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Riccati Technique and Asymptotic Behavior of Fourth-Order Advanced Differential Equations

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Abstract: In this paper, we deal with the oscillation of fourth-order nonlinear advanced differential equations of the form $\left(r(t)\left(y'''(t)\right)^{\alpha}\right)' + p(t) f\left(y'''(t)\right) + q(t) g\left(y\left(\sigma(t)\right)\right) = 0$. We provide oscillation criteria for this type of equations, and examples to illustrate the criteria.

Keywords: oscillatory solutions; fourth-order; advanced differential equations

1. Introduction

The present work deals with the investigation of the asymptotic properties of the solutions of fourth-order advanced differential equation of the form

$$\left(r(t)\left(y'''(t)\right)^{\alpha}\right)' + p(t) f\left(y'''(t)\right) + q(t) g\left(y\left(\sigma(t)\right)\right) = 0, \qquad (1)$$

where α is a quotient of odd positive integers. Throughout this work, we suppose that:

(S_1) $r,\ p,\ q \in C\left([t_0,\infty),[0,\infty)\right), r(t) > 0,\ q > 0, r'(t) + p(t) \geq 0,\ \sigma \in C\left([t_0,\infty),(0,\infty)\right),\ \sigma(t) \geq t,$ $\lim_{t\to\infty} \sigma(t) = \infty$ and under the condition

$$\int_{t_0}^{\infty} \left[\frac{1}{r(s)} \exp\left(-\int_{t_0}^{s} \frac{p(u)}{r(u)} du\right)\right]^{1/\alpha} ds < \infty. \qquad (2)$$

(S_2) $f,\ g \in (\mathbb{R},\mathbb{R}),\ f(u) \geq k_f u^{\alpha} > 0,\ g(u) \geq k_g u^{\alpha} > 0$ for $u \neq 0$ and k_f, k_g are constants.

By a solution of (1) we mean a function $y \in C'''[t_y,\infty)$, $t_y \geq t_0$, which has the property $r(t)\left(y'''(t)\right)^{\alpha} \in C^1[t_y,\infty)$, and satisfies (1) on $[t_y,\infty)$. We consider only those solutions y of (1) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$, for all $T > T_y$. We assume that (1) possesses such a solution. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[t_y,\infty)$, and otherwise it is called to be non-oscillatory. The Equation (1) is said to be oscillatory if all its solutions are oscillatory.

Advanced differential equations have applications in several real- world problems where the evolution rate depends not only on the present, but also on the future. In addition, differential equations in the form of (1) can have application in the mathematical modeling of engineering problems; see [1]. Oscillatory properties of differential equations are fairly well studied by authors in [2–27].

Baculikova [28] examined the oscillation of the second-order advanced equation

$$y''(t) + q(t)\,y(\tau(t)) = 0.$$

She used the generalized Riccati substitution, and established some new sufficient conditions for oscillation. Dzurina [29] established a new comparison principle for advanced canonical equations of the form

$$\left(r(t)\,y'(t)\right)' + q(t)\,y(\tau(t)) = 0.$$

Many authors, see [9,13,15,30,31] studied the oscillatory behavior of the higher-order advanced differential equation

$$\left(r(t)\left(y^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)\,y(\tau(t)) = 0 \tag{3}$$

and established the following results.

Theorem 1. *(See [31], Theorem 3.6). Let* $\int_t^\infty q(s)\,ds < \infty$ *such that*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} s^{n-2} \left(\int_s^\infty q(u)\,du\right)^{1/\alpha} ds > \frac{(n-2)!}{e}$$

and

$$\liminf_{t\to\infty} \int_t^{\tau(t)} (\tau(s)-s)^{n-2} \left(\int_s^\infty q(u)\,du\right)^{1/\alpha} ds > \frac{(n-2)!}{e},$$

then (3) is oscillatory.

Theorem 2. *(See [30]). If for all constants* $\theta \in (0,1)$,

$$\limsup_{t\to\infty} \frac{t^{n-1}}{(n-1)!} \left(\int_t^\infty q(s)\,ds + \frac{\theta\alpha}{2(n-2)!} \int_t^\infty s^{n-2} \left(\int_s^\infty q(u)\,du\right)^{(\alpha+1)/\alpha} ds\right)^{1/\alpha} > 1,$$

then (3) is oscillatory.

Theorem 3. *(See [13]) Let* $\alpha = 1$ *and* $f(y) = y$. *If there exists functions* $\rho, \vartheta \in C^1([t_0,\infty),(0,\infty))$ *such that*

$$\int_{t_0}^\infty \left(\rho(s)\,q(s)\,\frac{\mu}{2}\tau^2(s) - \frac{1}{4\rho(s)\,r(s)}\left[\frac{\rho'_+(s)}{\rho(s)} - \frac{p(s)}{r(s)}\right]^2\right) ds = \infty,$$

for some $\mu \in (0,1)$, *and*

$$\int_{t_0}^\infty \left[\vartheta(s) \int_s^\infty \left[\frac{1}{r(v)} \int_v^\infty q(v) \left(\frac{\sigma^2(v)}{v^2}\right) dv\right] dv - \frac{(\vartheta'(s))^2}{4\vartheta(s)}\right] ds = \infty,$$

then (1) is oscillatory.

Theorem 4. *(See [9]) If there exists functions* $\delta_1, \delta_2 \in C^1([t_0,\infty),(0,\infty))$. *Let the equations*

$$y'(t) + \delta_1(t)\,y(\sigma(t)) = 0$$

and

$$y'(t) + \delta_2(t)\,y(\sigma(t)) = 0,$$

are oscillatory, then the equation

$$y^{(4)}(t) + p(t)\, y'(t) + q(t)\, y(\sigma(t)) = 0,$$

is oscillatory.

Theorem 5. *(See [15]) If there exists function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{\infty} \left(H(t,s)\,\rho(s)\,q(s)\,k(s) - \left(\frac{h(t,s)}{p}\right)^p \frac{\rho(s)\,r(s)}{(H(t,s)\,G(s)\,k(s))^{p-1}} \right) ds = \infty,$$

then the equation

$$\left(r(t)\,\Phi\left(y^{(n-1)}(t)\right)\right)' + p(t)\,\Phi\left(y^{(n-1)}(t)\right) + q(t)\,\Phi(y(g(t))) = 0,$$

where $\Phi = |s|^{p-2} s$ is oscillatory.

1. By applying conditions in Theorem 1, we get

$$q_0 > 48/(e \ln 2),$$

2. By applying conditions in Theorem 2, we get

$$q_0 > 18.$$

From the above results it can be observed that [31] improves the results in [30]. The motivation in this paper is to complement the results in [15] and improve the results in [30,31] while obtaining some new oscillation criteria for (1).

The paper is organized as follows. In the next section, we will mention some auxiliary lemmas, and in Section 2 we will use the generalized Riccati transformation technique to give some sufficient conditions for the oscillation of (1). In the same section examples are given to illustrate our main results. The method used in this paper is different from that of [15], where they used the integral averaging technique.

Notation 1. *For convenience, we denote*

$$\begin{aligned}
\delta(t_0,t) &= \exp\left(\int_{t_0}^{t} \frac{p(u)}{r(u)} du\right), \\
\xi(t) &= \int_t^{\infty} \frac{ds}{(r(s)\,\delta(t_0,s))^{\frac{1}{\alpha}}}, \\
\phi(t) &= \frac{\rho'(t)}{\rho(t)} - \frac{k_f\, p(t)}{r(t)}, \\
\varphi(t) &= \frac{1}{\delta^{\frac{1}{\alpha}}(t_0,t)} - \frac{\xi(t)\, p(t)\, r^{1-\alpha/\alpha}(t)}{\alpha}
\end{aligned}$$

and

$$\tilde{\varphi}(t) = \frac{p(t)}{r(t)} + \frac{\alpha^{(\alpha+1)}\rho(t)\,\varphi^{\alpha+1}(t)\,\delta(t_0,t)}{\xi(t)\, r^{\frac{1}{\alpha}}(t)}.$$

2. Some Auxiliary Lemmas

We provide the following lemmas:

Lemma 1. *([32], Lemma 2.2.2) Let $y \in C^m([t_0,\infty),(0,\infty))$ such that it and its derivatives up to order $(m-1)$ are absolutely continuous and of constant sign in an interval (t_0,∞). If $y^{(m-1)}(t)\,y^{(m)}(t) \le 0$ for all $t \ge t_u$, m is a positive integer, then for every $\theta \in (0,1)$ there exists a constant $M > 0$ such that*

$$y(\theta t) \ge M t^{m-1} y^{(m-1)}(t),$$

for all sufficient large t.

Lemma 2. *([33], Lemma 2.1) Let $\alpha \ge 1$ be a ratio of two odd numbers. $C > 0$ and D are constants. Then*

$$Dy - Cy^{(\alpha+1)/\alpha} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{D^{\alpha+1}}{C^\alpha},\ C > 0.$$

Lemma 3. *([34], Lemma 2.2.3)Let $y \in C^m([t_0,\infty),(0,\infty))$. Assume that $y^{(m)}(t)$ is of a fixed sign, on $[t_0,\infty)$, m is a positive integer, $y^{(m)}(t)$ not identically zero and that there exists a $t_1 \ge t_0$ such that for all $t \ge t_1$,*

$$y^{(m-1)}(t)\,y^{(m)}(t) \le 0.$$

If we have $\lim_{t\to\infty} y(t) \ne 0$, then there exists $t_\lambda \ge t_0$ such that

$$y(t) \ge \frac{\lambda}{(m-1)!}t^{m-1}\left|y^{(m-1)}(t)\right|,$$

for every $\lambda \in (0,1)$ and $t \ge t_\lambda$.

Lemma 4. *([27], Theorem 2.1) Suppose that y is an eventually positive solution of (1). Then, there exist two possible cases:*

$$(\mathbf{N}_1)\ y(t) > 0,\ y'(t) > 0,\ y'''(t) > 0,\ y^{(4)}(t) < 0;$$
$$(\mathbf{N}_2)\ y(t) > 0,\ y''(t) > 0,\ y'''(t) < 0.$$

for $t \ge t_1$ where $t_1 \ge t_0$ is sufficiently large.

3. Oscillation Criteria

In this section, we establish new oscillation results for Equation (1) by using a generalized Riccati technique. First we prove the following two Lemmas:

Lemma 5. *Assume that $y(t)$ is an eventually positive solution of Equation (1) and that $(\mathbf{N}_1)$ holds. If*

$$\psi(t) := \rho(t)\frac{r(t)\,(y''')^\alpha(t)}{y^\alpha(t/2)}, \tag{4}$$

where $\rho \in C^1([t_0,\infty),(0,\infty))$ and $M > 0$ is constant, then

$$\psi'(t) \le -k_g\rho(t)\,q(t) + \phi(t)\,\psi(t) - \frac{\alpha M t^2}{2\,(r(t)\,\rho(t))^{1/\alpha}}\psi^{\frac{\alpha+1}{\alpha}}(t). \tag{5}$$

Proof. Let $y(t)$ be an eventually positive solution of Equation (1) and from Lemma 4, we see that $(\mathbf{N}_1)$ holds. From Lemma 1, we get

$$y'(t/2) \ge M t^2 y'''(t). \tag{6}$$

From the definition of $\psi(t)$, we see that $\psi(t) > 0$ for $t \geq t_1$, and

$$\begin{aligned}\psi'(t) &= \rho'(t)\frac{r(t)(y''')^{\alpha}(t)}{y^{\alpha}(t/2)} + \rho(t)\frac{(r(y''')^{\alpha})'(t)}{y^{\alpha}(t/2)} \\ &\quad -\alpha\rho(t)\frac{y'(t/2)r(t)(y''')^{\alpha}(t)}{2y^{\alpha+1}(t/2)}.\end{aligned}$$

Using (4) and (6), we obtain

$$\begin{aligned}\psi'(t) &\leq \frac{\rho'(t)}{\rho(t)}\psi(t) + \rho(t)\frac{(r(y''')^{\alpha})'(t)}{y^{\alpha}(t/2)} \\ &\quad -\alpha M t^2\rho(t)\frac{r(t)(y''')^{\alpha+1}(t)}{2y^{\alpha+1}(t/2)}.\end{aligned}$$

From (1), we get

$$\begin{aligned}\psi'(t) &\leq \frac{\rho'(t)}{\rho(t)}\psi(t) - k_f p(t)\frac{\psi(t)}{r(t)} \\ &\quad -k_g\rho(t)q(t)\frac{y^{\alpha}(\sigma(t))}{y^{\alpha}(t/2)} - \alpha M t^2\frac{\psi^{\frac{\alpha+1}{\alpha}}(t)}{2(\rho(t)r(t))^{1/\alpha}} \\ &\leq -k_g\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)} - k_f\frac{p(t)}{r(t)}\right)\psi(t) - \alpha M t^2\frac{\psi^{\frac{\alpha+1}{\alpha}}(t)}{2(\rho(t)r(t))^{1/\alpha}}.\end{aligned}$$

Hence, we obtain

$$\psi'(t) \leq -k_g\rho(t)q(t) + \phi(t)\psi(t) - \alpha M t^2\frac{\psi^{\frac{\alpha+1}{\alpha}}(t)}{2(\rho(t)r(t))^{1/\alpha}}.$$

The proof is complete. □

Lemma 6. *Assume that $y(t)$ is an eventually positive solution of Equation (1) and that $(\mathbf{N}_2)$ holds. Let $k_g > 1$ be constant. If*

$$\omega(t) := -\frac{r(t)(-y''')^{\alpha}(t)}{(y'')^{\alpha}(t)}, \tag{7}$$

then

$$\omega'(t) \leq \frac{k_f p(t)}{r(t)\xi^{\alpha}(t)\delta(t_0,t)} - k_g q(t)\left(\frac{\mu}{2}\sigma^2(t)\right)^{\alpha} - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}. \tag{8}$$

Proof. Since

$$\begin{aligned}\left(-r(t)\left(-y'''(t)\right)^{\alpha}\delta(t_0,t)\right)' &= \left(-r(t)\left(-y'''(t)\right)^{\alpha}\right)'\delta(t_0,t) \\ &\quad+\left(-r(t)\left(-y'''(t)\right)^{\alpha}\right)\delta(t_0,t)\frac{p(t)}{r(t)} \\ &= (-1)^{\alpha+1}\left(-p(t)f\left(y'''(t)\right)-q(t)g\left(y\left(\sigma(t)\right)\right)\right)\delta(t_0,t) \\ &\quad -p(t)\left(-y'''(t)\right)^{\alpha}\delta(t_0,t) \\ &\leq (-1)^{\alpha+1}\left(-k_f p(t)\left(y'''(t)\right)^{\alpha}-k_g q(t)y^{\alpha}(\sigma(t))\right)\delta(t_0,t) \\ &\quad -p(t)\left(-y'''(t)\right)^{\alpha}\delta(t_0,t) \\ &= \left(-p(t)\left(-y'''(t)\right)^{\alpha}\left(1-k_f\right)+k_g q(t)\left(-y^{\alpha}(\sigma(t))\right)\right)\delta(t_0,t) \\ &= (-1)^{\alpha}\left(-p(t)\left(y'''(t)\right)^{\alpha}\left(1-k_f\right)+k_g q(t)\left(y^{\alpha}(\sigma(t))\right)\right)\delta(t_0,t) \\ &\leq -k_g q(t)y^{\alpha}(\sigma(t))\delta(t_0,t)<0,\end{aligned}$$

we conclude that $-r(t)\left(-y'''(t)\right)^{\alpha}\delta(t_0,t)$ is decreasing. Thus, for $s \geq t \geq t_1$

$$\left(r(s)\delta(t_0,s)\right)^{1/\alpha}y'''(s) \leq \left(r(t)\delta(t_0,t)\right)^{1/\alpha}y'''(t). \tag{9}$$

Dividing both sides of (9) by $\left(r(s)\delta(t_0,s)\right)^{1/\alpha}$ and integrating the resulting inequality from t to u, we get

$$y''(u) \leq y''(t)+\left(r(s)\delta(t_0,s)\right)^{1/\alpha}y'''(t)\int_t^u \frac{ds}{\left(r(s)\delta(t_0,s)\right)^{1/\alpha}}.$$

Letting $u \to \infty$, we arrive that

$$0 \leq y''(t)+\left(r(t)\delta(t_0,t)\right)^{1/\alpha}y'''(t)\xi(t),$$

which yields

$$-\frac{y'''(t)}{y''(t)}\xi(t)\left(r(t)\delta(t_0,t)\right)^{1/\alpha} \leq 1.$$

Hence,

$$\frac{r(t)\left(y'''(t)\right)^{\alpha}}{\left(y''(t)\right)^{\alpha}} \geq \frac{-1}{\xi^{\alpha}(t)\delta(t_0,t)}.$$

From (7), we have

$$\omega(t) \geq \frac{-1}{\xi^{\alpha}(t)\delta(t_0,t)}. \tag{10}$$

From the definition of $\omega(t)$, we see that $\omega(t)<0$ for $t \geq t_1$, and

$$\omega'(t)=\frac{\left(-r(t)\left(-y'''(t)\right)^{\alpha}\right)'}{\left(y''(t)\right)^{\alpha}}-\alpha\frac{-r(t)\left(-y'''(t)\right)^{\alpha+1}}{\left(y''(t)\right)^{\alpha+1}}.$$

From (1) and (7), we get

$$\begin{aligned}\omega'(t) &= -k_f\frac{p(t)}{r(t)}\omega(t)-k_g q(t)\frac{y^{\alpha}(\sigma(t))}{\left(y''(t)\right)^{\alpha}}-\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}. \\ &= -k_f\frac{p(t)}{r(t)}\omega(t)-k_g q(t)\frac{y^{\alpha}(\sigma(t))}{\left(y''(\sigma(t))\right)^{\alpha}}\frac{\left(y''(\sigma(t))\right)^{\alpha}}{\left(y''(t)\right)^{\alpha}}-\alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}.\end{aligned} \tag{11}$$

From Lemma 3, we get that for a constant $\mu \in (0,1)$

$$y(t) \geq \frac{\mu}{2} t^2 y''(t). \tag{12}$$

Thus, from (10) and (12), we get

$$\omega'(t) \leq \frac{k_f p(t)}{r(t)\xi^{\alpha}(t)\delta(t_0,t)} - k_g q(t)\left(\frac{\mu}{2}\sigma^2(t)\right)^{\alpha} - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}.$$

The proof is complete. □

Theorem 6. *Assume that (2) holds. Let there exist positive functions $\rho, \vartheta \in C^1([t_0,\infty),(0,\infty))$ and $M > 0$ be a constant such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g\rho(s)q(s) - \left(\frac{2}{Ms^2}\right)^{\alpha}\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds = \infty. \tag{13}$$

If

$$\frac{\vartheta(t)}{\xi(t)(r(t)\delta(t_0,t))^{1/\alpha}} + \vartheta'(t) \leq 0 \tag{14}$$

and, for $\mu \in (0,1)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g q(s)\left(\frac{\mu\sigma^2(s)}{2}\frac{\vartheta(\sigma(s))}{\vartheta(s)}\xi(s)\right)^{\alpha}\delta(t_0,s) - \tilde{\varphi}(s)\right)ds = \infty, \tag{15}$$

then every solution of (1) is oscillatory.

Proof. Let y be a non-oscillatory solution of Equation (1) on the interval $[t_0,\infty)$. Without loss of generality, we can assume that $y(t)$ is eventually positive. Using Lemma 4, we have two cases $(\mathbf{N}_1)$ and $(\mathbf{N}_2)$. For case $(\mathbf{N}_1)$, from Lemma 5, we get that (5) holds. Using Lemma 2, and setting

$$D = \phi(t), \quad C = \alpha M t^2 / \left(2(r(t)\rho(t))^{1/\alpha}\right), \quad y = \psi,$$

we have:

$$\psi'(t) \leq -k_g\rho(t)q(t) + \left(\frac{2}{Mt^2}\right)^{\alpha}\frac{r(t)\rho(t)(\phi(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}. \tag{16}$$

Integrating from t_1 to t, we get:

$$\int_{t_1}^{t}\left(k_g\rho(s)q(s) - \left(\frac{2}{Ms^2}\right)^{\alpha}\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds \leq \psi(t_1),$$

which contradicts (13). For the case $(\mathbf{N}_2)$, from the proof of Lemma 6, we have:

$$\frac{y'''(t)}{y''(t)} \geq \frac{-1}{\xi(t)(r(t)\delta(t_0,t))^{1/\alpha}}.$$

Using the latter inequality and (14), we obtain

$$\begin{aligned}\left(\frac{y''(t)}{\vartheta(t)}\right)' &= \frac{y'''(t)\,\vartheta(t) - y''(t)\,\vartheta\prime(t)}{\vartheta^2(t)} \\ &\geq \frac{y''(t)}{\vartheta^2(t)}\left(\frac{\vartheta(t)}{\xi(t)\,(r(t)\,\delta(t_0,t))^{1/\alpha}} + \vartheta'(t)\right) \geq 0,\end{aligned}$$

which implies that $y''(t)/\vartheta(t)$ is nondecreasing. Hence, it follows from $\sigma(t) \geq t$ that

$$\frac{y''(\sigma(t))}{y''(t)} \geq \frac{\vartheta(\sigma(t))}{\vartheta(t)}.$$

Thus, by using (11) and (12), we get

$$\omega'(t) \leq \frac{k_f p(t)}{r(t)\,\xi^\alpha(t)\,\delta(t_0,t)} - k_g q(t)\left(\frac{\mu}{2}\sigma^2(t)\right)^\alpha \left(\frac{\vartheta(\sigma(t))}{\vartheta(t)}\right)^\alpha - \alpha\frac{\omega^{\frac{\alpha+1}{\alpha}}(t)}{r^{\frac{1}{\alpha}}(t)}. \tag{17}$$

Multiplying (17) by $\xi^\alpha(t)\,\delta(t_0,t)$ and integrating from t_1 to t, we get

$$\begin{aligned}&\xi^\alpha(t)\,\delta(t_0,t)\,\omega(t) - \xi^\alpha(t_1)\,\delta(t_0,t_1)\,\omega(t_1) - \int_{t_1}^t \frac{p(s)}{r(s)}ds \\ &+\alpha\int_{t_1}^t r^{\frac{-1}{\alpha}}(s)\,\xi^{\alpha-1}(s)\,\delta(t_0,s)\,\varphi(s)\,\omega(s)\,ds \\ &+\int_{t_1}^t k_g q(s)\left(\frac{\mu}{2}\sigma^2(s)\right)^\alpha \left(\frac{\vartheta(\sigma(s))}{\vartheta(s)}\right)^\alpha \xi^\alpha(s)\,\delta(t_0,s)\,ds \\ &+\alpha\int_{t_1}^t \frac{\omega^{\frac{\alpha+1}{\alpha}}(s)}{r^{\frac{1}{\alpha}}(s)}\xi^\alpha(s)\,\delta(t_0,s)\,ds \\ \leq\;& 0.\end{aligned}$$

Using Lemma 2 we set

$$C = \xi^\alpha(s)\,\delta(t_0,s)\,/r^{\frac{1}{\alpha}}(s)\,,\; D = \int_{t_1}^t r^{\frac{-1}{\alpha}}(s)\,\xi^{\alpha-1}(s)\,\delta(t_0,s)\,\varphi(s)\,,\; y = \omega(t)\,.$$

Thus, we get

$$\begin{aligned}&\xi^\alpha(t)\,\delta(t_0,t)\,\omega(t) - \xi^\alpha(t_1)\,\delta(t_0,t_1)\,\omega(t_1) - \int_{t_1}^t \frac{p(s)}{r(s)}ds \\ &+\int_{t_1}^t k_g q(s)\left(\frac{\mu}{2}\sigma^2(s)\right)^\alpha \left(\frac{\vartheta(\sigma(s))}{\vartheta(s)}\right)^\alpha \xi^\alpha(s)\,\delta(t_0,s)\,ds \\ &+\int_{t_1}^t \frac{\alpha^{(\alpha+1)}\rho(s)\,\varphi^{\alpha+1}(s)\,\delta(t_0,s)}{\xi(s)\,r^{\frac{1}{\alpha}}(t)}ds \\ \leq\;& 0.\end{aligned}$$

Hence, by using (10), we obtain

$$\int_{t_1}^t \left(k_g q(s)\left(\frac{\mu\sigma^2(s)}{2}\frac{\vartheta(\sigma(s))}{\vartheta(s)}\xi(s)\right)^\alpha \delta(t_0,s) - \tilde{\varphi}(s)\right)ds \leq \xi^\alpha(t)\,\delta(t_0,t)\,\omega(t_1) + 1,$$

which contradicts (15). Theorem 6 is proved. □

If we define the function

$$\omega(t) := \vartheta(t)\frac{y'(t)}{y(t)},$$

by using it into (1), we prove the following corollary:

Corollary 1. *Assume that (2) holds. If there exist positive functions $\rho, \vartheta \in C^1([t_0,\infty),(0,\infty))$ such that (14) is satisfied and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g\rho(s)q(s)-\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds=\infty, \tag{18}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g q(s)\left(\frac{\vartheta(\sigma(s))}{\vartheta(s)}\xi(s)\right)^{\alpha}\delta(t_0,s)-\tilde{\varphi}(s)\right)ds=\infty, \tag{19}$$

then every solution of (1) is oscillatory.

Remark 1. *By using similar methods to [22,23] , we can easily establish Philos-type, Hille and Nehari-type oscillation criteria for (1).*

Example 1. *Consider the differential equation*

$$\left(t^2\left(y'''(t)\right)\right)'+\frac{t}{2}y'''(t)+\frac{q_0}{t^2}y(2t)=0, \tag{20}$$

where $q_0>0$ is constant. Please note that $\alpha=1,\ t_0=1,\ r(t)=t^2,\ p(t)=t/2,\ q(t)=t,\ \sigma(t)=2t$. We now set $\rho(t)=t,\ k_f=k_g=1$, then:

$$\begin{aligned}\delta(t_0,t) &= \exp\left(\int_{t_0}^{t}\frac{p(u)}{r(u)}du\right)=t^{1/2},\ \xi(t)=\int_t^{\infty}\frac{ds}{(r(s)\delta(t_0,s))^{\frac{1}{\alpha}}}=\frac{2t^{-3/2}}{3},\\ \vartheta(t) &= \frac{2t^{-3/2}}{3},\ \varphi(t)=\frac{1}{\delta^{\frac{1}{\alpha}}(t_0,t)}-\frac{\xi(t)p(t)r^{1-\alpha/\alpha}(t)}{\alpha}=\frac{2t^{-1/2}}{3},\ \phi(t)=\frac{-1}{2t}\end{aligned}$$

and

$$\tilde{\varphi}(t)=\frac{p(t)}{r(t)}+\frac{\alpha^{(\alpha+1)}\rho(t)\varphi^{\alpha+1}(t)\delta(t_0,t)}{\xi(t)r^{\frac{1}{\alpha}}(t)}=\frac{2t^{-1/3}}{3}.$$

Thus, we get

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g\rho(s)q(s)-\left(\frac{2}{Ms^2}\right)^{\alpha}\frac{r(s)\rho(s)(\phi(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right)ds=\infty$$

and

$$\frac{\vartheta(t)}{\xi(t)(r(t)\delta(t_0,t))^{1/\alpha}}+\vartheta'(t)=0.$$

Furthermore, for $\mu\in(0,1)$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left(k_g q(s)\left(\frac{\mu\sigma^2(s)}{2}\frac{\vartheta(\sigma(s))}{\vartheta(s)}\xi(s)\right)^{\alpha}\delta(t_0,s)-\tilde{\varphi}(s)\right)ds=\infty.$$

Hence, by using Theorem 6, every solution of Equation (20) is oscillatory.

Example 2. *Consider the differential equation*

$$\left(t^2\left(y'(t)\right)\right)'+\frac{t}{2}y'(t)+q_0 y(2t)=0, \tag{21}$$

where $q_0 > 0$ is constant. Please note that $\alpha = 1,\ t_0 = 1,\ r(t) = t^2,\ p(t) = t/2,\ q(t) = q_0,\ \sigma(t) = 2t$. We now set $\rho(t) = k_f = k_g = 1$, then $\xi(t) = \vartheta(t) = 2t^{-3/2}/3$.

It is easy to observe that condition (18) holds and that condition (19) is satisfied for $q_0 > 2\sqrt{2}$. Hence, by using Corollary 1, every solution of Equation (21) is oscillatory if $q_0 > 2\sqrt{2}$.

4. Conclusions

In this paper, we provided new oscillation criteria for (1) by using a Riccati transformation*technique*. As a further extension of this article, one can consider the case of $z(t) = y(t) + p(t)\, y((\sigma(t)))$, and try to get some oscillation criteria of (1) by using a method of comparison with first-order delay equations and the integral averaging technique.

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