a(t)|u(t)|^p dt \right)^{1/p}$. We have

Lemma 2. *Assume that the assumptions (A) and (V) hold. Then, the inclusion $X \subset L_a^p(\mathbb{R})$ is continuous and compact.*

Proof of Lemma 2. The embedding $X \subset L_a^p(\mathbb{R})$ is continuous by the boundedness of the function a by (A). We show that the inclusion is compact. Let $\{u_j\} \subset X$ be a sequence such that $||u_j|| \leq M$ and $u_j \rightharpoonup u$ weakly in X. We'll show that $u_j \to u$ strongly in $L_a^p(\mathbb{R})$. Without loss of generality we can assume that $u = 0$, considering the sequence $\{u_j - u\}$. By (A) for any $\varepsilon > 0$, there exists $R > 0$, such that for $|t| \geq R$

$$
0 \leq a(t) \leq \frac{\varepsilon}{2(1 + M^p)}.
$$

Then
$$\int\limits_{|t|\geq R} a(t)|u_j(t)|^p dt \leq \frac{\varepsilon M^p}{2(1+M^p)}.$$

By Sobolev's imbedding theorem $u_j \to 0$ strongly in $C([-R,R])$ and there exists j_0 such that for $j > j_0$:
$$\int\limits_{|t|\leq R} a(t)|u_j(t)|^p dt < \frac{\varepsilon}{2(1+M^p)}.$$

Then, for $j > j_0$ we have $\int\limits_{\mathbb{R}} a(t)|u_j(t)|^p dt < \varepsilon$, which shows that $u_j \to 0$ strongly in $L^p_a(\mathbb{R})$. □

Lemma 3. *Let assumptions (A), $(F_1)-(F_3)$ and (V) hold. If $u_j \rightharpoonup u$ weakly in X, there exists a subsequence of the sequence $\{u_j\}$, still denoted by $\{u_j\}$ such that $f(t,u_j) \to f(t,u)$ in $L^p_a(\mathbb{R})$.*

Proof of Lemma 3. Let $u_j \rightharpoonup u$ weakly in X. By Banach-Steinhaus theorem there exists $M_1 > 0$, such that $||u_j|| \leq M_1$ and $||u|| \leq M_1$. By the elementary inequality for $a > 0, b > 0, p > 1$
$$(a+b)^p \leq 2^{p-1}(a^p + b^p),$$
and (F_2) we have
$$\begin{aligned} |f(t,u_j) - f(t,u)|^p &\leq 2^{p-1}(|f(t,u_j)|^p + |f(t,u)|^p) \\ &\leq 2^{p-1}|b(t)|^p(|u_j|^{p(q-1)} + |u|^{p(q-1)}). \end{aligned}$$

Let $0 < a(t) \leq A$. Then, by Hölder inequality and $b \in L^{\frac{p}{2-q}}(\mathbb{R})$ it follows that
$$\begin{aligned} & \int_{\mathbb{R}} a(t)|f(t,u_j(t)) - f(t,u(t))|^p dt \\ \leq\ & 2^{p-1}A\int_{\mathbb{R}} |b(t)|^p(|u_j|^{p(q-1)} + |u|^{p(q-1)})dt \\ \leq\ & 2^{p-1}A(\int_{\mathbb{R}} |b(t)|^{\frac{p}{2-q}})^{2-q}((\int_{\mathbb{R}} |u_j(t)|^p dt)^{q-1} + (\int_{\mathbb{R}} |u(t)|^p dt)^{q-1}) \\ \leq\ & 2^p A||b||^p_{L^{\frac{p}{2-q}}(\mathbb{R})} M_1^{p(q-1)}. \end{aligned}$$

By Lemma 2, $u_j \rightharpoonup u$ weakly in X implies that there exists a subsequence $\{u_j\}$, such that $u_j \to u$ strongly in $L^p_a(\mathbb{R})$. By analogous way as above we have that there exists $B > 0$, such that
$$\int_{\mathbb{R}} |f(t,u_j(t)) - f(t,u(t))|^p dt \leq B.$$

Let $\varepsilon > 0, R > 0$ are s.t. $0 < a(t) < \frac{\varepsilon}{2B}$ for $|t| \geq R$ by (A). Then
$$\int_{|t|\geq R} a(t)|f(t,u_j(t)) - f(t,u(t))|^p dt < \frac{\varepsilon}{2}. \quad (7)$$

Let $0 < a_R < a(t) \leq A$ for $|t| \leq R$. By $u_j \to u$ strongly in $L^p_a(\mathbb{R})$ it follows that
$$\int_{|t|\leq R} a(t)|u_j(t) - u(t)|^p dt \geq a_R \int_{|t|\leq R} |u_j(t) - u(t)|^p dt \to 0$$

and $u_j(t) - u(t) \to 0$ a.e. in $|t| \leq R$. Then, by Lebesque's dominated convergence theorem

$$I_R := \int_{|t|\leq R} a(t)|f(t,u_j(t)) - f(t,u(t))|^p dt \to 0.$$

Let j_0 is sufficiently large, such that for $j > j_0, 0 \leq I_R < \frac{\varepsilon}{2}$. Then by (7) for $j > j_0$ we have

$$\int_{\mathbb{R}} a(t)|f(t,u_j(t)) - f(t,u(t))|^p dt < \varepsilon,$$

which completes the proof. □

Next we have:

Lemma 4. *Under assumptions $(A), (F_1) - (F_3), (V)$ the functional $I \in C^1(X, \mathbb{R})$ and the identity (6) holds for all $u, v \in X$. holds.*

It can be proved in a standard way using Lemma 3 (see Yang [8], Tersian, Chaparova [6]).

Lemma 5. *Under assumptions $(A), (F_1) - (F_3)$ and (V) the functional I satisfies the (PS) condition.*

Proof of Lemma 5. Let $\{u_j\}$ be a sequence such that $\{I(u_j)\}$ is bounded in X and $I'(u_j) \to 0$ in X^*. Then, there exists a constant $C_1 > 0$, s.t.

$$||I(u_j)|| \leq C_1, \quad ||I'(u_j)||_{X^*} \leq C_1.$$

By (F_2) we have

$$\begin{aligned} C_1 + \frac{C_1}{q}||u_j|| &\geq \frac{1}{q} < I'(u_j), u_j > -I(u_j) \\ &= \left(\frac{1}{q} - \frac{1}{p}\right)||u_j||^p + \int_{\mathbb{R}} a(t)(F(t,u_j(t)) - \frac{1}{q} f(t,u_j(t))u_j(t))dt \\ &\geq \left(\frac{1}{q} - \frac{1}{p}\right)||u_j||^p. \end{aligned}$$

Then, $\{u_j\}$ is a bounded sequence in X and up to a subsequence, still denoted by $\{u_j\}$, $u_j \rightharpoonup u$ weakly in X. There exists $M_2 > 0$, such that $||u_j|| \leq M_2, ||u|| \leq M_2$. By Lemma 2, $u_m \to u$ in $L^2_a(\mathbb{R})$ and by Lemma 3, $f(t,u_m(t)) \to f(t,u(t))$ in $L^2_a(\mathbb{R})$. By Hölder inequality we have:

$$\begin{aligned} I_j &:= \int_{\mathbb{R}} a(t)(f(t,u_j(t)) - f(t,u(t)))(u_j(t) - u(t))dt \\ &= \int_{\mathbb{R}} a^{\frac{p-1}{p}}(t)(f(t,u_j(t)) - f(t,u(t)))a^{\frac{1}{p}}(t)(u_j(t) - u(t))dt \\ &\leq A^{\frac{p-1}{p}} \int_{\mathbb{R}} a(t)|u_j(t) - u(t)|^p dt \left(\int_{\mathbb{R}} |f(t,u_j(t)) - f(t,u(t))|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}}. \end{aligned}$$

As in the proof of Lemma 3, by assumption (F_2), $b \in L^{\frac{p}{p-q}}(\mathbb{R})$ and Hölder inequality we have for $p_1 = \frac{p}{p-1} > 1$:

$$\begin{aligned}
&\int_{\mathbb{R}} |f(t,u_j(t)) - f(t,u(t))|^{p_1} dt \\
\leq\ & 2^{p_1-1} \int_{\mathbb{R}} |b(t)|^{p_1}| \left(|u_j(t)|^{(q-1)p_1} + |u(t)|^{(q-1)p_1} \right) dt \\
\leq\ & 2^{p_1-1} \left(\int_{\mathbb{R}} |b|^{\frac{p}{p-q}} dt \right)^{\frac{p-q}{p-1}} \left(\left(\int_{\mathbb{R}} |u_j|^p dt \right)^{\frac{q-1}{p-1}} + \left(\int_{\mathbb{R}} |u|^p dt \right)^{\frac{q-1}{p-1}} \right) \\
\leq\ & 2^{p_1} ||b||^{p_1}_{L^{\frac{p}{p-q}}} M_2^{(q-1)p_1}.
\end{aligned}$$

Then, by $u_j \to u$ in $L^2_a(\mathbb{R})$ it follows that $I_j \to 0$ as $j \to \infty$. Next, we have

$$||u_j - u||^p \leq < I'(u_j) - I'(u), u_j - u > +I_j,$$

which shows that $u_j \to u$ in X. □

Next, we recall a minimization theorem which will be used in the proof of Theorem 1. (see [16], Theorem 2.7 of [13]).

Theorem 2. *(Minimization theorem) Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfying (PS) condition. If J is bounded below, then $c = \inf_E I$ is a critical value of J.*

We will use also the following generalization of Clark's theorem (see Rabinowitz [13], p. 53) due to Z. Liu and Z. Wang [17]:

Theorem 3. *(Generalized Clark's theorem, [17]) Let E be a Banach spa ce, $J \in C^1(E,\mathbb{R})$. Assume that J satisfies the (PS) condition, it is even, bounded from below and $J(0) = 0$. If for any $k \in \mathbb{N}$, there exists a $k-$dimensional subspace E^k of E and $\rho_k > 0$ such that $\sup_{E^k \cap S_{\rho_k}} J < 0$, where $S_\rho = \{u \in E,\ \|u\|_E = \rho\}$, then at least one of the following conclusions holds:*

1. *There exists a sequence of critical points $\{u_k\}$ satisfying $J(u_k) < 0$ for all k and $\lim_{k\to\infty} \|u_k\|_E = 0$.*
2. *There exists $r > 0$ such that for any $0 < \alpha < r$ there exists a critical point u such that $\|u\|_E = \alpha$ and $J(u) = 0$.*

Note that Theorem 3 implies the existence of infinitely many pairs of critical points $(u_k, -u_k)$, $u_k \neq 0$ of J, s.t. $J(u_k) \leq 0$, $\lim_{k\to+\infty} J(u_k) = 0$ and $\lim_{k\to+\infty} \|u_k\|_E = 0$.

Lemma 6. *Assume that assumptions (A), (F_2) and (V) hold. Then the functional I is bounded from below.*

Proof of Lemma 6. By (F_2) and the proof of Lemma 3 we have

$$|F(t,u)| \leq \frac{1}{q} b(t)|u|^q.$$

and

$$\begin{aligned}
I(u) &= \frac{1}{p}||u||^p - \int_{\mathbb{R}} a(t)F(t,u(t))dt \\
&\geq \frac{1}{p}||u||^p - \frac{A}{q} \int_{\mathbb{R}} b(t)|u(t)|^q dt \\
&\geq \frac{1}{p}||u||^p - \frac{A}{q} \left(\int_{\mathbb{R}} |b(t)|^{\frac{p}{p-q}} dt \right)^{\frac{p-q}{p}} \left(\int_{\mathbb{R}} |u(t)|^p dt \right)^{\frac{q}{p}} \\
&\geq \frac{1}{p}||u||^p - \frac{A}{q} ||b||_{L^{\frac{p}{p-q}}} ||u||^q.
\end{aligned}$$

By $p > q$ it follows that I is bounded from below functional. □

3. Proof of the Main Result

In this section we prove Theorem 1. The proof is based on the minimization Theorem 2 and multiplicity result Theorem 3. Their conditions are satisfied according to Lemmas 1–6.

Proof of Theorem 1. The functional I satisfies the assumptions of minimization Theorem 2. Let u_0 be the minimizer of I. Since $I(0) = 0$ to show that $u_0 \neq 0$, let us take $v \in W_0^{2,p}(J)$, where J is the interval from condition (F_3). Suppose that $||v||_\infty \leq 1$. Then for $\lambda > 0$ by (F_3)

$$\begin{aligned} I(\lambda v) &= \frac{\lambda^p}{p}||v||^p - \int_J a(t)F(t, \lambda v(t))dt \\ &\leq \frac{\lambda^p}{p}||v||^p - c\lambda^q \int_J a(t)|v(t)|^q dt. \end{aligned}$$

By $1 < q < p$ and the last inequality it follows for λ_0 sufficiently small and $\lambda_0 > \lambda > 0$ $I(\lambda v) < 0$. Then $I(u_0) = \min\{I(u) : u \in X\} < I(\lambda v) < 0$ and u_0 is a nonzero weak solution. Let the condition (F_4) holds additionally. We show that the functional I satisfies the assumptions of Theorem 3. We construct a sequence of finite dimensional subspaces $X_n \subset X$ and spheres $S_{r_n}^{n-1} \subset X_n$ with sufficiently small radius $r_n > 0$ such that $sup\{I(u) : u \in S_{r_n}^{n-1}\} < 0$. Let $J = (a, b) \subset \mathbb{R}$ and for $k \in \{1, 2, ..., n\}$ $J_k = (x_{k-1}, x_k)$, where $x_k = a + \frac{k}{n}(b - a)$. Next, we choose functions $v_k \in C_0^2(J_k)$ such that $||v_k||_\infty < \infty$ and $||v_k||_X = 1$.

Let X_n be the $n-$ dimensional subspace $X_n := span\{v_1, ..., v_k\} \subset X$ and

$$S_\rho^{n-1} := \{u \in X_n : ||u||_X = \rho\}.$$

For $u = \sum_{k=1}^n c_k v_k \in X_n$ we have

$$\begin{aligned} ||u||^p &= \int_{\mathbb{R}} \left(|u''(t)|^p - w\,|u'(t)|^p + V(t)\,|u(t)|^p\right) dt \\ &= \sum_{j=k}^n |c_k|^p \int_{J_k} (|v_k''(t)|^p - w|v_k'(t)|^p + V(t)|v_k(t)|^p))dt \\ &= \sum_{k=1}^n |c_k|^p. \end{aligned}$$

By analogous way for $\gamma_k = \int_{J_k} (|v_k(t)|^q dt > 0$ we have

$$||u||_n^q = \sum_{k=1}^n \gamma_k |c_k|^q \tag{8}$$

The space X_n is n-dimensional and the norms $||.||$ and $||.||_n$ are equivalent. There are positive constants d_{1n} and d_{2n} s.t.

$$d_{1n}||u|| \leq ||u||_n \leq d_{2n}||u||, \quad \forall u \in X_n. \tag{9}$$

Then, for $u \in X_n \bigcap S_1^{n-1}$

$$\begin{aligned} I(\lambda u) &= \frac{\lambda^p}{p}||u||^p - \sum_{k=1}^n \int_{J_k} a(t)F(t, \lambda c_k v_k(t))dt \\ &\leq \frac{\lambda^p}{p}||u||^p - c\lambda^q \sum_{k=1}^n |c_k|^q \int_{J_k} a(t)|v_k(t)|^q dt \\ &\leq \frac{\lambda^p}{p}||u||^p - c\lambda^q d_{1n}||u||^q \end{aligned}$$

By $1 < q < p$ and the last inequality it follows that $I(v) < 0$ for $v \in S^{\rho}_{n-1} := \{u \in X_n : ||u|| = \rho\}$. Finally, all assumptions of Theorem 3 are satisfied and by Remark 1 there exist infinitely many weak solutions $\{u_j\}$ of the problem (1), such that $I(\{u_j\}) \leq 0$ and $||u_j|| \to 0$. By imbedding $X \subset L^{\infty}(\mathbb{R})$ it follows that $||u_j||_{\infty} \to 0$ as $j \to \infty$ which completes the proof. □

4. Conlusions

In this paper, we obtained the existence of infinitely many homoclinic solutions of Equation (1) under conditions $(A), (F_1) - (F_4), (V)$ in the case $1 < q < 2 \leq p$. The equation is an extension of the stationary Fisher-Kolmogorov equation which appears in the phase transition models. The variational approach is applied based on the multiple critical point theorem due to Liu and Wang. It will be interesting to extend the result to the case $1 < q < p < 2$.

Author Contributions: Conceptualization, S.T.; methodology, S.T.; software, S.T.; validation, S.T.; formal Analysis, S.T.; writing—original draft preparation, S.T.; writing—review and editing, S.T.; visualization, S.T.; supervision, S.T.; funding acquisition, S.T. The author has read and agreed to the published version of the manuscript.

Funding: S.T. is partially supported by the Bulgarian National Science Fund under Project KP-06-N32/7 and bilateral agreement between BAS and Serbian Academy of Sciences and Arts (SASA), 2020-2022.

Acknowledgments: The author is thankful to the reviewer's remarks.

Conflicts of Interest: The author declares no conflict of interest.

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