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A Fixed-Point Approach to the Hyers–Ulam Stability of Caputo–Fabrizio Fractional Differential Equations

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Abstract: In this paper, we study Hyers–Ulam and Hyers–Ulam–Rassias stability of nonlinear Caputo–Fabrizio fractional differential equations on a noncompact interval. We extend the corresponding uniqueness and stability results on a compact interval. Two examples are given to illustrate our main results.

Keywords: Caputo–Fabrizio fractional differential equations; fixed-point theory; Hyers–Ulam stability

MSC: 26A33; 34D10; 45N05

1. Introduction

In 1940, Ulam posed a question concerning the stability of homomorphisms into metric groups, a question which is regarded as the origin of the problem of stability in the theory of functional equations. In 1941, Hyers [1] answered the problem for a linear functional equation on the Banach space and established a new concept on the stability of functional equation, now called Hyers–Ulam stability. In 1978, Rassias [2] introduced a new definition of generalized Hyers–Ulam stability by the constant ε by a variable, and obtained the stability of Hyers–Ulam–Rassias for functional equation. There is a rich literature on this topic for standard integer-order equations (see [3–17]). In addition, the same stability concepts are introduced to find approximate solutions to fractional differential equations, see [18,19] and the references therein.

In 2015, Caputo and Fabrizio [20] gave a new definition of fractional derivative with a smooth kernel. Losada and Nieto [21] introduced Caputo–Fabrizio fractional differential equation the newly developed Caputo–Fabrizio fractional derivative and obtained the existence and uniqueness results under some strong restriction. Baleanu et al. [22] obtained the approximate solution for some infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential equations. Goufo [23] used the fractional derivative of the newly developed Caputo–Fabrizio without singular kernel to establish the Korteweg–de Vries–Burgers equation with two perturbation levels. Atangana and Nieto [24] studied the numerical approximation of this new fractional derivative and established an improved RLC circuit model. Moore et al. [25] developed and analyzed a Caputo–Fabrizio fractional derivative model for the HIV epidemic which includes an antiretroviral treatment compartment. Dokuyucu et al. [26] applied the fractional derivative of Caputo–Fabrizio to model the cancer treatment by radiotherapy.

Recently, Başçı et al. [27] applied the Laplace transform method to study the Hyers–Ulam stability of the following linear differential equations with Caputo–Fabrizio fractional derivative (see Definition 1):

$$({}^{CF}\mathbb{D}^{\alpha}y)(t) = f(t),\ 0 < \alpha < 1,$$

and

$$({}^{CF}\mathbb{D}^{\alpha}y)(t) - \lambda y(t) = f(t),\ 0 < \alpha < 1.$$

Meanwhile, Liu et al. [4] presented the Hyers–Ulam stability of linear differential equations with two term Caputo–Fabrizio derivatives as follows

$$({}^{CF}\mathbb{D}^{\alpha}y)(t) - \lambda({}^{CF}\mathbb{D}^{\beta}y)(t) = u(t),\ 0 < \alpha,\ \beta < 1,$$

and applied fixed-point theorems to derive the existence and uniqueness of solution to nonlinear equations as follows

$$({}^{CF}\mathbb{D}^{\alpha}f)(t) = g(t, f(t)),\ 0 < \alpha < 1, \tag{1}$$

and obtained the generalized Hyers–Ulam–Rassias stability via the Gronwall's inequality.

Observing that ([4], Theorem 3) adopted the generalized Banach fixed-point theorem instead of the standard Banach contraction mapping and weakened the condition $a_{\alpha}L + b_{\alpha}TL < 1$ in ([21], Theorem 1) to $a_{\alpha}L < 1$ where $k > 0$ denoted by the Lipschtiz constant of g, T denoted by the step of the interval and

$$a_{\cdot} = \frac{2(1-\cdot)}{(2-\cdot)M(\cdot)},\ b_{\cdot} = \frac{2\cdot}{(2-\cdot)M(\cdot)}. \tag{2}$$

and $M(\cdot)$ denotes a normalization constant depending on $\cdot$.

Based on the above observation, we apply a new fixed-point approach to show the existence and uniqueness and stability for (1) on a compact interval to a noncompact interval $J = [\tau_0, \tau_0 + k), k > 0$.

2. Preliminaries

Definition 1 (see [20])**.** *Let $0 < \gamma < 1$, the Caputo–Fabrizio fractional derivative of order γ for a function f can be written as*

$${}^{CF}\mathbb{D}^{\gamma}f(\tau) = \frac{(2-\gamma)M(\gamma)}{2(1-\gamma)}\int_a^{\tau}\exp(-\frac{\gamma}{1-\gamma}(\tau - s))f'(s)ds,\ \tau > a,$$

where $M(\gamma)$ is a normalization constant depending on γ. Please note that $({}^{CF}\mathbb{D}^{\gamma})(f) = 0$ if and only if f is a constant function.

Definition 2 (see [21] or ([4], Definition 2))**.** *Let $0 < \gamma < 1$. The Caputo–Fabrizio fractional integral of order γ for a function f is defined as*

$${}^{CF}I^{\gamma}f(\tau) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}f(\tau) + \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_a^{\tau}f(s)ds,\ \tau > a.$$

Let Ω be a nonempty set, we present the following definition of generalized metric on Ω.

Definition 3 (see [3])**.** *A function $\rho : \Omega \times \Omega \to [0, \infty]$ is called a generalized metric on Ω if and only if ρ satisfies*

(i) $\rho(\tau_1, \tau_2) = 0$ *if and only if* $\tau_1 = \tau_2$*;*
(ii) $\rho(\tau_1, \tau_2) = \rho(\tau_2, \tau_1)$ *for all* $\tau_1, \tau_2 \in \Omega$*;*
(iii) $\rho(\tau_1, \tau_3) \leq \rho(\tau_1, \tau_2) + \rho(\tau_2, \tau_3)$ *for all* $\tau_1, \tau_2, \tau_3 \in \Omega$*;*

Theorem 1 (see [28])**.** *Let* (Ω, ρ) *is a generalized complete metric space. Suppose* $P : \Omega \to \Omega$ *is a strictly contractive operator with the Lipschitz constant* $K < 1$*. If there exists a nonnegative integer* l *such that* $\rho(P^{l+1}\tau, P^l\tau) < \infty$ *for some* $\tau \in \Omega$*, then the followings are true:*

(i) *The sequence* $\{P^n\tau\}$ *converges to a fixed point* τ^* *of* P*;*
(ii) τ^* *is the unique fixed point of* P *in*

$$\Omega^* = \{\tilde{\tau} \in \Omega \mid \rho(P^l\tau, \tilde{\tau}) < \infty\};$$

(iii) *If* $\tilde{\tau} \in \Omega^*$*, then*

$$\rho(\tilde{\tau}, \tau^*) \leq \frac{1}{1-K}\rho(P\tilde{\tau}, \tilde{\tau}).$$

Definition 4 (see [4])**.** *Let* $g : J \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Equation* (7) *is Hyers–Ulam stable if there exists a real number* $N > 0$*, such that for each* $\epsilon > 0$ *and for any solution* $f \in C(J, \mathbb{R})$ *of*

$$|^{CF}\mathbb{D}^{\gamma} f(\tau) - g(\tau, f(\tau))| \leq \epsilon, \ \forall\, \tau \in J, \tag{3}$$

there exists a solution $h \in C(J, \mathbb{R})$ *of* (1) *with*

$$|f(\tau) - h(\tau)| \leq N\epsilon, \ \forall\, \tau \in J.$$

Definition 5 (see [4])**.** *Let* $\phi : J \to \mathbb{R}_+$ *and* $g : J \times \mathbb{R} \to \mathbb{R}$ *be continuous functions. Equation* (7) *is generalized Hyers–Ulam–Rassias stable with respect to* $\phi \in C(J, \mathbb{R}_+)$*, if there exists a constant* $c_{f,\phi} > 0$ *such that for any solution* $f \in C(J, \mathbb{R})$ *of*

$$|^{CF}\mathbb{D}^{\gamma} f(\tau) - g(\tau, f(\tau))| \leq \phi(\tau), \ \forall\, \tau \in J, \tag{4}$$

there exists a solution $h \in C(J, \mathbb{R})$ *of* (1) *with*

$$|f(\tau) - h(\tau)| \leq c_{f,\phi}\phi(\tau), \ \forall\, \tau \in J.$$

3. Main Results

Throughout this section, we denote the set Y of all continuous functions on J by

$$Y := \{g : J \to \mathbb{R} \mid g \text{ is continuous}\} = C(J, \mathbb{R}) \tag{5}$$

Lemma 1 (see ([3], Theorem 3.1))**.** *Define the function* $d : Y \times Y \to [0, \infty]$ *with*

$$d(f, g) := \inf\{M \in [0, \infty] \mid |f(\tau) - g(\tau)| \leq M\psi(\tau), \forall\, \tau \in J\}$$

where $\psi : J \to [0, \infty)$ *is a given continuous function. Then* (Y, d) *is a generalized complete metric space.*

We give the following conditions:

$[A_1]$ The function $g : J \times \mathbb{R} \to \mathbb{R}$ is continuous and locally Lipschitz in τ.
$[A_2]$ There exists a constant $L > 0$ such that

$$|g(\tau, y_1) - g(\tau, y_2)| \leq L|y_1 - y_2|, \ \ \forall\, y_1, y_2 \in \mathbb{R}, \ \tau \in J.$$

Now, we prove the Hyers–Ulam stability of (7).

Theorem 2. *Assume that* $[A_1]$ *and* $[A_2]$ *and* $|a_\gamma| < 1/(L+1)$ *hold. If the function* $h : J \to \mathbb{R}$ *is continuously differentiable and satisfies*

$$|({}^{CF}\mathbb{D}^\gamma h)(\tau) - g(\tau, h(\tau))| \le \epsilon \tag{6}$$

for all $\tau \in J$ *and for some* $\epsilon > 0$*, then there exists a unique solution* $f(\tau)$ *of*

$$({}^{CF}\mathbb{D}^\gamma f)(\tau) = g(\tau, f(\tau)),\ 0 < \gamma < 1, \tag{7}$$

satisfying

$$|h(\tau) - f(\tau)| \le (L+1)(|a_\gamma| + |b_\gamma|k)\epsilon \tag{8}$$

for all $\tau \in J$*, where* a_γ *and* b_γ *are defined in* (2)*.*

Proof. We introduce a function $d_1 : Y \times Y \to [0, \infty]$, where Y defined by (5) with

$$d_1(f, g) := \inf\{M \in [0, \infty] \mid |f(\tau) - g(\tau)|e^{-K(\tau-\tau_0)} \le M,\ \forall\, \tau \in J\}, \tag{9}$$

where $K = \frac{(L+1)|b_\gamma|}{1-(L+1)|a_\gamma|} > 0$ and a_γ, b_γ are given in (2)

Let $\psi(\cdot) = e^{K(\cdot-\tau_0)}$ in Lemma 1, we obtain (Y, d_1) is a generalized complete metric space.

Next, we consider the operator $P : Y \to Y$ as follows:

$$(Pf)(\tau) := f_0 + a_\gamma g(\tau, f(\tau)) + b_\gamma \int_{\tau_0}^{\tau} g(s, f(s))ds,\ \ \tau \in J. \tag{10}$$

for any $f, g \in Y$, where $f_0 = f(\tau_0)$. Please note that any fixed point of P solves (7). Indeed, the function $u - a_\gamma g(\tau, u) = v$ in (10) is invertible, it is increasing. We denote its inverse $u = G(\tau, v)$, and G is globally Lipschitz in v and locally Lipschitz in τ by our assumptions. So, any fixed point of (10) satisfies

$$f(\tau) = G(\tau, b_\gamma \int_{\tau_0}^{\tau} g(s, f(s))ds + f_0). \tag{11}$$

Now clearly the function $\tau \to b_\gamma \int_{\tau_0}^{\tau} g(s, f(s))ds + f_0$ is locally Lipschitz in τ, we see that the composition function $\tau \to G(\tau, b_\gamma \int_{\tau_0}^{\tau} g(s, f(s))ds + f_0)$ is also locally Lipschitz in τ. So, any fixed point $f(\tau)$ of (10) is a locally Lipschitz function, and thus it is locally absolute continuous on J. So really (10) gives solutions of (7). As a matter of fact, we need just that $u - a_\gamma g(\tau, u) = v$ is invertible, i.e., $u - a_\gamma g(\tau, u)$ is strictly monotonic in u, and we can extend our results for more general case. We shall consider (11) instead of (10).

We prove that Pf is continuous. Let $\tau_1, \tau_2 \in J$, and $\tau_1 < \tau_2$, we have

$$
\begin{aligned}
& |Pf(\tau_1) - Pf(\tau_2)| \\
= \ & |a_\gamma g(\tau_1, f(\tau_1)) + b_\gamma \int_{\tau_0}^{\tau_1} g(s, f(s))ds - a_\gamma g(\tau_2, f(\tau_2)) - b_\gamma \int_{\tau_0}^{\tau_2} g(s, f(s))ds| \\
\leq \ & |a_\gamma||g(\tau_1, f(\tau_1)) - g(\tau_2, f(\tau_2))| + |b_\gamma|| \int_{\tau_0}^{\tau_1} g(s, f(s))ds - \int_{\tau_0}^{\tau_2} g(s, f(s))ds| \\
\leq \ & |a_\gamma||g(\tau_1, f(\tau_1)) - g(\tau_1, f(\tau_2))| + |a_\gamma||g(\tau_1, f(\tau_2)) - g(\tau_2, f(\tau_2))| + |b_\gamma|| \int_{\tau_1}^{\tau_2} g(s, f(s))ds| \\
\leq \ & |a_\gamma||g(\tau_1, f(\tau_1)) - g(\tau_1, f(\tau_2))| + |a_\gamma||g(\tau_1, f(\tau_2)) - g(\tau_2, f(\tau_2))| \\
& + |b_\gamma|(\int_{\tau_1}^{\tau_2} |g(s, f(s)) - g(s, 0)|ds + \int_{\tau_1}^{\tau_2} |g(s, 0)|ds) \\
\leq \ & |a_\gamma||g(\tau_1, f(\tau_1)) - g(\tau_1, f(\tau_2))| + |a_\gamma||g(\tau_1, f(\tau_2)) - g(\tau_2, f(\tau_2))| \\
& + |b_\gamma|(L\|f\|_{C(J,\mathbb{R})}(\tau_2 - \tau_1) + \|g\|_{C(J,\mathbb{R})}(\tau_2 - \tau_1)).
\end{aligned}
$$

Then, for all $f \in Y$, as $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero (due to $[A_1]$ and $f \in Y$). Thus, Pf is continuous, i.e., $Pf \in Y$ for all $f \in Y$.

Then, we have

$$
|(Pf_0)(\tau) - f_0(\tau)|e^{-K(\tau-\tau_0)} \leq \|Pf_0 - f_0\|_{C(J,\mathbb{R})} \max\{1, e^{-Kk}\} < \infty,
$$

for all $f_0 \in Y$, and $\tau \in J$. Therefore, by (9), we obtain $d_1(Pf_0, f_0) < \infty$, $f_0 \in Y$.

Similarly, we have

$$
|(f_0)(\tau) - f(\tau)|e^{-K(\tau-\tau_0)} \leq \|f_0 - f\|_{C(J,\mathbb{R})} \max\{1, e^{-Kk}\} < \infty,
$$

for all $f \in Y$, and $\tau \in J$, which implies that

$$
d_1(f_0, f) < \infty, \ \forall \ f \in Y,
$$

that is $\{f \in Y \mid d_1(f_0, f) < \infty\} = Y$.

Next, we show that P is strictly contractive on Y. For any $l, n \in Y$, we get

$$
\begin{aligned}
& |(Pl)(\tau) - (Pn)(\tau)| \\
\leq \ & |a_\gamma||g(\tau, l(\tau)) - g(\tau, n(\tau))| + |b_\gamma| \int_{\tau_0}^{\tau} |g(s, l(s)) - g(s, n(s))|ds \\
\leq \ & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma| \int_{\tau_0}^{\tau} |l(s) - n(s)|ds \\
\leq \ & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma| \int_{\tau_0}^{\tau} |l(s) - n(s)|e^{-K(s-\tau_0)}e^{K(s-\tau_0)}ds \\
\leq \ & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma|d_1(l, n) \int_{\tau_0}^{\tau} e^{K(s-\tau_0)}ds \\
\leq \ & L|a_\gamma||l(\tau) - n(\tau)| + \frac{L|b_\gamma|}{K}d_1(l, n)(e^{K(\tau-\tau_0)} - 1) \\
\leq \ & L|a_\gamma||l(\tau) - n(\tau)| + \frac{L|b_\gamma|}{K}d_1(l, n)e^{K(\tau-\tau_0)}
\end{aligned}
$$

for all $\tau \in J$. Thus, for any $l, n \in Y$ and all $\tau \in J$, we have

$$\begin{aligned} |(Pl)(\tau) - (Pn)(\tau)|e^{-K(\tau-\tau_0)} &\leq L|a_\gamma||l(\tau) - n(\tau)|e^{-K(\tau-\tau_0)} + \frac{L|b_\gamma|}{K} d_1(l,n) \\ &\leq L|a_\gamma|d_1(l,n) + \frac{L|b_\gamma|}{K} d_1(l,n) \\ &= L(|a_\gamma| + \frac{|b_\gamma|}{K}) d_1(l,n) \\ &= \frac{L}{L+1} d_1(l,n). \end{aligned}$$

Hence, we obtain

$$d_1(Pl, Pn) \leq \frac{L}{L+1} d_1(l,n).$$

Therefore, P is strictly contractive on Y.

When $k = 1$ and $Y = \Omega^*$, the operator P satisfies all the conditions of Theorem 1.

On the other hand, by (6), we have

$$-\epsilon \leq ({}^{CF}\mathbb{D}^\gamma h)(\tau) - g(\tau, h(\tau)) \leq \epsilon \;\; \forall\, \tau \in J.$$

Similar to the approach in ([4], Theorem 2), we can obtain

$$|h(\tau) - h_0 - a_\gamma g(\tau, h(\tau)) - b_\gamma \int_{\tau_0}^{\tau} g(s, f(s))ds| \leq \epsilon(|a_\gamma| + |b_\gamma|k) \tag{12}$$

for all $\tau \in J$. From (10), (12) is equivalent to

$$|h(\tau) - (Ph)(\tau)| \leq \epsilon(|a_\gamma| + |b_\gamma|k). \tag{13}$$

Multiply both sides of (13) by $e^{-K(\tau-\tau_0)}$,

$$|h(\tau) - (Ph)(\tau)|e^{-K(\tau-\tau_0)} \leq \epsilon(|a_\gamma| + |b_\gamma|k)e^{-K(\tau-\tau_0)} \leq M := \epsilon(|a_\gamma| + |b_\gamma|k)\max\{1, e^{-Kk}\}$$

for all $\tau \in J$. Then

$$d_1(Ph, h) \leq \epsilon(|a_\gamma| + |b_\gamma|k)e^{-K(\tau-\tau_0)}.$$

By Theorem 1, there exists a unique solution $f : J \to \mathbb{R}$ of (7) satisfying

$$d_1(h, f) \leq \frac{1}{1 - L/(L+1)} d_1(Ph, h) \leq (L+1)\epsilon(|a_\gamma| + |b_\gamma|k)e^{-K(\tau-\tau_0)}, \;\; \tau \in J,$$

by (9), we have

$$|h(\tau) - f(\tau)|e^{-K(\tau-\tau_0)} \leq (L+1)\epsilon(|a_\gamma| + |b_\gamma|k)e^{-K(\tau-\tau_0)}, \;\; \tau \in J,$$

which implies that (8) holds. □

Remark 1. *From Definition 4, (8) shows (7) is Hyers–Ulam stable with the constant* $N = (L+1)(|a_\gamma| + |b_\gamma|k)$ *provided that* $0 < k < +\infty$. *Of course, (7) is not Hyers–Ulam stable if* $k = +\infty$. *Theorem 2 covers the result in ([27], Theorem 2.6) and shows that the condition* $0 < \lambda < \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$ *can be removed.*

Now we will prove the Hyers–Ulam–Rassias stability of (7).

Theorem 3. *Assume that* $[A_1]$ *and* $[A_2]$ *and* $|a_\gamma| < 1/(L+1)$ *hold. If a continuously differentiable function* $h : J \to \mathbb{R}$ *satisfies*

$$|({}^{CF}\mathbb{D}^{\gamma}h)(\tau) - g(\tau, h(\tau))| \leq G(\tau) \tag{14}$$

for all $\tau \in J$ *and for some* $G : J \to (0, \infty)$ *is a nondecreasing continuous function satisfying*

$$\left| \int_{\tau_0}^{\tau} G(s)ds \right| \leq F_G G(\tau), \;\; F_G > 0, \tag{15}$$

for all $\tau \in J$*, then there exists a unique solution* $f(\tau)$ *of* (7) *satisfying*

$$|h(\tau) - f(\tau)| \leq (L+1)(a_\gamma + b_\gamma F_G)G(\tau) \tag{16}$$

for all $\tau \in J$.

Proof. We introduce a function $d_2 : Y \times Y \to [0, \infty]$, where Y defined by (5) with

$$d_2(f, g) := \inf\{M \in [0, \infty] \mid |f(\tau) - g(\tau)|e^{-K(\tau-\tau_0)} \leq MG(\tau), \forall \, \tau \in J, \; K \in \mathbb{R}\} \tag{17}$$

Let $\psi(\cdot) = e^{K(\cdot-\tau_0)}G(\cdot)$ in the Lemma 1, (Y, d_2) is a generalized complete metric space.

Consider $P : Y \to Y$ defined in (10). Similar to the method of Theorem 2, we can conclude that $d_2(Pf_0, f) < \infty$ for each $f_0 \in X$ and $\{f \in Y \mid d_2(f_0, f) < \infty\} = Y$.

Next, we prove that P is strictly contractive on Y. Note

$$\begin{aligned}
\int_{\tau_0}^{\tau} G(s)e^{K(s-\tau_0)}ds &\leq G(\tau)\int_{\tau_0}^{\tau} e^{K(s-\tau_0)}ds \\
&= \frac{1}{K}G(\tau)\int_{\tau_0}^{\tau} de^{K(s-\tau_0)} \\
&\leq \frac{1}{K}G(\tau)(e^{K(\tau-\tau_0)} - 1) \\
&\leq \frac{1}{K}G(\tau)e^{K(\tau-\tau_0)}
\end{aligned}$$

for all $\tau \in J$.

For any $l, n \in Y$, let $M_{l,n} \in [0, \infty]$ be an arbitrary constant with $d_2(l, n) \leq M_{l,n}$, by (17), we obtain

$$|l(\tau) - n(\tau)|e^{-K(\tau-\tau_0)} \leq M_{l,n}G(\tau), \quad \textit{for all } \tau \in J.$$

Then, for each $l, n \in Y$, we have

$$\begin{aligned}
&|(Pl)(\tau) - (Pn)(\tau)| \\
\leq\; & |a_\gamma||g(\tau, l(\tau)) - g(\tau, n(\tau))| + |b_\alpha| \int_{\tau_0}^{\tau} |g(s, l(s)) - g(s, n(s))|ds \\
\leq\; & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma| \int_{\tau_0}^{\tau} |l(s) - n(s)|ds \\
\leq\; & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma| \int_{\tau_0}^{\tau} |l(s) - n(s)|e^{-K(s-\tau_0)}e^{K(s-\tau_0)}ds \\
\leq\; & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma|M_{l,n} \int_{\tau_0}^{\tau} G(s)e^{K(s-\tau_0)}ds \\
\leq\; & L|a_\gamma||l(\tau) - n(\tau)| + L|b_\gamma|M_{l,n}\frac{1}{K}G(\tau)e^{K(\tau-\tau_0)}
\end{aligned}$$

for all $\tau \in J$. Thus, for any $l, n \in \Upsilon$ and all $\tau \in J$, we have

$$\begin{aligned} |(Pl)(\tau) - (Pn)(\tau)|e^{-K(\tau-\tau_0)} &\leq L|a_\gamma||l(\tau) - n(\tau)|e^{-K(\tau-\tau_0)} + \frac{L|b_\gamma|}{K} M_{l,n} G(\tau) \\ &\leq L|a_\gamma| M_{l,n} G(\tau) + \frac{L|b_\gamma|}{K} M_{l,n} G(\tau) \\ &= L(|a_\gamma| + \frac{|b_\gamma|}{K}) M_{l,n} G(\gamma) \\ &= \frac{L}{L+1} M_{l,n} G(\tau), \end{aligned}$$

that is, $d_2(Pl, Pn) \leq \frac{L}{L+1} M_{l,n}$, $\forall\, \tau \in J$. Hence, we obtain

$$d_2(Pl, Pn) \leq \frac{L}{L+1} d_2(l, n), \;\; \forall\, \tau \in J.$$

Therefore, P is strictly contractive on Υ. When $k = 1$ and $\Upsilon = \Omega^*$, the operator P satisfies all the conditions of Theorem 1.

On the other hand, by (14), we have

$$-G(\tau) \leq ({}^{CF}\mathbb{D}^\gamma h)(\tau) - g(\tau, h(\tau)) \leq G(\tau), \;\; \forall\, \tau \in J.$$

By simple computation, we can obtain

$$\begin{aligned} & |h(\tau) - h_0 - a_\gamma g(\tau, h(\tau)) - b_\gamma \int_{\tau_0}^{\tau} g(s, f(s)) ds| \\ \leq\;& |a_\gamma| G(\tau) + |b_\gamma| \int_{\tau_0}^{\tau} G(s) ds \\ \leq\;& (|a_\gamma| + |b_\gamma| F_G) G(\tau), \;\; \forall\, \tau \in J. \end{aligned}$$

This yields that

$$|h(\tau) - (Ph)(\tau)| \leq (|a_\gamma| + |b_\gamma| F_G) G(\tau), \;\; \forall\, \tau \in J. \tag{18}$$

Multiply both sides of (18) by $e^{-K(\tau-\tau_0)}$, then,

$$|h(\tau) - (Ph)(\tau)|e^{-K(\tau-\tau_0)} \leq (|a_\gamma| + |b_\gamma| F_G) G(\tau) e^{-K(\tau-\tau_0)}, \;\; \forall\, \tau \in J.$$

Then

$$d_2(Ph, h) \leq (|a_\gamma| + |b_\gamma| F_G) G(\tau) e^{-K(\tau-\tau_0)}, \;\; \forall\, \tau \in J.$$

By Theorem 1, there exists a unique solution $f : J \to \mathbb{R}$ of (7) satisfying

$$d_2(h, f) \leq \frac{1}{1 - L/(L+1)} d_2(Ph, h) \leq (L+1)(|a_\gamma| + |b_\gamma| F_G) G(\tau) e^{-K(\tau-\tau_0)}, \quad \forall\, \tau \in J.$$

By (17), we have

$$|h(\tau) - f(\tau)|e^{-K(\tau-\tau_0)} \leq (L+1)(|a_\gamma| + |b_\gamma| F_G) G(\tau) e^{-K(\tau-\tau_0)}, \quad \forall\, \tau \in J,$$

which implies (16) holds. The proof is complete. □

Remark 2. *By the Definition 5, (16) shows (7) is generalized Hyers–Ulam–Rassias stable with the constant* $c_{f,G} = (L+1)(|a_\gamma| + |b_\gamma| F_G)$*. Theorem 3 extend the result in ([27], Corollary 2.8) and also shows that the condition* $0 < \lambda < \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}$ *can be removed.*

Remark 3. *Compared to ([4], Theorems 3 and 5), we extend the existence and uniqueness result and the generalized Hyers–Ulam–Rassias stability result for* (1) *on the noncompact interval and also remove the condition* $L|a_\alpha| < 1$ *from the assumptions.*

4. Examples

Assume that $M(\cdot)$ in Definition 1 is the solution of the following equation:

$$\frac{2(1-\cdot)}{(2-\cdot)M(\cdot)} + \frac{2\cdot}{(2-\cdot)M(\cdot)} = 1.$$

Then one can derive an explicit formula $M(\cdot) = \frac{2}{2-\cdot}$ (see ([21], p. 89)).

Example 1. *We consider the following equation:*

$$({}^{CF}\mathbb{D}^{\gamma} f)(\tau) - \lambda f(\tau) = g(\tau),\ \tau \in [0,k),\ \ k > 0, \tag{19}$$

and let $g(\tau, f(\tau)) = g(\tau) + \lambda f(\tau)$*. Obviously,* $|g(\tau, f_1(\tau)) - g(\tau, f_2(\tau))| = |\lambda||f_1(\tau) - f_2(\tau)|, \tau \in [0,k)$ *and the Lipschitz condition holds with the Lipschitz constant* $L = |\lambda|$*. Then,* (19) *is Hyers–Ulam stable on* J*, for all* $\lambda \in \mathbb{R}$ *and* $\alpha \in (0,1)$*.*

Now, let $\gamma = \frac{1}{2}, \lambda = -2, f(0) = 0$*, and* $g(\tau) = 4\tau - 4 + 4e^{-\tau} - \frac{1}{2}e^{-2\tau} + 2\tau^2$*. We consider the following equation:*

$$({}^{CF}\mathbb{D}^{\frac{1}{2}} f)(\tau) + 2f(\tau) = g(\tau),\ \tau \in [0,k),\ \ k > 0. \tag{20}$$

Let $h(\tau) = \tau^2$*, for* $\epsilon = \frac{1}{2}$ *by simple calculation, we have*

$$({}^{CF}\mathbb{D}^{\frac{1}{2}} h)(\tau) = 4\tau - 4 + 4e^{-\tau}$$

then

$$|({}^{CF}\mathbb{D}^{\frac{1}{2}} h)(\tau) + 2f(\tau) - g(\tau)| = \frac{1}{2}e^{-2\tau} \le \frac{1}{2} = \epsilon,\ \tau \in [0,k),\ \ k > 0.$$

Integrating (20) *from 0 to* τ*, we get*

$$f(\tau) = \tau^2 - \frac{1}{12}e^{-2\tau} + \frac{1}{12}e^{-\frac{1}{2}\tau}$$

then

$$\begin{aligned} |h(\tau) - f(\tau)| &= |\frac{1}{12}e^{-2\tau} - \frac{1}{12}e^{-\frac{1}{2}\tau}| = \frac{1}{12}e^{-\frac{1}{2}\tau}|1 - e^{-\frac{3}{2}\tau}| \\ &\le \frac{1}{6} \times \frac{1}{2} = \frac{1}{6}\epsilon. \end{aligned} \tag{21}$$

So (20) *is Hyers–Ulam stable (see Figure 1). Please note that the condition* $\lambda > 0$ *in ([27], Theorem 2.6) is not required here, and moreover,* (20) *is Hyers–Ulam stable, too.*

On the other hand, (21) *implies that* (20) *is also Hyers–Ulam stable even for* $\tau = +\infty$*, which shows that ([27], Remark 2.7) is not suitable.*

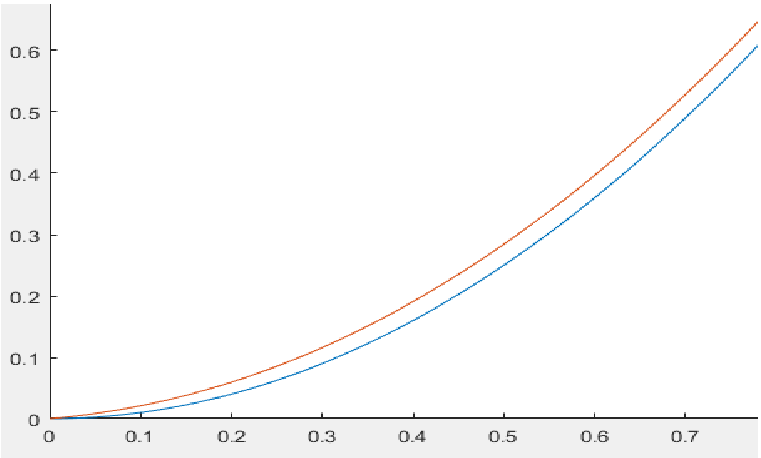

Figure 1. The exact and approximated solutions of the differential equation (20) are shown by the red and blue lines, respectively.

Example 2. *We consider the following fractional problem*

$$({}^{CF}\mathbb{D}^{\frac{1}{3}} f)(\tau) = \frac{5}{1+e^{\tau}} \frac{|f|}{1+|f|},\ \tau \in [0, +\infty), \tag{22}$$

and the inequality

$$|({}^{CF}\mathbb{D}^{\frac{1}{3}} f)(\tau) - \frac{5}{1+e^{\tau}} \frac{|f|}{1+|f|}| \leq G(\tau),\ \tau \in [0, +\infty).$$

Let $g(\tau, f(\tau)) = \frac{5}{1+e^{\tau}} \frac{|f|}{1+|f|}$, $(\tau, f) \in [0, +\infty) \times \mathbb{R}$. *Obviously* $[A_1]$ *holds. For any* $\tau \in [0, +\infty)$ *and* $f_1, f_2 \in \mathbb{R}$, *we have*

$$\begin{aligned} |g(\tau, f_1) - g(\tau, f_2)| &= \frac{5}{1+e^{\tau}} \left| \frac{|f_1|}{1+|f_1|} - \frac{|f_2|}{1+|f_2|} \right| \leq \frac{5|f_1 - f_2|}{(1+|f_1|)(1+|f_2|)} \\ &\leq 5|f_1 - f_2|. \end{aligned}$$

Then the condition $[A_2]$ *hold and* $L = 5$ *and* $k_\alpha = 5$ *in ([4], Theorem 5).*

Let $G(\tau) = e^{\tau} \in C([0, +\infty), (0, +\infty))$ *and* $\int_0^{\tau} G(s)ds = \int_0^{\tau} e^s ds = e^{\tau} - 1 \leq e^{\tau}$. (15) *holds for* $F_G = 1 > 0$. *Therefore, in view of Theorem 3,* (22) *is generalized Hyers–Ulam–Rassias stable.*

Here $\gamma = \frac{1}{3}$, *by calculation, we have* $M(\frac{1}{3}) = \frac{6}{5}$, $a_{\frac{1}{3}} = \frac{24}{25}$. *Then* $a_\gamma k_f = \frac{24}{25} \times 5 = \frac{24}{5} > 1$. *Thus* $a_\alpha k_f < 1$ *condition of Theorem 5 in [4] does not hold in this problem. Thus, ([4], Theorem 5) does not work even on* $[0, 2]$.

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