mathematics

Article

Improved Conditions for Oscillation of Functional Nonlinear Differential Equations

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Received: 17 March 2020; Accepted: 3 April 2020; Published: 9 April 2020

Abstract: The aim of this work is to study oscillatory properties of a class of fourth-order delay differential equations. New oscillation criteria are obtained by using generalized Riccati transformations. This new theorem complements and improves a number of results reported in the literature. Some examples are provided to illustrate the main results.

Keywords: oscillatory solutions; nonoscillatory solutions; fourth-order; delay differential equations; riccati transformation

1. Introduction

In this article, we investigate the asymptotic behavior of solutions of the fourth-order differential equation

$$\left(b\left(x\right)\left(w'''\left(x\right)\right)^{\kappa}\right)' + \sum_{i=1}^{j} q_i\left(x\right) f\left(w\left(\vartheta_i\left(x\right)\right)\right) = 0,\ x \geq x_0. \tag{1}$$

Throughout this paper, we assume the following conditions hold:

(Z_1) κ are quotient of odd positive integers;

(Z_2) $b \in C^1\left([x_0,\infty),\mathbb{R}\right),\ b\left(x\right) > 0,\ b'\left(x\right) \geq 0$ and under the condition

$$\int_{x_0}^{\infty} \frac{1}{b^{1/\kappa}\left(x\right)} \mathrm{d}x = \infty. \tag{2}$$

(Z_3) $q_i \in C[x_0,\infty),\ q\left(x\right) > 0,\ i = 1,2,\ldots,j,$

(Z_4) $\vartheta_i \in C[x_0,\infty),\ \vartheta_i\left(x\right) \leq x,\ \lim_{x\to\infty} \vartheta_i\left(x\right) = \infty;\ i = 1,2,..,j,$

(Z_5) $f \in C\left(\mathbb{R},\mathbb{R}\right)$ such that

$$f\left(x\right)/x^{\kappa} \geq \ell > 0,\ \text{for } x \neq 0. \tag{3}$$

Definition 1. *The function $y \in C^3[\nu_y,\infty),\ \nu_y \geq \nu_0,$ is called a solution of equation (1), if $b\left(x\right)\left(w'''\left(x\right)\right)^{\kappa} \in C^1[x_w,\infty)$, and $w\left(x\right)$ satisfies (1) on $[x_w,\infty)$.*

Definition 2. *A solution of (1) is called oscillatory if it has arbitrarily large zeros on* $[x_w, \infty)$, *and otherwise is called to be nonoscillatory.*

Definition 3. *Equation (1) is said to be oscillatory if all its solutions are oscillatory.*

Differential equations arise in modeling situations to describe population growth, biology, economics, chemical reactions, neural networks, and in aeromechanical systems, etc.; see [1].

More and more scholars pay attention to the oscillatory solution of functional differential equations, see [2–5], especially for the second/third-order, see [6–8], or higher-order equations see [9–17]. With the development of the oscillation for the second-order equations, researchers began to study the oscillation for the fourth-order equations, see [18–25].

In the following, we show some previous results in the literature which related to this paper: Moaaz et al. [21] studied the fourth-order nonlinear differential equations with a continuously distributed delay

$$\left(b\left(x\right)\left(\left(w\left(x\right)\right)'''\right)^{\alpha}\right)' + \int_a^c q\left(x,\xi\right) f\left(w\left(g\left(x,\xi\right)\right)\right) d\xi = 0, \tag{4}$$

by means of the theory of comparison with second-order delay equations, the authors established some oscillation criteria of (4) under the condition

$$\int_{x_0}^{\infty} \frac{1}{b^{1/\kappa}\left(x\right)} \mathrm{d}x < \infty. \tag{5}$$

Cesarano and Bazighifan [22] considered Equation (4), and established some new oscillation criteria by means of Riccati transformation technique.

Agarwal et al. [9] and Baculikova et al. [10] studied the equation

$$\left(\left(w^{(n-1)}\left(x\right)\right)^{\kappa}\right)' + q\left(x\right) f\left(w\left(\vartheta\left(x\right)\right)\right) = 0 \tag{6}$$

and established some new sufficient conditions for oscillation.

Theorem 1 (See [9])**.** *If there exists a positive function* $g \in C^1\left(\left[x_0,\infty\right),\left(0,\infty\right)\right)$, *and* $\theta > 1$ *is a constant such that*

$$\limsup_{x\to\infty} \int_{x_0}^{x} \left(g\left(s\right) q\left(s\right) - \lambda\theta \frac{\left(g'\left(s\right)\right)^{\kappa+1}}{\left(g\left(s\right)\vartheta^{n-2}\left(s\right)\vartheta'\left(s\right)\right)^{\kappa}}\right) \mathrm{d}s = \infty, \tag{7}$$

where $\lambda := \left(1/\left(\kappa+1\right)\right)^{\kappa+1}\left(2\left(n-1\right)!\right)^{\kappa}$, *then every solution of (6) is oscillatory.*

Theorem 2 (See [10])**.** *Let* $f\left(x^{1/\kappa}\right)/x \geq 1$ *for* $0 < x \leq 1$ *such that*

$$\lim_{x\to\infty}\inf \int_{\vartheta_i(x)}^{x} q\left(s\right) f\left(\frac{\varsigma}{\left(n-1\right)!}\frac{\vartheta^{n-1}\left(s\right)}{b^{1/\kappa}\left(\vartheta\left(s\right)\right)}\right) \mathrm{d}s > \frac{1}{\mathrm{e}} \tag{8}$$

for some $\varsigma \in (0,1)$, *then every solution of (6) is oscillatory.*

To prove this, we apply the previous results to the equation

$$w^{(4)}\left(x\right) + \frac{c_0}{x^4} w\left(\frac{9}{10}x\right) = 0,\ x \geq 1, \tag{9}$$

then we get that (9) is oscillatory if

The condition	(7)	(8)
The criterion	$c_0 > 60$	$c_0 > 28.7$

From above, we see that [10] improved the results in [9].

The motivation in studying this paper is complementary and improves the results in [9,10].

The paper is organized as follows. In Section 2, we state some lemmas, which will be useful in the proof of our results. In Section 3, by using generalized Riccati transformations, we obtain a new oscillation criteria for (1). Finally, some examples are considered to illustrate the main results.

For convenience, we denote

$$\delta(x) := \int_x^\infty \frac{1}{b^{1/\kappa}(s)} \mathrm{d}s, \; F_+(x) := \max\{0, F(x)\},$$

$$\psi(x) := g(x)\left(\ell \sum_{i=1}^{j} q_i(x)\left(\frac{\vartheta_i^3(x)}{x^3}\right)^{\kappa} + \frac{\varepsilon\beta_1^{(1+\kappa)/\kappa} x^2 - 2\beta_1\kappa}{2b^{\frac{1}{\kappa}}(x)\,\delta^{\kappa+1}(x)}\right),$$

$$\phi(x) := \frac{g'_+(x)}{g(x)} + \frac{(\kappa+1)\,\beta_1^{1/\kappa}\varepsilon x^2}{2b^{\frac{1}{\kappa}}(x)\,\delta(x)}, \; \phi^*(x) := \frac{\xi'_+(x)}{\xi(x)} + \frac{2\beta_2}{\delta(x)},$$

and

$$\psi^*(x) := \xi(x)\left(\int_x^\infty \left(\frac{\ell}{b(v)} \int_v^\infty \sum_{i=1}^{j} q_i(s)\frac{\vartheta_i^\kappa(s)}{s^\kappa} \mathrm{d}s\right)^{1/\kappa} \mathrm{d}v + \frac{\beta_2^2 - \beta_2 b^{\frac{-1}{\kappa}}(x)}{\delta^2(x)}\right),$$

where β_1, β_2 are constants and $g, \xi \in C^1([x_0,\infty), (0,\infty))$.

Remark 1. *We define the generalized Riccati substitutions*

$$\pi(x) := g(x)\left(\frac{b(x)\,(w''')^\kappa(x)}{w^\kappa(x)} + \frac{\beta_1}{\delta^\kappa(x)}\right), \tag{10}$$

and

$$\varpi(x) := \xi(x)\left(\frac{w'(x)}{w(x)} + \frac{\beta_2}{\delta(x)}\right). \tag{11}$$

2. Some Auxiliary Lemmas

Next, we begin with the following lemmas.

Lemma 1 ([8])**.** *Let β be a ratio of two odd numbers, $V > 0$ and U are constants. Then,*

$$P^{(\beta+1)/\beta} - (P-Q)^{(\beta+1)/\beta} \le \frac{1}{\beta} Q^{1/\beta}\left[(1+\beta)P - Q\right], \; PQ \ge 0, \; \beta \ge 1$$

and

$$Uw - Vw^{(\beta+1)/\beta} \le \frac{\beta^\beta}{(\beta+1)^{\beta+1}} \frac{U^{\beta+1}}{V^\beta}.$$

Lemma 2 ([15])**.** *Suppose that $h \in C^n([x_0,\infty), (0,\infty))$, $h^{(n)}$ is of a fixed sign on $[x_0,\infty)$, $h^{(n)}$ not identically zero, and there exists a $x_1 \ge x_0$ such that*

$$h^{(n-1)}(x)\,h^{(n)}(x) \le 0,$$

for all $x \geq x_1$. *If we have* $\lim_{x\to\infty} h(x) \neq 0$, *then there exists* $x_\beta \geq x_1$ *such that*

$$h(x) \geq \frac{\beta}{(n-1)!} x^{n-1} \left| h^{(n-1)}(x) \right|,$$

for every $\beta \in (0,1)$ *and* $x \geq x_\beta$.

Lemma 3 ([19]). *If the function u satisfies* $u^{(j)} > 0$ *for all* $j = 0, 1, \ldots, n$, *and* $u^{(n+1)} < 0$, *then*

$$\frac{n!}{x^n} u(x) - \frac{(n-1)!}{x^{n-1}} \frac{d}{dx} u(x) \geq 0.$$

3. Oscillation Criteria

In this section, we shall establish some oscillation criteria for Equation (1).

Upon studying the asymptotic behavior of the positive solutions of (1), there are only two cases:

$$\begin{array}{llll} \text{Case (1)}: & w^{(r)}(x) > 0 & \text{for} & r = 0,1,2,3. \\ \text{Case (2)}: & w^{(r)}(x) > 0 & \text{for} & r = 0,1,3 \text{ and } w''(x) < 0. \end{array}$$

Moreover, from Equation (1) and condition (3), we have that $\left(b(x)\left(w'''(x)\right)^\kappa\right)'$. In the following, we will first study each case separately.

Lemma 4. *Assume that w be an eventually positive solution of (1) and* $w^{(r)}(x) > 0$ *for all* $r = 1, 2, 3$. *If we have the function* $\pi \in C^1[x, \infty)$ *defined as (10), where* $g \in C^1([x_0, \infty), (0, \infty))$, *then*

$$\pi'(x) \leq -\psi(x) + \phi(x)\pi(x) - \frac{\kappa \varepsilon x^2}{2\left(b(x) g(x)\right)^{1/\kappa}} \pi^{\frac{\kappa+1}{\kappa}}(x), \tag{12}$$

for all $x > x_1$, *where* x_1 *is large enough.*

Proof. Let w be an eventually positive solution of (1) and $w^{(r)}(x) > 0$ for all $r = 1, 2, 3$. Thus, from Lemma 2, we get

$$w'(x) \geq \frac{\varepsilon}{2} x^2 w'''(x), \tag{13}$$

for every $\varepsilon \in (0,1)$ and for all large x. From (10), we see that $\pi(x) > 0$ for $x \geq x_1$, and

$$\begin{aligned} \pi'(x) &= g'(x)\left(\frac{b(x)\left(w'''\right)^\kappa(x)}{w^\kappa(x)} + \frac{\beta_1}{\delta^\kappa(x)}\right) + g(x)\frac{\left(b\left(w'''\right)^\kappa\right)'(x)}{w^\kappa(x)} \\ &\quad -\kappa g(x)\frac{w^{\kappa-1}(x)\, w'(x)\, b(x)\left(w'''\right)^\kappa(x)}{w^{2\kappa}(x)} + \frac{\kappa\beta_1 g(x)}{b^{\frac{1}{\kappa}}(x)\,\delta^{\kappa+1}(x)}. \end{aligned}$$

Using (13) and (10), we obtain

$$\begin{aligned} \pi'(x) &\leq \frac{g'_+(x)}{g(x)}\pi(x) + g(x)\frac{\left(b(x)\left(w'''(x)\right)^\kappa\right)'}{w^\kappa(x)} \\ &\quad -\kappa g(x)\frac{\varepsilon}{2}x^2\frac{b(x)\left(w'''(x)\right)^{\kappa+1}}{w^{\kappa+1}(x)} + \frac{\kappa\beta_1 g(x)}{b^{\frac{1}{\kappa}}(x)\,\delta^{\kappa+1}(x)} \\ &\leq \frac{g'(x)}{g(x)}\pi(x) + g(x)\frac{\left(b(x)\left(w'''(x)\right)^\kappa\right)'}{w^\kappa(x)} \\ &\quad -\kappa g(x)\frac{\varepsilon}{2}x^2 b(x)\left(\frac{\pi(x)}{g(x)\,b(x)} - \frac{\beta_1}{b(x)\,\delta^\kappa(x)}\right)^{\frac{\kappa+1}{\kappa}} + \frac{\kappa\beta_1 g(x)}{b^{\frac{1}{\kappa}}(x)\,\delta^{\kappa+1}(x)}. \end{aligned} \tag{14}$$

Using Lemma 1 with $P = \pi(x) / (g(x) b(x))$, $Q = \beta_1 / (b(x) \delta^{\kappa}(x))$ and $\beta = \kappa$, we get

$$\left(\frac{\pi(x)}{g(x)b(x)} - \frac{\beta_1}{b(x)\delta^{\kappa}(x)}\right)^{\frac{\kappa+1}{\kappa}} \geq \left(\frac{\pi(x)}{g(x)b(x)}\right)^{\frac{\kappa+1}{\kappa}} - \frac{\beta_1^{1/\kappa}}{\kappa b^{\frac{1}{\kappa}}(x)\delta(x)}\left((\kappa+1)\frac{\pi(x)}{g(x)b(x)} - \frac{\beta_1}{b(x)\delta^{\kappa}(x)}\right). \quad (15)$$

From Lemma 3, we have that $w(x) \geq \frac{x}{3} w'(x)$ and hence

$$\frac{w(\vartheta_i(x))}{w(x)} \geq \frac{\vartheta_i^3(x)}{x^3}. \quad (16)$$

From (1), (14), and (15), we obtain

$$\pi'(x) \leq \frac{g'_+(x)}{g(x)}\pi(x) - \ell g(x)\sum_{i=1}^{j} q_i(x)\left(\frac{\vartheta_i^3(x)}{x^3}\right)^{\kappa} - \kappa g(x)\frac{\varepsilon}{2}x^2 b(x)\left(\frac{\pi(x)}{g(x)b(x)}\right)^{\frac{\kappa+1}{\kappa}} - \kappa g(x)\frac{\varepsilon}{2}x^2 b(x)\left(\frac{-\beta_1^{1/\kappa}}{\kappa b^{\frac{1}{\kappa}}(x)\delta(x)}\left((\kappa+1)\frac{\pi(x)}{g(x)b(x)} - \frac{\beta_1}{b(x)\delta^{\kappa}(x)}\right)\right) + \frac{\kappa\beta_1 g(x)}{b^{\frac{1}{\kappa}}(x)\delta^{\kappa+1}(x)}.$$

This implies that

$$\pi'(x) \leq \left(\frac{g'_+(x)}{g(x)} + \frac{(\kappa+1)\beta_1^{1/\kappa}\varepsilon x^2}{2b^{\frac{1}{\kappa}}(x)\delta(x)}\right)\pi(x) - \frac{\kappa\varepsilon x^2}{2b^{1/\kappa}(x)g^{1/\kappa}(x)}\pi^{\frac{\kappa+1}{\kappa}}(x) - g(x)\left(\ell\sum_{i=1}^{j} q_i(x)\left(\frac{\vartheta_i^3(x)}{x^3}\right)^{\kappa} + \frac{\varepsilon\beta_1^{(1+\kappa)/\kappa}x^2 - 2\beta_1\kappa}{2b^{\frac{1}{\kappa}}(x)\delta^{\kappa+1}(x)}\right).$$

Thus,

$$\pi'(x) \leq -\psi(x) + \phi(x)\pi(x) - \frac{\kappa\varepsilon x^2}{2(b(x)g(x))^{1/\kappa}}\pi^{\frac{\kappa+1}{\kappa}}(x).$$

The proof is complete. □

Lemma 5. *Assume that w is an eventually positive solution of (1), $w^{(r)}(x) > 0$ for $r = 1,3$ and $w''(x) < 0$. If we have the function $\varpi \in C^1[x,\infty)$ defined as (11), where $\xi \in C^1([x_0,\infty),(0,\infty))$, then*

$$\varpi'(x) \leq -\psi^*(x) + \phi^*(x)\varpi(x) - \frac{1}{\xi(x)}\varpi^2(x), \quad (17)$$

for all $x > x_1$, where x_1 is large enough.

Proof. Let w be an eventually positive solution of (1), $w^{(r)} > 0$ for $r = 1,3$ and $w''(x) < 0$. From Lemma 3, we get that $w(x) \geq x w'(x)$. By integrating this inequality from $\vartheta_i(x)$ to x, we get

$$w(\vartheta_i(x)) \geq \frac{\vartheta_i(x)}{x} w(x).$$

Hence, from (3), we have

$$f(w(\vartheta_i(x))) \geq \ell\frac{\vartheta_i^{\kappa}(x)}{x^{\kappa}} w^{\kappa}(x). \quad (18)$$

Integrating (1) from x to u and using $w'(x) > 0$, we obtain

$$\begin{aligned} b(u)\left(w'''(u)\right)^{\kappa} - b(x)\left(w'''(x)\right)^{\kappa} &= -\int_x^u \sum_{i=1}^{j} q_i(s) f\left(w\left(\vartheta_i(s)\right)\right) ds \\ &\leq -\ell w^{\kappa}(x) \int_x^u \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s. \end{aligned}$$

Letting $u \to \infty$, we see that

$$b(x)\left(w'''(x)\right)^{\kappa} \geq \ell w^{\kappa}(x) \int_x^{\infty} \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s$$

and so

$$w'''(x) \geq w(x) \left(\frac{\ell}{b(x)} \int_x^{\infty} \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s \right)^{1/\kappa}.$$

Integrating again from x to ∞, we get

$$w''(x) \leq -w(x) \int_x^{\infty} \left(\frac{\ell}{b(v)} \int_v^{\infty} \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s \right)^{1/\kappa} \mathrm{d}v. \tag{19}$$

From the definition of $\varpi(x)$, we see that $\varpi(x) > 0$ for $x \geq x_1$. By differentiating, we find

$$\varpi'(x) = \frac{\xi'(x)}{\xi(x)} \varpi(x) + \xi(x) \frac{w''(x)}{w(x)} - \xi(x) \left(\frac{\varpi(x)}{\xi(x)} - \frac{\beta_2}{\delta(x)} \right)^2 + \frac{\xi(x)\beta_2}{b^{1/\kappa}(x)\delta^2(x)}. \tag{20}$$

Using Lemma 1 with $P = \varpi(x)/\xi(x)$, $Q = \beta_2/\delta(x)$ and $\beta = 1$, we get

$$\left(\frac{\varpi(x)}{\xi(x)} - \frac{\beta_2}{\delta(x)} \right)^2 \geq \left(\frac{\varpi(x)}{\xi(x)} \right)^2 - \frac{\beta_2}{\delta(x)} \left(\frac{2\varpi(x)}{\xi(x)} - \frac{\beta_2}{\delta(x)} \right). \tag{21}$$

From (1), (20), and (21), we obtain

$$\begin{aligned} \varpi'(x) \leq\ & \frac{\xi'(x)}{\xi(x)} \varpi(x) - \xi(x) \int_x^{\infty} \left(\frac{\ell}{b(v)} \int_v^{\infty} \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s \right)^{1/\kappa} \mathrm{d}v \\ & -\xi(x) \left(\left(\frac{\varpi(x)}{\xi(x)} \right)^2 - \frac{\beta_2}{\delta(x)} \left(\frac{2\varpi(x)}{\xi(x)} - \frac{\beta_2}{\delta(x)} \right) \right) + \frac{\beta_2 \xi(x)}{b^{\frac{1}{\kappa}}(x)\delta^2(x)}. \end{aligned}$$

This implies that

$$\begin{aligned} \varpi'(x) \leq\ & \left(\frac{\xi'_+(x)}{\xi(x)} + \frac{2\beta_2}{\delta(x)} \right) \varpi(x) - \frac{1}{\xi(x)} \varpi^2(x) \\ & -\xi(x) \left(\int_x^{\infty} \left(\frac{\ell}{b(v)} \int_v^{\infty} \sum_{i=1}^{j} q_i(s) \frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}} \mathrm{d}s \right)^{1/\kappa} \mathrm{d}v + \frac{\beta_2^2 - \beta_2 b^{\frac{-1}{\kappa}}(x)}{\delta^2(x)} \right). \end{aligned}$$

Thus,

$$\varpi'(x) \leq -\psi^*(x) + \phi^*(x)\varpi(x) - \frac{1}{\xi(x)} \varpi^2(x).$$

The proof is complete. □

Lemma 6. *Assume that w is an eventually positive solution of (1). If there exists a positive function $g \in C([x_0, \infty))$ such that*

$$\int_{x_0}^{\infty} \left(\psi(s) - \left(\frac{2}{\varepsilon s^2} \right)^{\kappa} \frac{b(s) g(s) (\phi(s))^{\kappa+1}}{(\kappa+1)^{\kappa+1}} \right) \mathrm{d}s = \infty, \tag{22}$$

for some $\varepsilon \in (0,1)$, then w does not fulfill Case (1).

Proof. Assume that w is an eventually positive solution of (1). From Lemma 4, we get that (12) holds. Using Lemma 1 with

$$U = \phi(x), \ V = \kappa \varepsilon x^2 / \left(2 (b(x) g(x))^{1/\kappa} \right) \text{ and } x = \pi,$$

we get

$$\pi'(x) \leq -\psi(x) + \left(\frac{2}{\varepsilon x^2} \right)^{\kappa} \frac{b(x) g(x) (\phi(x))^{\kappa+1}}{(\kappa+1)^{\kappa+1}}. \tag{23}$$

Integrating from x_1 to x, we get

$$\int_{x_1}^{x} \left(\psi(s) - \left(\frac{2}{\varepsilon s^2} \right)^{\kappa} \frac{b(s) g(s) (\phi(s))^{\kappa+1}}{(\kappa+1)^{\kappa+1}} \right) \mathrm{d}s \leq \pi(x_1),$$

for every $\varepsilon \in (0, 1)$, which contradicts (22). The proof is complete. □

Lemma 7. *Assume that w is an eventually positive solution of (1), $w^{(r)}(x) > 0$ for $r = 1, 3$ and $w''(x) < 0$. If there exists a positive function $\xi \in C([x_0, \infty))$ such that*

$$\int_{x_0}^{\infty} \left(\psi^*(s) - \frac{1}{4} \xi(s) (\phi^*(s))^2 \right) \mathrm{d}s = \infty, \tag{24}$$

then w does not fulfill Case (2).

Proof. Assume that w is an eventually positive solution of (1). From Lemma 5, we get that (17) holds. Using Lemma 1 with

$$U = \phi^*(x), \ V = 1/\xi(x), \kappa = 1 \text{ and } x = \varpi,$$

we get

$$\pi'(x) \leq -\psi^*(x) + \frac{1}{4} \xi(x) (\phi^*(x))^2. \tag{25}$$

Integrating from x_1 to x, we get

$$\int_{x_1}^{x} \left(\psi^*(s) - \frac{1}{4} \xi(s) (\phi^*(s))^2 \right) \mathrm{d}s \leq \pi(x_1),$$

which contradicts (24). The proof is complete. □

Theorem 3. *Assume that there exist positive functions $g, \xi \in C([x_0, \infty))$ such that (22) and (24) hold, for some $\varepsilon \in (0,1)$. Then, every solution of (1) is oscillatory.*

When putting $g(x) = x^3$ and $\xi(x) = x$ into Theorem 3, we get the following oscillation criteria:

Corollary 1. *Let (2) hold. Assume that*

$$\limsup_{x\to\infty}\int_{x_1}^{x}\left(\varphi(s)-\left(\frac{2}{\varepsilon s^2}\right)^{\kappa}\frac{b(s)\,g(s)\,(\tilde{\varphi}(s))^{\kappa+1}}{(\kappa+1)^{\kappa+1}}\right)\mathrm{d}s=\infty, \tag{26}$$

for some $\varepsilon\in(0,1)$. *If*

$$\limsup_{x\to\infty}\int_{x_1}^{x}\left(\varphi_1(s)-\frac{1}{4}\xi(s)\,(\tilde{\varphi}_1(s))^2\right)\mathrm{d}s=\infty, \tag{27}$$

where

$$\begin{aligned}\varphi(x)\ &:\ =x^3\left(\ell\sum_{i=1}^{j}q_i(x)\left(\frac{\vartheta_i^3(x)}{x^3}\right)^{\kappa}+\frac{\varepsilon\beta_1^{(1+\kappa)/\kappa}x^2-2\beta_1\kappa}{2b^{\frac{1}{\kappa}}(x)\,\delta^{\kappa+1}(x)}\right)\\ \tilde{\varphi}(x)\ &:\ =\frac{3}{x}+\frac{(\kappa+1)\,\beta_1^{1/\kappa}\varepsilon x^2}{2b^{\frac{1}{\kappa}}(x)\,\delta(x)},\ \ \tilde{\varphi}_1(x):=\frac{1}{x}+\frac{2\beta_2}{\delta(x)}\end{aligned}$$

and

$$\varphi_1(x):=x\left(\int_x^{\infty}\left(\frac{\ell}{b(v)}\int_v^{\infty}\sum_{i=1}^{j}q_i(s)\frac{\vartheta_i^{\kappa}(s)}{s^{\kappa}}\mathrm{d}s\right)^{1/\kappa}\mathrm{d}v+\frac{\beta_2^2-\beta_2 b^{\frac{-1}{\kappa}}(x)}{\delta^2(x)}\right),$$

then every solution of (1) is oscillatory.

Example 1. *Consider a differential equation*

$$w^{(4)}(x)+\frac{c_0}{x^4}w\left(\frac{1}{2}x\right)=0,\ x\geq 1, \tag{28}$$

where $c_0>0$ *is a constant. Note that* $\kappa=b(x)=1,\ q(x)=c_0/x^4$ *and* $\vartheta(x)=x/2$. *Hence, we have*

$$\delta(x_0)=\infty,\ \varphi(s)=\frac{c_0}{8s}.$$

If we set $\ell=\beta_1=1$, *then condition (26) becomes*

$$\begin{aligned}\limsup_{x\to\infty}\int_{x_1}^{x}\left(\varphi(s)-\left(\frac{2}{\varepsilon s^2}\right)^{\kappa}\frac{b(s)\,g(s)\,(\tilde{\varphi}(s))^{\kappa+1}}{(\kappa+1)^{\kappa+1}}\right)\mathrm{d}s\ &=\ \limsup_{x\to\infty}\int_{x_1}^{x}\left(\frac{c_0}{8s}-\frac{9}{2s}\right)ds\\ &=\ \infty\ \ \textit{if}\ \ c_0>36.\end{aligned}$$

Therefore, from Corollary 1, the solutions of Equation (28) are all oscillatory if $c_0>36$.

Remark 2. *We compare our result with the known related criteria for oscillations of this equation as follows:*

1. *By applying Condition (7) in [9] on Equation (28) where* $\theta=2$, *we get*

$$c_0>432.$$

2. *By applying Condition (8) in [10] on Equation (28) where* $\varsigma=1/2$, *we get*

$$c_0>51.$$

Therefore, our result improves results [9,10].

Remark 3. *By applying Condition (26) in Equation (9), we find*

$$c_0 > 6.17.$$

Therefore, our result improves results [9,10].

4. Conclusions

In this article, we study the oscillatory behavior of a class of nonlinear fourth-order differential equations and establish sufficient conditions for oscillation of a fourth-order differential equation by using Riccati transformation. Furthermore, in future work, we get some Hille and Nehari type and Philos type oscillation criteria of (1).

Author Contributions: O.B.: Writing original draft, and writing review and editing. M.P.: Formal analysis, writing review and editing, funding and supervision. All authors have read and agreed to the published version of the manuscript.

Funding: The authors received no direct funding for this work.

Acknowledgments: The authors thank the reviewers for for their useful comments, which led to the improvement of the content of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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