

Article

Dynamical Behavior of a Modified Leslie–Gower One Prey–Two Predators with Competition

Dian Savitri ^{1,2}, Agus Suryanto ^{1,*} , Wuryansari Muharini Kusumawinahyu ¹ and Abadi ² 

¹ Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Brawijaya, Malang 65145, Indonesia; dian11januari@student.ub.ac.id (D.S.); wmuharini@ub.ac.id (W.M.K.)

² Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Negeri Surabaya, Surabaya 60231, Indonesia; abadi@unesa.ac.id

* Correspondence: suryanto@ub.ac.id

Received: 25 February 2020; Accepted: 19 April 2020; Published: 28 April 2020



Abstract: We study the dynamics of a modified Leslie–Gower one prey–two predators model with competition between predator populations. The model describes complex dynamics in the permanence, global stability and bifurcation. It is shown that there are eight possible equilibrium states. Two equilibrium states, i.e., the extinction of all of the species state and the extinction of both predators state are always unstable, while the other equilibrium states are conditionally locally and globally asymptotically stable. We also analyzed numerically the effect of competition between predators. Our numerical simulations showed that the competition rate of the second-predator may induce the transcritical bifurcation, the saddle-node bifurcation as well as the bi-stability phenomenon. Such numerical results are consistent with the analytical results.

Keywords: modified Leslie–Gower; two predator–one prey interaction; stability analysis; Lyapunov function; saddle-node bifurcation; transcritical bifurcation; bi-stability

1. Introduction

The model of population interaction has attracted great interest of many scholars. One of the important mathematical models is the model of the predator–prey interaction [1]. A well-known model that describes the predator–prey interaction is the Leslie–Gower model [2]. In this model, the predator is assumed to grow logistically, where its carrying capacity is proportional to the density of prey. In the case of prey scarcity, predators can consume other populations but their growth will be bounded by the fact that their main prey is not abundantly available. To account for such a situation, Aziz-Alaoui and Okiye [3] proposed a modified Leslie–Gower model by introducing a constant that measures the environment protection for the predator. Since then, many researchers have studied the modified Leslie–Gower models with various types of functional responses [4,5], harvesting [6–9], Allee effect [10–13], etc.

The model of predator–prey interaction is one of the food chain building block models that involves two species. Although predator–prey interaction models have been widely studied or developed, many ecological and modeling problems remain open. One of the developments that has been made includes adding another prey or another predator. For example, some scholars have proposed mathematical models that describe the interaction between two preys and one predator and have studied dynamics such as the stability of equilibrium states to determine conditions of coexistence in ecosystems, species extinction, existence of stable periodic solutions or transitions between prey and predator [14–18]. There are also many mathematical models describing the interaction of one prey with two predators. Sarwandi et al. [19] studied the effects of a prey refuge on the two competing predators. Sayekti et al. [20] considered the effects of prey harvesting on the interaction of one prey

and two predators. They found that coexistence can be obtained if the prey harvesting rate is less than the prey intrinsic growth rate. We notice that all above mentioned models are derived based on the Lotka–Volterra predator–prey model. Recently, Alebraheem and Abu-Hasan [21–23] introduced mathematical models of one prey–two predators interactions, which are based on the Leslie–Gower model. Hence, they considered that the two predators grow following a logistic model, where the carrying capacity is only dependent on the prey density.

Recently, we [24] revisited the model of Alebraheem and Abu-Hasan [23]. We assumed in [24] that without competition, the interaction of prey with the two predators follows the Leslie–Gower model with a Holling type I functional response. If competition is also considered, then the model is

$$\begin{cases} \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \beta_1xy - \beta_2xz \\ \frac{dy}{dt} = r_1y\left(1 - \frac{y}{x}\right) - \alpha_1yz \\ \frac{dz}{dt} = r_2z\left(1 - \frac{z}{x}\right) - \alpha_2yz, \end{cases} \tag{1}$$

where $x(t)$, $y(t)$ and $z(t)$ represent the population density of the prey, the predator-1 and the predator-2 at time t , respectively and r , r_1 and r_2 are, respectively, the intrinsic growth rate of x , y and z . The carrying capacity of prey is k . Parameters β_1 and β_2 are the maximum value of the per capita reduction rate of x due to y and z , respectively. Parameters α_1 and α_2 denote the competition rate of predator-1 and predator-2, respectively. All parameters in the model are positive. As opposed to the model in [23], the intrinsic growth rates of both predators in system (1) are constant. In [23], the intrinsic growth rates of both predators are proportional to the density of prey. The dynamical behavior of system (1) was studied by Savitri et al. [24]. They only focused on local stability analysis and show some phase portraits by numerical simulations; transcritical bifurcation and saddle-node bifurcation have not been observed.

As explained by Aziz-Alaoui and Okiye [3], the Leslie–Gower term (y/x) in the system (1) describes the reduced predator population due to scarcity of its favorite foods. In this article it is assumed that if there is a severe scarcity of prey, predators can consume other populations, but their growth will be bounded by the fact that their favorite food (x) is not available in large quantities. To take into account such a situation, following Aziz-Alaoui and Okiye [3], we add a constant in the denominator to obtain a modified Leslie–Gower one prey–two predators model with competition between predators.

$$\begin{cases} \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \beta_1xy - \beta_2xz \\ \frac{dy}{dt} = r_1y\left(1 - \frac{y}{x+k_1}\right) - \alpha_1yz \\ \frac{dz}{dt} = r_2z\left(1 - \frac{z}{x+k_2}\right) - \alpha_2yz, \end{cases} \tag{2}$$

where k_1 and k_2 are measures of environmental protection for predator-1 and predator-2, respectively.

In this article we investigate the dynamics of system (2), which includes the permanence, the existence of equilibrium states and local and global stability of equilibrium states. We also discuss the bifurcation numerically using MatCont. This article is organized in the following manner. In Section 2, we give some preliminary theorems and show that all solutions of the model are always non-negative, permanent and bounded. We analyze the existence of the equilibrium states, and the local and global stability in Section 3. In Section 4 we show numerical simulations to illustrate the analytical results and explore transcritical bifurcation, saddle-node bifurcation and bistability. Finally, we end with a brief conclusion in Section 5.

2. Non-Negativity, Boundedness and Permanence

2.1. Non-Negativity

Notice that model (2) describes the interaction of three populations, and therefore, for biological relevance, we require that the solutions of this model must be non-negative. The non-negativity of solutions are assured by the following theorem:

Theorem 1. *All solutions of system (2) with initial values $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$ are always non-negative.*

Proof. From system (2) we show that

$$\begin{cases} x(t) = x(0)e^{\int_0^t (r(1-\frac{x(\tau)}{k})-\beta_1y(\tau)-\beta_2z(\tau))d\tau} \\ y(t) = y(0)e^{\int_0^t (r_1(1-\frac{y(\tau)}{x(\tau)+k_1})-\alpha_1z(\tau))d\tau} \\ z(t) = z(0)e^{\int_0^t (r_2(1-\frac{z(\tau)}{x(\tau)+k_2})-\alpha_2y(\tau))d\tau} \end{cases} \tag{3}$$

Because exponential function $e^u \geq 0$ for any real number u , and the initial values are positive, i.e., $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$, system (3) shows that $x(t) \geq 0$, $y(t) \geq 0$ and $z(t) \geq 0$. \square

2.2. Boundedness

In nature, resources for every population are always limited such that the population should also be bounded. The boundedness of population prey and predators as solution of system (2) is demonstrated by utilizing the following comparison lemma:

Lemma 1. *Suppose that $a > 0$, $b > 0$ and $x(0) > 0$. Then for $\frac{dx}{dt} \leq x(b - ax)$, we have*

$$\limsup_{t \rightarrow +\infty} x(t) \leq \frac{b}{a},$$

and for $\frac{dx}{dt} \geq x(b - ax)$, we have

$$\liminf_{t \rightarrow +\infty} x(t) \geq \frac{b}{a}.$$

[25].

Theorem 2. *All solutions of system (2) with initial values $x(0) > 0$, $y(0) > 0$ and $z(0) > 0$ are always bounded above.*

Proof. From the first equation of system (2), we have

$$\frac{dx}{dt} \leq rx\left(1 - \frac{x}{k}\right).$$

Since $r > 0$ and $k > 0$, Lemma 1 gives

$$\limsup_{t \rightarrow +\infty} x(t) \leq k \equiv L_1. \tag{4}$$

Then, for any sufficiently small $\varepsilon > 0$, there is a $t_1 \geq 0$ such that

$$x(t) \leq k + \varepsilon, \text{ for } t \geq t_1 \tag{5}$$

Based on this result and the second equation of system (2) we have

$$\frac{dy}{dt} \leq r_1 y \left(1 - \frac{y}{x + k_1} \right) \leq r_1 y \left(1 - \frac{y}{k + \varepsilon + k_1} \right), \quad t \geq t_1. \tag{6}$$

From Lemma 1 we obtain

$$\limsup_{t \rightarrow +\infty} y(t) \leq k + \varepsilon + k_1.$$

Taking $\varepsilon \rightarrow 0$, we get

$$\limsup_{t \rightarrow +\infty} y(t) \leq k + k_1 \equiv L_2.$$

Finally, the third equation of system (2) leads to

$$\frac{dz}{dt} \leq r_2 z \left(1 - \frac{z}{x + k_2} \right) \leq r_2 z \left(1 - \frac{z}{k + \varepsilon + k_2} \right), \quad t \geq t_1.$$

Again, Lemma 1 gives

$$\limsup_{t \rightarrow +\infty} z(t) \leq k + \varepsilon + k_2$$

If we let $\varepsilon \rightarrow 0$, then

$$\limsup_{t \rightarrow +\infty} z(t) \leq k + k_2 \equiv L_3.$$

□

2.3. Permanence

In this part, we show the permanence of system (2). System (2) is said to be permanent if there exist positive constants l and L such that

$$l \leq \min \left\{ \liminf_{t \rightarrow +\infty} x(t), \liminf_{t \rightarrow +\infty} y(t), \liminf_{t \rightarrow +\infty} z(t) \right\} \\ \leq \max \left\{ \limsup_{t \rightarrow +\infty} x(t), \limsup_{t \rightarrow +\infty} y(t), \limsup_{t \rightarrow +\infty} z(t) \right\} \leq L.$$

Theorem 3. System (2) with initial values $x(0) > 0, y(0) > 0$ and $z(0) > 0$ is permanent if $\frac{(\beta_1(k+k_1) + \beta_2(k+k_2))}{r} < 1, \frac{\alpha_1(k+k_2)}{r_1} < 1$ and $\frac{\alpha_2(k+k_1)}{r_2} < 1$.

Proof. From Theorem 2, we have that for any sufficiently small $\varepsilon > 0$, there are a $t_2, t_3 \geq 0$ such that

$$y(t) \leq k + k_1 + \varepsilon, \quad \text{for } t \geq t_2$$

$$z(t) \leq k + k_2 + \varepsilon, \quad \text{for } t \geq t_3.$$

Let $\hat{T} = \max\{t_1, t_2, t_3\}$. Using Theorem 2 and the first equation in system (2), we can show that for any $t \geq \hat{T}$,

$$\frac{dx}{dt} = rx \left(1 - \frac{x}{k} \right) - \beta_1 xy - \beta_2 xz \\ \geq rx \left(1 - \frac{x}{k} \right) - \beta_1 (k + k_1 + \varepsilon)x - \beta_2 (k + k_2 + \varepsilon)x \\ = xr \left(\left(1 - \frac{\beta_1(k + k_1 + \varepsilon) + \beta_2(k + k_2 + \varepsilon)}{r} \right) - \frac{x}{k} \right).$$

By applying Lemma 1 and taking $\varepsilon \rightarrow 0$ we obtain

$$\lim_{t \rightarrow +\infty} \inf x(t) \geq l_1$$

where $l_1 = \left(1 - \frac{\beta_1(k+k_1)+\beta_2(k+k_2)}{r}\right)k$.

Similarly, using the second equation in system (2), we get for any $t \geq \hat{T}$,

$$\begin{aligned} \frac{dy}{dt} &= r_1y\left(1 - \frac{y}{x+k_1}\right) - \alpha_1yz \\ &\geq r_1y\left(1 - \frac{y}{l_1+k_1+\varepsilon}\right) - \alpha_1(k+k_2+\varepsilon)y \\ &= r_1y\left(1 - \frac{\alpha_1(k+k_2+\varepsilon)}{r_1} - \frac{y}{l_1+k_1+\varepsilon}\right). \end{aligned}$$

By using Lemma 1 and letting $\varepsilon \rightarrow 0$ we obtain

$$\lim_{t \rightarrow +\infty} \inf y(t) \geq l_2$$

where

$$l_2 = \left(1 - \frac{\alpha_1(k+k_2)}{r_1}\right)(l_1+k_1).$$

As in the previous case, the third equation in system (2) gives

$$\begin{aligned} \frac{dz}{dt} &= r_2z\left(1 - \frac{z}{x+k_2}\right) - \alpha_2yz \\ &\geq r_2z\left(1 - \frac{z}{l_1+k_2+\varepsilon}\right) - \alpha_2(k+k_1+\varepsilon)z \\ &= r_2z\left(1 - \frac{\alpha_2(k+k_1+\varepsilon)}{r_2} - \frac{r_2z}{l_1+k_2+\varepsilon}\right), \end{aligned}$$

and therefore (by Lemma 1 and taking $\varepsilon \rightarrow 0$) we get

$$\lim_{t \rightarrow +\infty} \inf z(t) \geq l_3$$

where

$$l_3 = \left(1 - \frac{\alpha_2(k+k_1)}{r_2}\right)(l_1+k_2).$$

Observe that if $\frac{\beta_1(k+k_1)+\beta_2(k+k_2)}{r} < 1$, $\frac{\alpha_1(k+k_2)}{r_1} < 1$ and $\frac{\alpha_2(k+k_1)}{r_2} < 1$ then $l_i > 0$, $i = 1, 2, 3$. Hence, by choosing $l \leq \min\{l_1, l_2, l_3\}$ and $L \geq \max\{L_1, L_2, L_3\}$, we can easily see that system (2) is permanent. This completes the proof. \square

3. Equilibrium States and Their Stability

3.1. Equilibrium State

By solving the following system of equations, we can show that system (2) has eight feasible positive equilibrium states:

$$rx\left(1 - \frac{x}{k}\right) - \beta_1xy - \beta_2xz = 0$$

$$r_1 y \left(1 - \frac{y}{x + k_1} \right) - \alpha_1 y z = 0$$

$$r_2 z \left(1 - \frac{z}{x + k_2} \right) - \alpha_2 y z = 0.$$

All equilibrium states and their existence conditions are shown in Table 1. We remark that x^* in the equilibrium state $E_8 = (x^*, y^*, z^*)$ is the positive solution of the cubic equation

$$Ax^3 + x^2 + Cx + D = 0 \tag{7}$$

where

$$A = r\alpha_1\alpha_2.$$

$$B = r\alpha_1k\alpha_2 + rkk_2\alpha_2\beta_2 + rk_2\alpha_1\alpha_2 - r\alpha_1\alpha_2 + kr_2\alpha_1\beta_1.$$

$$C = -rkk_1\alpha_1\alpha_2 + rkk_1\alpha_2\beta_2 - rkk_1\alpha_1\alpha_2 + \alpha_2r\beta_2kk_2 + \alpha_2rk_1k_2\alpha_1 - \alpha_2rkk_2\alpha_1 - rr_1r_2 + r_2kk_1\alpha_1\beta_1 - r_1r_2k\beta_2 - r_2kk_2\alpha_1\beta_1.$$

$$D = r_1\alpha_2kk_1k_2\beta_2 - r\alpha_1\alpha_2kk_1k_2 + rr_1r_2k - r_1r_2kk_1\beta_1 + r_2kk_1k_2\alpha_1\beta_1 - r_1r_2kk_2\beta_2.$$

Table 1. Equilibrium states and their existence conditions.

Equilibrium State	Existence Condition
$E_1 = (0, 0, 0)$	-
$E_2 = (k, 0, 0)$	-
$E_3 = (0, 0, k_2)$	-
$E_4 = (0, k_1, 0)$	-
$E_5 = \left(\frac{k(r-\beta_2k_2)}{k\beta_2+r}, 0, \frac{r(k+k_2)}{k\beta_2+r} \right)$	$\beta_2k_2 < r$
$E_6 = \left(\frac{k(r-\beta_1k_1)}{k\beta_1+r}, \frac{r(k_1+k)}{k\beta_1+r}, 0 \right)$	$\beta_1k_1 < r$
$E_7 = \left(0, \frac{k_1r_2(\alpha_1k_2-r_1)}{\alpha_1\alpha_2k_1k_2-r_1r_2}, \frac{k_2r_1(\alpha_2k_1-r_2)}{\alpha_1\alpha_2k_1k_2-r_1r_2} \right)$	$(\alpha_1k_2 > r_1, \alpha_2k_1 > r_2 \text{ and } \alpha_1\alpha_2k_1k_2 > r_1r_2)$ or $(\alpha_1k_2 < r_1, \alpha_2k_1 < r_2 \text{ and } \alpha_1\alpha_2k_1k_2 < r_1r_2)$
$E_8 = (x^*, y^*, z^*)$	See comments in text

The existence of positive root x^* of a cubic equation can be easily derived by Cardano’s criteria. The detail of Cardano’s criteria can be seen, for example, in [10] and is not discussed here. Furthermore, the values of y^* and z^* are respectively given by

$$y^* = \frac{r_2(x^* + k_1)(a_1(x^* + k_2) - r_1)}{a_1a_2(x^* + k_1)(x^* + k_2) - r_1r_2}$$

and

$$z^* = \frac{r_1(x^* + k_2)(a_2(x^* + k_1) - r_2)}{a_1a_2(x^* + k_1)(x^* + k_2) - r_1r_2}.$$

We also require that y^* and z^* be positive.

3.2. Local Stability of Equilibrium States

The local behavior of system (2) is investigated by considering its linear approximation around each equilibrium state. The variational matrix of the linearized system around a point (x, y, z) is given by

$$J(x, y, z) = \begin{bmatrix} -\beta_1y - \beta_2z - \frac{2x}{k} + 1 & -\beta_1x & -\beta_2x \\ \frac{r_1y^2}{(x+k_1)^2} & r_1 - \frac{2r_1y}{x+k_1} - \alpha_1z & -\alpha_1y \\ \frac{r_2z^2}{(x+k_2)^2} & -\alpha_2z & r_2 - \frac{2r_2z}{x+k_2} - \alpha_2y \end{bmatrix}.$$

The stability of an equilibrium state $(\hat{x}, \hat{y}, \hat{z})$ is determined by the eigenvalues of the variational matrix evaluated at the corresponding equilibrium state, i.e., $J(\hat{x}, \hat{y}, \hat{z})$.

Theorem 4.

- (i) Equilibrium states $E_1 = (0, 0, 0)$ and $E_2 = (k, 0, 0)$ are unstable.
- (ii) If $r < k_2\beta_2$ and $r_1 < \alpha_1k_2$ then equilibrium state $E_3 = (0, 0, k_2)$ is locally asymptotically stable.
- (iii) If $r < \beta_1k_1$ and $r_2 < \alpha_2k_1$ then equilibrium state $E_4 = (0, k_1, 0)$ is locally asymptotically stable.

Proof.

- (i) All eigenvalues of the variational matrix $J(E_1)$ are $\lambda_1 = r > 0, \lambda_1 = r_1 > 0, \lambda_2 = r_2 > 0$. Hence, E_1 is unstable. The variational matrix at $E_2 = (k, 0, 0)$ has the following eigenvalues: $\lambda_1 = -r < 0, \lambda_2 = r_1 > 0$ and $\lambda_3 = r_2 > 0$. Since there are positive eigenvalues, E_2 is also unstable (saddle-point).
- (ii) The eigenvalues of variational matrix at $E_3 = (0, 0, k_2)$ are $\lambda_1 = -r_2 < 0, \lambda_2 = -\beta_2k_2 + r, \lambda_3 = -\alpha_1k_2 + r_1$. If $r < k_2\beta_2$ and $r_1 < \alpha_1k_2$ then we have $\lambda_2 < 0$ and $\lambda_3 < 0$. This completes the proof.
- (iii) The variational matrix at point $E_4 = (0, k_1, 0)$ has eigenvalues $\lambda_1 = -r_1 < 0, \lambda_2 = -\beta_1k_1 + r$ and $\lambda_3 = -\alpha_2k_1 + r_2$. Clearly, if $r < \beta_1k_1$ and $r_2 < \alpha_2k_1$ then E_4 is locally asymptotically stable. \square

Theorem 5. Let $x_5 = \frac{k(r-\beta_2k_2)}{k\beta_2+r} > 0$ and $z_5 = \frac{r(k_2+k)}{k\beta_2+r} > 0$. The equilibrium state $E_5 = (x_5, 0, z_5)$ is locally asymptotically stable if $r_1(k\beta_2 + r) < \alpha_1r(k + k_2)$.

Proof. See Appendix A. \square

Theorem 6. Let $x_6 = \frac{k(r-\beta_1k_1)}{k\beta_1+r} > 0$ and $y_6 = \frac{r(k_1+k)}{k\beta_1+r} > 0$. The equilibrium state $E_6 = (x_6, y_6, 0)$ is locally asymptotically stable if $r_2(k\beta_1 + r) < \alpha_2r(k + k_1)$.

Proof. See Appendix B. \square

Theorem 7. Let $y_7 = \frac{k_1r_2(\alpha_1k_2-r_1)}{\alpha_1\alpha_2k_1k_2-r_1r_2} > 0$ and $z_7 = \frac{k_2r_1(\alpha_2k_1-r_2)}{\alpha_1\alpha_2k_1k_2-r_1r_2} > 0$. The equilibrium state $E_7 = (0, y_7, z_7)$ is locally asymptotically stable if $r < \beta_1y_7 + \beta_2z_7$ and $r_1r_2 > \alpha_1\alpha_2k_1k_2$.

Proof. See Appendix C. \square

Theorem 8. The coexistence equilibrium state $E_8 = (x^*, y^*, z^*)$ is locally asymptotically stable if $A_3 > 0$, and $A_1A_2 - A_3 > 0$, where

$$A_1 = \left(\frac{rx^*}{k} + \frac{r_1y^*}{x^* + k_1} + \frac{r_2z^*}{x^* + k_2} \right)$$

$$A_2 = \left(\frac{rr_1x^*y^*}{k(x^* + k_1)} + \frac{\beta_1r_1x^*y^{*2}}{(x^* + k_1)^2} + \frac{r_1r_2y^*z^*}{(x^* + k_1)(x^* + k_2)} + \alpha_1\alpha_2y^*z^* + \frac{rr_2x^*z^*}{k(x^* + k_2)} + \frac{\beta_2r_2x^*z^{*2}}{(x^* + k_1)^2} \right) > 0$$

$$A_3 = \frac{\beta_2r_1r_2x^*y^*z^{*2}}{(x^* + k_1)(x^* + k_2)^2} + \frac{\alpha_1\alpha_2rx^*y^*z^*}{k} + \frac{\beta_1r_1r_2x^*y^{*2}z^*}{(x^* + k_1)^2(x^* + k_2)} + \frac{rr_1r_2x^*y^*z^*}{k(x^* + k_1)(x^* + k_2)} - \frac{\alpha_1\beta_1r_2x^*y^*z^{*2}}{(x^* + k_2)^2} - \frac{\alpha_2\beta_2r_1x^*y^{*2}z^*}{(x^* + k_1)^2}.$$

Proof. See Appendix D. \square

Remark 1.

- (i) We notice that model (1) has only four equilibrium states (see [24]). This is a consequence of the use of the Leslie–Gower model, which does not allow the extinction of prey, and therefore equilibrium states E_1, E_3, E_4 and E_7 do not exist in the model (1). In the model (2) we introduced environmental protection for both predators, which are denoted by k_1 and k_2 , respectively. Both constants cause model (2) to have an additional four equilibrium states where each density of prey equilibrium states is zero, i.e., E_1, E_3, E_4 and E_7 .
- (ii) From Theorems 4–8, we can classify the stability of each equilibrium state, as seen in Table 2. The extinction of all of the species state and the extinction of the both predators state are always unstable, while other equilibrium states are conditionally stable. This can be interpreted as meaning that the two predators cannot become extinct at the same time, even though there are no prey in the environment. This phenomenon can be understood from model (2) that predator-1 and predator-2 have environmental protection. We can see in Table 2 that the environmental protection is very important in determining the stability of the equilibrium points.

Table 2. The type stability and stability condition of all of the equilibrium states

Equilibrium State	The Type Stability	Condition of the Stability
E_1	unstable (saddle-point)	-
E_2	unstable (saddle-point)	-
E_3	locally asymptotically stable	$r < k_2\beta_2$ and $r_1 < \alpha_1k_2$
E_4	locally asymptotically stable	$r < \beta_1k_1$ and $r_2 < \alpha_2k_1$
E_5	locally asymptotically stable	$r_1(k\beta_2 + r) < \alpha_1r(k + k_2)$
E_6	locally asymptotically stable	$r_2(k\beta_1 + r) < \alpha_2r(k + k_1)$
E_7	locally asymptotically stable	$r < \beta_1y_7 + \beta_2z_7$ and $r_1r_2 > \alpha_1\alpha_2k_1k_2$
E_8	locally asymptotically stable	$A_3 > 0$, and $A_1A_2 - A_3 > 0$

3.3. Global Stability of Equilibrium States

In the following we present the global stability analysis of some equilibrium states. The global stability is performed by choosing an appropriate Lyapunov function for each equilibrium state.

Theorem 9. The equilibrium state $E_3 = (0, 0, k_2)$ of the system (2) is globally asymptotically stable if $\alpha_1l_3 > r_1$ and $1 + \frac{k_2}{l_1+k_2} < \frac{l_3}{k+k_2}$.

Proof. We assume that $\alpha_1l_3 > r_1$ and construct a Lyapunov function $V_1 = ay + (z - k_2 - k_2 \ln \frac{z}{k_2})$, where $a = \frac{\alpha_2k_2(k+k_1)}{r_1l_2+(k+k_1)(\alpha_1l_3-r_1)} > 0$. The time derivative of this Lyapunov function is

$$\begin{aligned} \frac{dV_1}{dt} &= a\left[r_1y\left(1 - \frac{y}{x+k_1}\right) - \alpha_1yz\right] + (z - k_2)\left[r_2\left(1 - \frac{z}{x+k_2}\right) - \alpha_2y\right] \\ &\leq \left(ar_1 + \alpha_2k_2 - a\alpha_1z - \frac{ar_1y}{x+k_1}\right)y + r_2\left(1 + \frac{k_2}{x+k_2} - \frac{z}{x+k_2}\right)z \\ &\leq \left(ar_1 + \alpha_2k_2 - a\alpha_1l_3 - \frac{ar_1l_2}{k+k_1}\right)y + r_2\left(1 + \frac{k_2}{l_1+k_2} - \frac{l_3}{k+k_2}\right)z \\ &= r_2\left(1 + \frac{k_2}{l_1+k_2} - \frac{l_3}{k+k_2}\right)z. \end{aligned}$$

It is clear that if $1 + \frac{k_2}{l_1+k_2} < \frac{l_3}{k+k_2}$ then $\frac{dV_1}{dt} \leq 0$. We observe that $\frac{dV_1}{dt} = 0$ if and only if $x = 0, y = 0$ and $z = k_2$. Hence, according to La Salle’s invariant principle, the equilibrium state $E_3 = (0, 0, k_2)$ is globally asymptotically stable if $\alpha_1l_3 > r_1$ and $1 + \frac{k_2}{l_1+k_2} < \frac{l_3}{k+k_2}$. □

Theorem 10. The equilibrium $E_4 = (0, k_1, 0)$ of the system (2) is globally asymptotically stable if $\alpha_2 l_2 > r_2$ and $1 + \frac{k_1}{l_1 + k_1} < \frac{l_2}{k + k_1}$.

Proof. The theorem can be proven using a similar procedure as in the Proof of Theorem 9, i.e., by choosing a Lyapunov function $V_2 = (y - k_1 - k_1 \ln \frac{y}{k_1}) + bz$. The detail of the Proof can be seen in Appendix E. □

Theorem 11. The coexistence equilibrium $E_8 = (x^*, y^*, z^*)$ of system (2) is globally asymptotically stable if $r_1 \geq \frac{1}{2}(\alpha_1(x^* + k_1) + c_2\alpha_2(x^* + k_2))$ and $r_2 \geq \frac{1}{2c_2}(\alpha_1(x^* + k_1) + c_2\alpha_2(x^* + k_2))$ where

$$c_1 = \begin{cases} \frac{r_1 L_2}{\beta_1(l_1 + k_1)}, & \text{if } (x - x^*)(y - y^*) \geq 0 \\ \frac{r_1 l_2}{\beta_1(k + k_1)}, & \text{if } (x - x^*)(y - y^*) < 0 \end{cases} \quad \text{and } c_2 = \begin{cases} \frac{c_1 \beta_2(l_1 + k_2)}{r_2 L_3}, & \text{if } (x - x^*)(z - z^*) \geq 0 \\ \frac{c_1 \beta_2(k + k_2)}{r_2 l_3}, & \text{if } (x - x^*)(z - z^*) < 0. \end{cases}$$

Proof. The theorem can be proven by defining a Lyapunov function $V_3 = V_{31} + V_{32} + V_{33}$, where $V_{31} = c_1(x - x^* + x \ln \frac{x}{x^*})$, $V_{32} = (x^* + k_1)(y - y^* + y \ln \frac{y}{y^*})$ and $V_{33} = c_2(x^* + k_3)(z - z^* + z \ln \frac{z}{z^*})$. Detail of the Proof can be seen in Appendix F. □

4. Numerical Simulations

In this section we solve model (2) using the 4th-order Runge–Kutta method with parameter values given in Table 3. The numerical simulations are performed to study the dynamics of model (2), particularly to investigate the change of dynamical behavior due to the change of the competition rate of predator-2 (α_2). To give a view on the dynamical behavior, we consider four different cases ($\alpha_2 = 0.2, \alpha_2 = 0.4, \alpha_2 = 0.5, \alpha_2 = 0.6$) as the consequence of the previous stability analysis. Such dynamical behavior is depicted in Figure 1, which shows some phase portraits of model (2) for four different values of α_2 .

Table 3. Values of parameters.

Parameter	r	r_1	r_2	k	k_1	k_2	β_1	β_2	α_1
Value	1.8	0.8	1.1	2.8	1.8	1.4	0.5	1.3	0.5

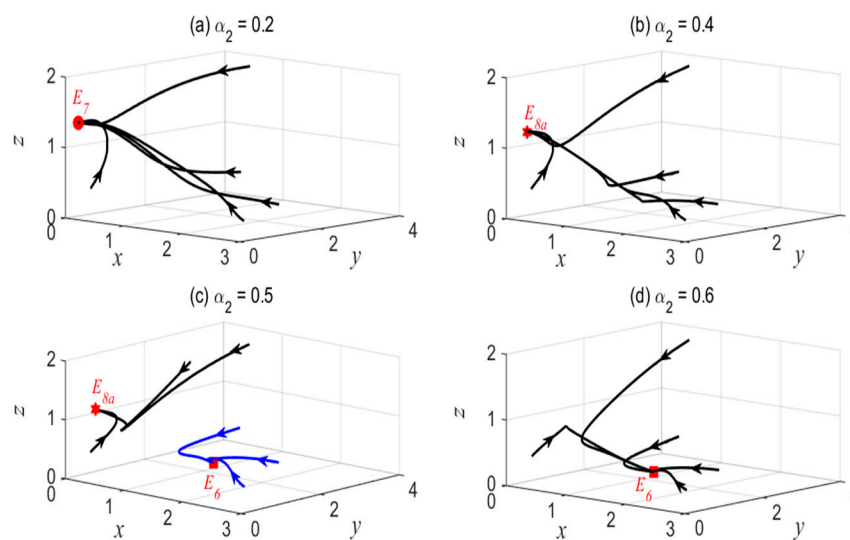


Figure 1. Phase portraits of model (2) for different value of competition rate of predator-2 (α_2). (a) $\alpha_2 = 0.2$, (b) $\alpha_2 = 0.4$, (c) $\alpha_2 = 0.5$, (d) $\alpha_2 = 0.6$.

Case (1). When $\alpha_2 = 0.2$, it is found that $0.88 = r_1r_2 > \alpha_1\alpha_2k_1k_2 = 0.252$, and therefore according to Theorem 7, $E_7 = (0.0, 0.31529, 1.31974)$ is asymptotically stable. In this case, no other equilibrium state is stable (see Figure 1a).

Case (2). If we increase the competition rate of predator-2 such that $\alpha_2 = 0.4$, then we have a unique coexistence equilibrium state, namely $E_8 = (0.03305, 0.47462, 1.18577)$. E_8 is asymptotically stable since the Routh–Hurwitz criterion in Theorem 8 is satisfied. Our numerical simulation shown in Figure 1b confirms this stability property.

Case (3). If we take $\alpha_2 = 0.5$, then $3.52 = r_2(kb_1 + r) < r\alpha_2(k + k_2) = 4.14$. From Theorem 6 we know that $E_6 = (0.7875, 2.5875, 0.0)$ is asymptotically stable. Further inspection shows that there are coexistence equilibrium states, namely $E_{8a} = (0.10985, 0.59518, 1.101378)$ and $E_{8b} = (0.56274, 1.68736, 0.45735)$. Using Theorem 8, we can show that E_{8a} is asymptotically stable, while E_{8b} is unstable. Hence, for the case of $\alpha_2 = 0.5$, model (2) exhibits a bi-stability phenomenon as there are two stable equilibrium states. The bi-stability phenomenon can be observed clearly in our numerical simulation depicted in Figure 1c.

Case (4). Finally, we take $\alpha_2 = 0.6$ and show the result of our numerical simulation in Figure 1d. From Figure 1d, it is observed that only $E_6 = (0.7875, 2.5875, 0.0)$ is stable. This can be understood from the fact that only the stability condition in Theorem 7 is satisfied ($3.52 = r_2(kb_1 + r) < r\alpha_2(k + k_2) = 4.96$), while the stability conditions for other equilibrium states are not fulfilled.

From previous numerical simulations, we observe that the dynamical property of model (2) changes as we vary the value of α_2 . To understand the dynamics of model (2) with parameters values in Table 3, we check the existence of equilibrium states and their local stability properties using the previous theoretical results. From the existence conditions, it is found that our model has the following equilibrium states: $E_1 = (0, 0, 0)$, $E_2 = (2.8, 0, 0)$, $E_3 = (0, 0, 1.4)$, $E_4 = (0, 1.8, 0)$ and $E_6 = (0.7875, 2.5875, 0)$. The extinction of the predator-1 equilibrium state (E_5) does not exist, while the prey extinction equilibrium state (E_7) exists if $\alpha_2 < 0.6111$. Based on Theorems 4–7, we can check that E_1 , E_2 , E_3 and E_4 are unstable. E_7 is stable if $\alpha_2 < \alpha_2^{*1} = 0.330499$, and the stability of E_6 is achieved whenever $\alpha_2 > \alpha_2^{*2}$.

The density of the prey population in the coexistence equilibrium state is determined by the cubic Equation (7). We applied the well-known Cardano formula in MAPLE Software to verify all possible real roots of Equation (7) as well as to check the existence of the coexistence equilibrium state. For biological reasons, the coexistence equilibrium state exists if the densities of prey (x^*), predator-1 (y^*) and predator-2 (z^*) are positive. Based on the cubic Equation (7), there are three possible roots. We denote those possible roots as x_8^* , x_{8a}^* and x_{8b}^* and the respective coexistence equilibrium states as $E_8 = (x_8^*, y_8^*, z_8^*)$, $E_{8a} = (x_{8a}^*, y_{8a}^*, z_{8a}^*)$ and $E_{8b} = (x_{8b}^*, y_{8b}^*, z_{8b}^*)$. We found that one of roots of Equation (7), say x_8^* , is a real number if $\alpha_2 < \alpha_2^{*3} = 0.560599$. Under this condition, y_8^* and z_8^* are positive, but x_8^* is positive only if $\alpha_2 > \alpha_2^{*1} = 0.330499$. Hence, E_8 exists if $\alpha_2^{*1} < \alpha_2 < \alpha_2^{*3}$. The second root (x_{8a}^*) is a positive real number if $\alpha_2 < \alpha_2^{*3}$. However, $y_{8a}^* > 0$ and $z_{8a}^* > 0$ can be achieved only if $\alpha_2 > \alpha_2^{*2} = 0.425121$, so that E_{8a} exists if $\alpha_2^{*1} < \alpha_2 < \alpha_2^{*3}$. Finally, it is found that the third root (x_{8b}^*) is also a real number whenever $\alpha_2^{*3} < \alpha_2 < 13.7915378$. However, $x_{8b}^* < 0$ and therefore E_{8b} is not biologically feasible. The stability of the coexistence equilibrium point can be checked by applying the Routh–Hurwitz criteria (see Theorem 8). Due to the explicit forms of the coexistence equilibrium points (x_8^* and x_{8a}^*) being very complicated, we investigated the stability of coexistence equilibrium points numerically using a numerical continuation package MatCont. The numerical continuation was performed not only to study the stability of the coexistence equilibrium points, but also to examine all possible equilibrium states and their stability properties. In Figure 2 we plot the equilibrium state density of predator-1 (y) as the value of α_2 is changed. It is noticed that the equilibrium state density of prey (x) and that of predator-2 (z) are similar to this graph. It appears that our theoretical results are confirmed by the results of numerical continuation. The detail of the dynamics shown in Figure 2 could be explained as follows.

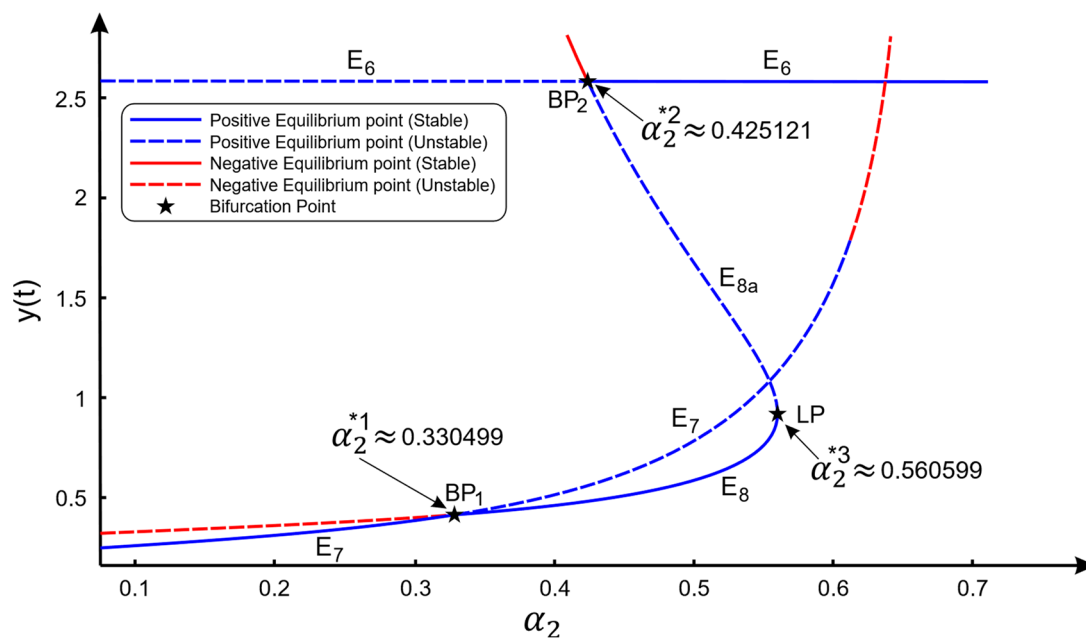


Figure 2. Equilibrium state density of predator-1 as function of competition rate of predator-2 (α_2).

4.1. Transcritical Bifurcation

From Figure 1a,b we observe the exchange of stability between the prey extinction equilibrium state (E_7) and the coexistence equilibrium state (E_8), indicating there occurs a transcritical bifurcation caused by α_2 . The transcritical bifurcation is more clearly seen in Figure 2. For a relatively small α_2 , only the equilibrium state E_7 is stable. As α_2 increases, there appears a branch point (BP_1) at α_2^{*1} . This BP_1 indicates the appearance of transcritical bifurcation, i.e., there is a change of stability between the extinction of the prey equilibrium state (E_7) and the coexistence equilibrium state (E_8). When $\alpha_2 < \alpha_2^{*1}$, E_7 is asymptotically stable while E_8 is unstable. On the other hand, if $\alpha_2^{*1} < \alpha_2 < \alpha_2^{*2}$, then E_7 is unstable while E_8 becomes asymptotically stable. This shows that when the competition rate of predator-2 is relatively low, the predator-2 can grow well, while if the competition rate of predator-2 is large enough, then the growth of predators is reduced and therefore predation is decreased. As a result, prey can survive, which is shown by the stability of the coexistence equilibrium state (E_8).

Another transcritical bifurcation may also occur between E_6 and E_{8a} , where the bifurcation point is at BP_2 (or $\alpha_2 = \alpha_2^{*2}$). We remark that those phenomena of transcritical bifurcations are only possible mathematically. Indeed, E_8 does not biologically exist for $\alpha_2 < \alpha_2^{*1}$, since in this case the value of x_8^* is negative. Similarly, E_{8a} does not exist for biological reasons for $\alpha_2 < \alpha_2^{*2}$. Thus, the transcritical bifurcations in our case are not biologically feasible.

4.2. Saddle-Node Bifurcation

As seen in Figure 2, if $\alpha_2^{*1} < \alpha_2 < \alpha_2^{*2}$, then there exists a unique coexistence equilibrium state (E_8), which is asymptotically stable (see also Figure 1b). A further observation shows that for $\alpha_2^{*2} < \alpha_2 < \alpha_2^{*3}$ there are two coexistence equilibrium states. One of them is asymptotically stable while the other one is unstable. The stability of one of these coexistence equilibrium states can also be seen in Figure 1c. Then, these two coexistence equilibrium states collide at the limit point (LP), i.e., at $\alpha_2 = \alpha_2^{*3}$. If $\alpha_2 > \alpha_2^{*3}$, then the two coexistence equilibrium states disappear and only the extinction of predator-2 equilibrium state is stable. Hence there occurs a saddle-node bifurcation driven by α_2 .

4.3. Bi-Stability

Another interesting dynamics that can be observed in Figure 2 is the appearance of two different stable equilibrium states. This interesting occurrence is known as the bi-stability phenomenon.

The bi-stability phenomenon occurs when $\alpha_2^{*2} < \alpha_2 < \alpha_2^{*3}$. In this case, both extinction of the predator-2 equilibrium state (E_6) and the coexistence equilibrium (E_8) are asymptotically stable. Here the dynamics of model (2) are very sensitive to the initial values. For some relatively large initial values of predator-2, we have a situation where the solutions converge to the coexistence equilibrium state (E_8), as shown by the black-trajectories in Figure 1c. Hence, all three species (prey, predator-1 and predator-2) can coexist. On the other hand, whenever the initial predator-2 is relatively small, then the solutions converge to the extinction of the predator-2 equilibrium state (E_6) (see blue-trajectories in Figure 1c).

5. Conclusions

We have discussed a model that describes the interaction of one prey–two predators with competition between the two predators. The model is derived from the modified Leslie–Gower predator–prey model. The model is proven to be permanent and all solutions of the model are always non-negative and bounded. We found that the proposed model has eight equilibrium states. There are two equilibrium states that are always unstable, namely the extinction of all of the species and the extinction of the two predators. Other equilibrium states are conditionally stable. We have also shown the global stability of some equilibrium states by defining suitable Lyapunov functions. The results of dynamical analysis were confirmed by our numerical simulations. Moreover, our numerical simulations have shown that the proposed model exhibits rich dynamics as we observed the occurrence of transcritical bifurcation, saddle-node bifurcation and bi-stability phenomenon, which are driven by the competition rate of predator-2 species.

Author Contributions: Conceptualization, A.S.; Data curation, D.S.; Formal analysis, D.S. and A.S.; Funding acquisition, A.S.; Investigation, D.S. and A.S.; Methodology, A.S.; Project administration, A.S.; Software, D.S. and A.S.; Supervision, A.S., W.M.K. and A.; Validation, W.M.K.; Visualization, D.S. and A.; Writing—original draft, D.S.; Writing—review & editing, A.S., W.M.K. and A. All authors have read and approved the final manuscript.

Funding: This research received no external funding.

Acknowledgments: This research and APC were funded by the Directorate of Research and Community Service, the Directorate General of Strengthening Research and Development, and the Ministry of Research, Technology and Higher Education (Brawijaya University), Indonesia, via Doctoral Dissertation Research, in accordance with the Research Contract No. 055/SP2H/LT/DRPM/2019, dated 11 March 2019. The authors thank Hasan S. Panigoro (University of Brawijaya and State University of Gorontalo) for discussion about the continuation method.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

Appendix A. Proof of Theorem 5

If we evaluate the variational matrix at E_5 , then we get

$$J(E_5) = \begin{bmatrix} -\frac{rx_5}{k} & -\beta_1x_5 & -\beta_2x_5 \\ 0 & r_1 - \alpha_1z_5 & 0 \\ \frac{r_2z_5^2}{(x_5+k_2)^2} & -\alpha_2z_5 & -\frac{r_2z_5}{x_5+k_2} \end{bmatrix}.$$

One of the eigenvalues of $J(E_5)$ is $\lambda_1 = r_1 - \alpha_1z_5$, while the others (say λ_2 and λ_3) are determined by $\lambda^2 - T_1\lambda + D_1 = 0$, where $T_1 = -\left(\frac{rx_5}{k} + \frac{r_2z_5}{x_5+k_2}\right) < 0$ and $D_1 = \left(\frac{rr_2x_5z_5}{k(x_5+k_2)} + \frac{r_2\beta_2x_5z_5^2}{(x_5+k_2)^2}\right) > 0$. Since $T_1 < 0$ and $D_1 > 0$, we have $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus, E_5 is locally asymptotically stable if $\lambda_1 = r_1 - \alpha_1z_5 = r_1 - \alpha_1\frac{r(k_2+k)}{k\beta_2+r} < 0$ or when $r_1(k\beta_2 + r) < \alpha_1r(k + k_2)$.

Appendix B. Proof of Theorem 6

The variational matrix at E_6 is

$$J(E_6) = \begin{bmatrix} -\frac{rx_6}{k} & -\beta_1 x_6 & -\beta_2 x_6 \\ \frac{r_1 y_6^2}{(x_6+k_1)^2} & -\frac{r_1 y_6}{x_6+k_1} & -\alpha_1 y_6 \\ 0 & 0 & r_2 - \alpha_2 y_6 \end{bmatrix}.$$

The eigenvalues of $J(E_6)$ are $\lambda_1 = r_2 - \alpha_2 y_6$ and the roots of $\lambda^2 - T_2 \lambda + D_2 = 0$, where $T_2 = -\left(\frac{rx_6}{k} + \frac{r_1 y_6}{x_6+k_1}\right) < 0$ and $D_2 = \left(\frac{r r_1 x_6 y_6}{k(x_6+k_1)} + \frac{r_1 \beta_1 x_6 y_6^2}{(x_6+k_1)^2}\right) > 0$. Clearly the last two eigenvalues are negative and λ_1 is negative if $r_2 < \frac{\alpha_2 r(k+k_1)}{k\beta_1+r}$. This completes the proof.

Appendix C. Proof of Theorem 7

The variational matrix evaluated at E_7 is

$$J(E_7) = \begin{bmatrix} -(\beta_1 y_7 + \beta_2 z_7 - r) & 0 & 0 \\ \frac{r_1 r_2^2 (\alpha_1 k_2 - r_1)^2}{(\alpha_1 \alpha_2 k_1 k_2 - r_1 r_2)^2} & -\frac{r_1 y_7}{k_1} & -\alpha_1 y_7 \\ \frac{r_2 r_1^2 (\alpha_2 k_1 - r_2)^2}{(\alpha_1 \alpha_2 k_1 k_2 - r_1 r_2)^2} & -\alpha_2 z_7 & -\frac{r_2 z_7}{k_2} \end{bmatrix}.$$

The first eigenvalue of $J(E_7)$ is $\lambda_1 = -(\beta_1 y_7 + \beta_2 z_7 - r)$ and other eigenvalues are determined by quadratic equation $\lambda^2 - T_0 \lambda + D_0 = 0$, where $T_0 = -\left(\frac{r_1 y_7}{k_1} + \frac{r_2 z_7}{k_2}\right) < 0$ and $D_0 = \left(\frac{r_1 r_2}{k_1 k_2} - \alpha_1 \alpha_2\right) y_7 z_7$. $T_0 < 0$, E_7 is stable if $\lambda_1 < 0$ and $D_0 > 0$, that is, when $r < \beta_1 y_7 + \beta_2 z_7$ and $r_1 r_2 > \alpha_1 \alpha_2 k_1 k_2$. Hence, we have the theorem.

Appendix D. Proof of Theorem 8

By evaluating the variational matrix at E_8 , we obtain

$$J(E_8) = \begin{bmatrix} -\frac{rx^*}{k} & -\beta_1 x^* & -\beta_2 x^* \\ \frac{r_1 (y^*)^2}{(x^*+k_1)^2} & -\frac{r_1 y^*}{x^*+k_1} & -\alpha_1 y^* \\ \frac{r_2 (z^*)^2}{(x^*+k_2)^2} & -\alpha_2 z^* & -\frac{r_2 z^*}{x^*+k_2} \end{bmatrix}.$$

The characteristic equation of $J(E_8)$ is given by $\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0$, where A_1, A_2 and A_3 are given as above. By applying the Routh–Hurwitz criteria, the real parts of all characteristic roots are negative if $A_1 > 0, A_3 > 0$, and $A_1 A_2 - A_3 > 0$. It is clear that $A_1 > 0$. Hence, E_8 is locally asymptotically stable if $A_3 > 0$, and $A_1 A_2 - A_3 > 0$.

Appendix E. Proof of Theorem 10

Assume that $\alpha_2 l_2 > r_2$ and consider the following Lyapunov function $V_2 = \left(y - k_1 - k_1 \ln \frac{y}{k_1}\right) + bz$ where $b = \frac{\alpha_1 k_1 (k+k_2)}{r_2 l_3 + (k+k_2)(\alpha_2 l_2 - r_2)} > 0$. If we take the time derivative of V_2 , then we get

$$\begin{aligned} \frac{dV_2}{dt} &= (y - k_1) \left[r_1 \left(1 - \frac{y}{x+k_1}\right) - \alpha_1 z \right] + b \left[r_2 z \left(1 - \frac{z}{x+k_2}\right) - \alpha_2 y z \right] \\ &\leq r_1 \left(1 + \frac{k_1}{x+k_1} - \frac{y}{x+k_1}\right) y + \left(br_2 + \alpha_1 k_1 - b\alpha_2 y - \frac{br_2 z}{x+k_2} \right) z \\ &\leq r_1 \left(1 + \frac{k_1}{l_1+k_1} - \frac{l_2}{k+k_1}\right) y + \left(br_2 + \alpha_1 k_1 - b\alpha_2 l_2 - \frac{br_2 l_3}{k+k_2} \right) z \end{aligned}$$

$$= r_1 \left(1 + \frac{k_1}{l_1 + k_1} - \frac{l_2}{k + k_1} \right) y.$$

It is found that if $1 + \frac{k_1}{l_1 + k_1} < \frac{l_2}{k + k_1}$ then $\frac{dV_2}{dt} \leq 0$. Moreover, $\frac{dV_2}{dt} = 0$ if and only if $x = 0$, $y = k_1$ and $z = 0$. By applying La Salle’s invariant principle, we find that the equilibrium state $E_4 = (0, k_1, 0)$ is globally asymptotically stable if $\alpha_2 l_2 > r_2$ and $1 + \frac{k_1}{l_1 + k_1} < \frac{l_2}{k + k_1}$.

Appendix F. Proof of Theorem 11

Consider the following Lyapunov function $V_3 = V_{31} + V_{32} + V_{33}$, where $V_{31} = c_1(x - x^* + x \ln \frac{x}{x^*})$, $V_{32} = (x^* + k_1)(y - y^* + y \ln \frac{y}{y^*})$ and $V_{33} = c_2(x^* + k_2)(z - z^* + z \ln \frac{z}{z^*})$. The time derivative of V_{31} , V_{32} and V_{33} along the solutions of system (2) are, respectively, as follows.

$$\begin{aligned} \frac{dV_{31}}{dt} &= c_1(x - x^*) \left(r \left(1 - \frac{x}{k} \right) - \beta_1 y - \beta_2 z \right) = c_1(x - x^*) \left(\frac{r}{k}(x^* - x) + \beta_1(y^* - y) + \beta_2(z^* - z) \right) \\ &= -\frac{c_1 r}{k}(x - x^*)^2 - c_1 \beta_1(x - x^*)(y - y^*) - c_1 \beta_2(x - x^*)(z - z^*). \\ \frac{dV_{32}}{dt} &= (x^* + k_1)(y - y^*) \left(r_1 - \frac{r_1 y}{x + k_1} - \alpha_1 z \right) \\ &= (x^* + k_1)(y - y^*) \left(\frac{r_1 y^*}{x^* + k_1} - \frac{r_1 y}{x + k_1} + \alpha_1(z^* - z) \right) \\ &= -r_1(y - y^*)^2 + \frac{r_1 y}{x + k_1}(x - x^*)(y - y^*) - \alpha_1(x^* + k_1)(y - y^*)(z - z^*). \\ \frac{dV_{33}}{dt} &= c_2(x^* + k_2)(z - z^*) \left(r_2 - \frac{r_2 z}{x + k_2} - \alpha_2 y \right) \\ &= c_2(x^* + k_2)(z - z^*) \left(\frac{r_2 z^*}{x^* + k_2} - \frac{r_2 z}{x + k_2} + \alpha_2(y^* - y) \right) \\ &= -c_2 r_2(z - z^*)^2 + \frac{c_2 r_2 z}{x + k_2}(x - x^*)(z - z^*) - c_2 \alpha_2(x^* + k_2)(y - y^*)(z - z^*). \end{aligned}$$

Hence, the derivative of V_3 with respect to t is given by

$$\begin{aligned} \frac{dV_3}{dt} &= \frac{dV_{31}}{dt} + \frac{dV_{32}}{dt} + \frac{dV_{33}}{dt} \\ &= -\frac{c_1 r}{k}(x - x^*)^2 - r_1(y - y^*)^2 - c_2 r_2(z - z^*)^2 + \left(\frac{r_1 y}{x + k_1} - c_1 \beta_1 \right) (x - x^*)(y - y^*) \\ &\quad + \left(\frac{c_2 r_2 z}{x + k_2} - c_1 \beta_2 \right) (x - x^*)(z - z^*) - (\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))(y - y^*)(z - z^*). \end{aligned}$$

We note that if $(x - x^*)(y - y^*) \geq 0$ then

$$\left(\frac{r_1 y}{x + k_1} - c_1 \beta_1 \right) (x - x^*)(y - y^*) \leq \left(\frac{r_1 l_2}{l_1 + k_1} - c_1 \beta_1 \right) (x - x^*)(y - y^*),$$

while if $(x - x^*)(y - y^*) < 0$ then $\left(\frac{r_1 y}{x + k_1} - c_1 \beta_1 \right) (x - x^*)(y - y^*) \leq \left(\frac{r_1 l_2}{k + k_1} - c_1 \beta_1 \right) (x - x^*)(y - y^*)$. Similarly, if we have $(x - x^*)(z - z^*) \geq 0$ then

$$\left(\frac{c_2 r_2 z}{x + k_2} - c_1 \beta_2 \right) (x - x^*)(z - z^*) \leq \left(\frac{c_2 r_2 l_3}{l_1 + k_2} - c_1 \beta_2 \right) (x - x^*)(z - z^*),$$

while if $(x - x^*)(z - z^*) < 0$ then $(\frac{c_2 r_2 z}{x+k_2} - c_1 \beta_2)(x - x^*)(z - z^*) \leq (\frac{c_2 r_2 l_3}{k+k_2} - c_1 \beta_2)(x - x^*)(z - z^*)$. Thus, if we take

$$c_1 = \begin{cases} \frac{r_1 L_2}{\beta_1(l_1+k_1)}, & \text{if } (x - x^*)(y - y^*) \geq 0 \\ \frac{r_1 l_2}{\beta_1(k+k_1)}, & \text{if } (x - x^*)(y - y^*) < 0 \end{cases}$$

and

$$c_2 = \begin{cases} \frac{c_1 \beta_2(l_1+k_2)}{r_2 L_3}, & \text{if } (x - x^*)(z - z^*) \geq 0 \\ \frac{c_1 \beta_2(k+k_2)}{r_2 l_3}, & \text{if } (x - x^*)(z - z^*) < 0, \end{cases}$$

then we obtain

$$\begin{aligned} \frac{dV_3}{dt} &\leq -\frac{c_1 r}{k}(x - x^*)^2 - r_1(y - y^*)^2 - c_2 r_2(z - z^*)^2 - (\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))(y - y^*)(z - z^*) \\ &\leq -\frac{c_1 r}{k}(x - x^*)^2 - r_1(y - y^*)^2 - c_2 r_2(z - z^*)^2 + (\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))\left(\frac{(y - y^*)^2 + (z - z^*)^2}{2}\right) \\ &= -\frac{c_1 r}{k}(x - x^*)^2 - \left(r_1 - \frac{1}{2}(\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))\right)(y - y^*)^2 \\ &\quad - \left(c_2 r_2 - \frac{1}{2}(\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))\right)(z - z^*)^2. \end{aligned}$$

It is obvious that if $r_1 \geq \frac{1}{2}(\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))$ and $r_2 \geq \frac{1}{2c_2}(\alpha_1(x^* + k_1) + c_2 \alpha_2(x^* + k_2))$ then $\frac{dV_3}{dt} \leq 0$. We can also check that $\frac{dV_3}{dt} = 0$ if and only if $x = x^*, y = y^*$ and $z = z^*$. From La Salle’s invariant principle, it is concluded that the equilibrium state $E_8 = (x^*, y^*, z^*)$ is globally asymptotically stable.

References

1. Berryman, A.A. The origins and evolution of predator-prey theory. *Ecology* **1992**, *73*, 1530–1535. [CrossRef]
2. Leslie, P.H.; Gower, J.C. The properties of a stochastic model for the predator-prey type of interaction between two species. *Biometrika* **1960**, *47*, 219–234. [CrossRef]
3. Aziz-Alaoui, M.A.; Okiye, M.D. Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes. *Appl. Math. Lett.* **2003**, *16*, 1069–1075. [CrossRef]
4. Yu, S. Global asymptotic stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes. *Discrete Dyn. Nat. Soc.* **2012**, *2012*, 208167. [CrossRef]
5. Yu, S. Global asymptotic stability a modified Leslie-Gower with Beddington-DeAngelis functional response. *Adv. Differ. Equ.* **2014**, *84*, 1–14. [CrossRef]
6. Zhang, N.; Chen, F.; Su, Q.; Wu, T. Dynamics behaviors of a harvesting Leslie-Gower predator-prey model. *Discrete Dyn. Nat. Soc.* **2011**, *2011*, 473949. [CrossRef]
7. Gupta, R.P.; Chandra, P. Bifurcation analysis of modified Leslie-Gower predator prey model with Michaelis-Menten type prey harvesting. *J. Math. Anal. Appl.* **2013**, *398*, 278–295. [CrossRef]
8. Darti, I.; Suryanto, A. Stability preserving non-standard finite difference scheme for a harvesting Leslie-Gower predator-prey model. *J. Differ. Equ. Appl.* **2015**, *21*, 528–534. [CrossRef]
9. Darti, I.; Suryanto, A. Dynamics preserving nonstandard Finite difference method for the modified Leslie-Gower predator-prey model with Holling type II functional responses. *Far East. J. Math. Sci.* **2016**, *998*, 719–733. [CrossRef]
10. Cai, Y.; Zhao, C.; Wang, W.; Wang, J. Dynamics of Leslie-Gower predator-prey model with additive Allee effect. *Appl. Math. Model.* **2015**, *39*, 2092–2106. [CrossRef]
11. Crow, Y.; Jang, S. Asymptotic dynamics of the Leslie-Gower competition system with Allee effects and stocking. *Adv. Differ. Equ.* **2015**, *2*, 1–19. [CrossRef]
12. Pal, P.J.; Mandal, P.K. Bifurcation analysis of a modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and strong Allee effect. *Math. Comput. Simulat.* **2014**, *97*, 123–146. [CrossRef]

13. Indrajaya, D.; Suryanto, A.; Alghofari, A.R. Dynamics of modified Leslie-Gower predator-prey model with Beddington-DeAngelis functional response and additive Allee effect. *Int. J. Ecol. Dev.* **2003**, *31*, 60–71.
14. Kar, T.K.; Batabyal, A. Persistence and stability of a two prey one predator system. *Int. J. Eng. Sci. Technol.* **2010**, *2*, 174–190. [[CrossRef](#)]
15. Tripathi, J.P.; Abbas, S.; Thakur, M. Local and global stability analysis of a two prey one predator model with help. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 3284–3297. [[CrossRef](#)]
16. Deka, B.D.; Patra, A.; Tushar, J.; Dubey, B. Stability and Hopf-bifurcation in a general Gauss type two-prey and one-predator system. *Appl. Math. Model.* **2016**, *40*, 5793–5818. [[CrossRef](#)]
17. Reddy, K.S.; Srinivas, M.N.; Kumar, C.V.P. Dynamics of a two prey and one predator system with time interruption and random fluctuations. *Pac. Sci. Rev. A Natur. Sci. Eng.* **2016**, *18*, 150–156. [[CrossRef](#)]
18. Raymond, C.; Hugo, A.; Kung'aro, M. Modeling dynamics of prey-predator fishery model with harvesting: A bioeconomic model. *J. Appl. Math.* **2019**, *2019*, 2601648. [[CrossRef](#)]
19. Sarwandi, S.; Mandal, P.K.; Ray, S. Dynamical behaviour of a two-predator model with prey refuge. *J. Biol. Phys.* **2013**, *39*, 701–722. [[CrossRef](#)]
20. Sayekti, I.M.; Malik, M.; Aldila, D. One-Prey Two-Predator Model with Prey Harvesting in a Food Chain Interaction. *AIP Conf. Proc.* **2017**, *1862*, 030124. [[CrossRef](#)]
21. Alebraheem, J.; Abu-Hasan, Y. Persistence of predator in a two predator-one prey model with non-periodic solution. *Appl. Math. Sci.* **2012**, *6*, 943–956.
22. Alebraheem, J.; Abu-Hasan, Y. Dynamics of a two predator–one prey system. *Comput. Appl. Math.* **2014**, *33*, 767–780. [[CrossRef](#)]
23. Alebraheem, J.; Abu-Hasan, Y. The effects of capture efficiency on the coexistence of a predator in a two predators—One prey model. *J. Appl. Sci.* **2011**, *11*, 3717–3724. [[CrossRef](#)]
24. Savitri, D.; Suryanto, A.; Kusumawinahyu, W.M.; Abadi. A dynamics behaviour of two predators and one prey interaction with competition between predators. *IOP Conf. Ser. Mater. Sci. Eng.* **2019**, *546*, 052069. [[CrossRef](#)]
25. Chen, F.D. On a nonlinear non-autonomous predator-prey model with diffusion and distributed delay. *J. Comput. Appl. Math.* **2005**, *180*, 33–49. [[CrossRef](#)]



© 2020 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<http://creativecommons.org/licenses/by/4.0/>).